Predicting Extreme Events for Passive Scalar Turbulence through Reduced-Order Models

Di Qi and Andrew Majda

Center for Atmosphere Ocean Science
Courant Institute of Mathematical Sciences
New York University

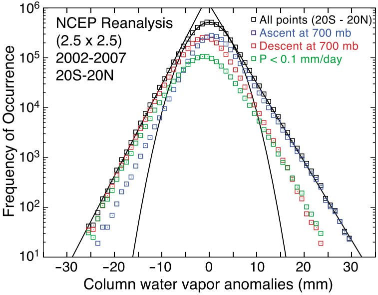
SIAM Conference on Applications of Dynamical Systems

May 24, 2017

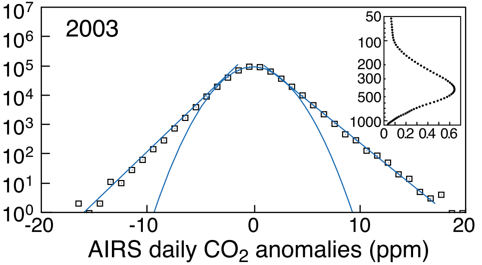
Long tails of natural tracers from observation

Intermittency is a key feature of turbulent flows and correctly taking it into account in turbulence models is of critical importance.

- Passive tracer distributions exhibit approximately exponential tails.
- Such tails are indeed ubiquitous in observation, model, and reanalysis data sets for a variety of tracers.[1]



(a) Distribution of water vapor anomalies



(b) Distribution of CO2 anomalies

[1] *Neelin et al, Geophys. Res. Lett., 2010*

Simplified passive scalar models

- **Simplified models in tracer turbulence:**
  - Understanding the fundamental mechanisms that can potentially lead to such behavior is a question of great theoretical interest.
  - The analysis of simple systems with a Gaussian core can shed some light on similar behaviors observed in more complex models.

- **Major Questions**:
  - What structure is needed for a velocity field so that the PDF for a passive scalar exhibits a transition from a Gaussian PDF to a broader than Gaussian shape.
  - How to develop explicit reduced-order models with unambiguous behavior for intermittency of scalar PDFs.

Outline

1. Turbulent diffusion models with mean gradient formulation
   - Elementary models with intermittency for passive scalars
   - Rigorous intermittency in a random resonance regime
2. Two-layer baroclinic turbulence as advection flow
   - Two-layer baroclinic turbulence in ocean and atmosphere regimes
   - A reduced-order stochastic model with consistency and sensitivity
3. Predicting intermittent PDFs with low-order stochastic models
   - Model calibration in Gaussian velocity field
   - Prediction skill of low-order stