Vortex structure in p-wave superconductors

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Results obtained with: S. Alama, X. Lamy

Ginzburg–Landau models

- Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain. Here we take $n = 2$.
- For order parameter (or wave function) $\eta \in H^1(\Omega; \mathbb{C}^k)$, we define the Ginzburg–Landau energy

$$E(\eta) = \int_\Omega \left\{ e_{kin}(\nabla\eta) + \kappa^2 e_{pot}(\eta) \right\} dx$$

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- κ is the Ginzburg–Landau parameter (large.)
- Expect minimizing η to take values in Σ, while minimizing the gradient energy.
- For topological reasons, this may not be possible, so in the limit $\kappa \to \infty$ singularities are formed: Vortices!

The *p*-wave functional, I

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Used to describe Sr-Ru superconductors with ferromagnetic "spin triplet" interactions. *[Sigrist, Heeb-Agterberg, Ashby-Kallin]*

$$e_{kin}(\eta) = |\nabla\eta_+|^2 + |\nabla\eta_-|^2 + (\Pi_-\eta_+ \cdot \Pi_+\eta_-) + \nu\,(\Pi_+\eta_+ \cdot \Pi_-\eta_-)$$

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- Note: broken charge symmetry $\eta \not\leftrightarrow \overline{\eta}$!

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- $\Sigma = (\mathbb{S}^1 \times 0) \cup (0 \times \mathbb{S}^1)$, so expect:
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  - (or vice-versa)

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  - $|\eta_-(x)| \simeq 1$ "dominant" component, with quantized degree;
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  - (or vice-versa)
  - Seek entire solutions on $\mathbb{R}^2$ with prescribed degree in "dominant" η_-, to describe the core structure of vortices.

Euler–Lagrange equations

Recall operators $\Pi_+ = \partial_x + i\partial_y$, $\Pi_- = -\Pi_+^* = \partial_x - i\partial_y$.
Critical points of E satisfy:

$$\left.\begin{array}{l} 2\Delta\eta_- + [\Pi_-^2 + \nu\Pi_+^2]\eta_+ = \kappa^2\left(2\eta_-(|\eta_-|^2 - 1) + 4\eta_-|\eta_+|^2 + 2\nu\overline{\eta}_-\eta_+^2\right) \\ 2\Delta\eta_+ + [\Pi_+^2 + \nu\Pi_-^2]\eta_- = \kappa^2\left(2\eta_+(|\eta_+|^2 - 1) + 4\eta_+|\eta_-|^2 + 2\nu\overline{\eta}_+\eta_-^2\right) \end{array}\right\}$$

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- Note: we do not expect half-trivial solutions, $\eta = (\eta_+, 0)$ or $(0, \eta_-)$!!

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- The system is elliptic, in the sense of *Legendre–Hadamard.*
  - Write $\eta = (u_1, u_2, u_3, u_4)$, real vector.
  - The left-hand side may be written in operator form:

$$(\mathcal{L}\eta)_\alpha = \sum_{i,j=1}^{2}\sum_{\alpha,\beta=1}^{4} A_{\alpha\beta}^{ij}\partial_i\partial_j u^\beta$$

with $$\sum_{i,j,\alpha,\beta} A_{\alpha\beta}^{ij}\xi_i\xi_j\tau^\alpha\tau^\beta \geq c|\xi|^2|\tau|^2, \quad \forall\xi\in\mathbb{R}^2, \tau\in\mathbb{R}^4.$$

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- If solutions exist, they are smooth.

The Dirichlet Problem

Recall: $e_{kin}(\eta, A) = |\nabla \eta_+|^2 + |\nabla \eta_-|^2 + (\Pi_- \eta_+ \cdot \Pi_+ \eta_-) + \nu (\Pi_+ \eta_+ \cdot \Pi_- \eta_-)$

Let $\Omega \subset \mathbb{R}^2$ bounded domain, $g_\pm \in H^{1/2}(\partial\Omega)$ given, and

$$W := \{\eta \in H^1(\Omega; \mathbb{C}^2) : \ \eta_\pm = g_\pm \text{ on } \partial\Omega\}.$$

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Theorem

Assume $W \cap Z = \emptyset$. Then there exists a minimizer of $E(\eta)$ in W, which is a smooth solution of the EL system with the given Dirichlet BC.

In particular, in $\Omega = B_R$, $\exists$ solutions with any given degrees $n_\pm \in \mathbb{Z}$,

$$\eta_\pm = \alpha_\pm e^{in_\pm\theta} \text{ on } \partial B_R,$$

provided $\alpha_- \neq -\alpha_+$ or one of $n_\pm \neq \pm1$, eg, for Σ-valued BC!

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$$\begin{aligned} E(\eta) - E(\omega \cdot \eta) = & \int \left[(1-\omega^{n_+-n_--2})\Pi_-\eta_+\right] \cdot (\Pi_+\eta_-) \\ & + \nu \int \left[(1-\omega^{n_+-n_-+2})\Pi_+\eta_+\right] \cdot (\Pi_-\eta_-) \\ & + \kappa^2\nu \int \left[(1-\omega^{2(n_+-n_-)})\eta_+^2\right] \cdot (\eta_-^2). \end{aligned}$$

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- Energy invariant iff $\nu = 0$ and $n_+ = n_- + 2$.

When $\nu \neq 0$, solutions are expected to have square symmetry.

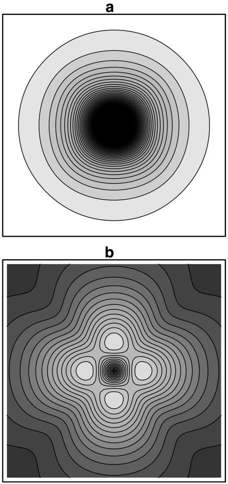

FIG. 2. Contour plot of GL calculations for the absolute values of the dominant η_- (a) and the admixed η_+ component (b) for the parameters κ=2.5 and ν=−0.3. The contours are 0.99, 0.975, . . . for (a) and 0.03, 0.045, . . . ,0.225 for (b).

Heeb–Agterberg, Phys Rev B59, 1999

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So we assume $\nu = 0$, $n_+ = n_- + 2$, with ansatz:

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- Solve in B_R (existence with Σ-valued Dirichlet BC!), and let $R \to \infty$.

$$\begin{aligned}
&\Delta_r f_- - \frac{n_-^2}{r^2} f_- + \frac{1}{2}\left(\Delta_r f_+ + \frac{n_-(n_- + 2)}{r^2} f_+ + 2\frac{n_- + 1}{r} f_+' \right) \\
&\qquad = f_-(|f_-|^2 - 1) + 2 f_- f_+^2, \\
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&f_-(R) = 1,\ f_+(R) = 0.
\end{aligned}$$

Vortex solutions, II

So we assume $\nu = 0$, $n_+ = n_- + 2$, with ansatz:

$$\eta_\pm = f_\pm(r)e^{in_\pm\theta}, \text{ with } f_+(r) \to 0,\ f_-(r) \to 1,\ r \to \infty.$$

- As in classical G-L, energy of nontrivial entire solutions diverges.
- Solve in B_R (existence with Σ-valued Dirichlet BC!), and let $R \to \infty$.

$$\begin{aligned}
&\Delta_r f_- - \frac{n_-^2}{r^2} f_- + \frac{1}{2}\left(\Delta_r f_+ + \frac{n_-(n_-+2)}{r^2} f_+ + 2\frac{n_-+1}{r} f_+'\right) \\
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- Crucial estimate via Pohozaev identity; available when $n_- = -1, n_+ = +1$ only.

Case $\nu = 0$ and $n_- = -1, n_+ = +1$

Degrees $n_- = -1, n_+ = +1$ are most relevant for the physics (expected stability) and make the equations much more tractable:

$$\Delta_r f_- - \frac{1}{r^2} f_- + \frac{1}{2}\left(\Delta_r f_+ - \frac{1}{r^2} f_+\right) = f_-(|f_-|^2 - 1) + 2 f_- f_+^2,$$
$$\Delta_r f_+ - \frac{1}{r^2} f_+ + \frac{1}{2}\left(\Delta_r f_- - \frac{1}{r^2} f_-\right) = f_+(|f_+|^2 - 1) + 2 f_+ f_-^2,$$

with $f_-(r) \to 1,\ f_+(r) \to 0$ as $r \to \infty$.

Theorem

There exists a smooth entire equivariant solution $\eta = (\eta_-, \eta_+) = (f_-(r)e^{-i\theta}, f_+(r)e^{+i\theta})$ *with* $f_-(r) \to 1$ *and* $f_+(r) \to 0$ *as* $r \to \infty$. *Moreover, as* $r \to +\infty$,

$$f_- = 1 - \frac{1}{2r^2} - \frac{7}{4r^4} + O(r^{-6}), \qquad f_+ = -\frac{1}{2r^2} - \frac{13}{4r^4} + O(r^{-6}). \tag{1}$$

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- If so, then we can show $|\eta_-|^2 + |\eta_+|^2 \leq 1$, as is expected on physical grounds. But this does not follow from standard variational or maximum principle arguments.

Perturbative approach

We introduce a parameter $t \in [0,1]$, to connect our system to G-L. (See also Han–Lin, Kim–Phillips)

For equivariant solutions $\eta_- = f_-(r)e^{-i\theta}$, $\eta_+ = f_+(r)e^{i\theta}$, E-L system:

$$\Delta_r f_- - \frac{1}{r^2} f_- + \frac{t}{2}(\Delta_r f_+ - \frac{1}{r^2} f_+) = f_-(f_-^2 - 1) + 2 f_- f_+^2,$$
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- At $t = 0$, f^0 is nondegenerate (linearly stable) [Uses Mironescu '95]
- Implicit Function Theorem plus sharp asymptotic estimates yields result:

Theorem

There exists t_0 such that for all $t \in (0, t_0)$ there exist smooth bounded solutions (f_-^t, f_+^t) such that:

(a) $f_-^t(0) = 0 = f_+^t(0)$;

(b) $f_-^t(r) \to 1$, $f_+^t(r) \to 0$ *as* $r \to \infty$;

(c) $0 < f_-^t(r) < 1$, $f_+^t(r) < 0$ *for all* $r \in (0, \infty)$;

(d) *As* $r \to \infty$,

$$f_-^t = 1 - \frac{1}{2r^2} - \frac{5t^2 + 9}{8r^4} + O(r^{-6}), \qquad f_+^t = t\left[-\frac{1}{2r^2} - \frac{13}{4r^4} + O(r^{-6})\right].$$

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Many, many questions remain open!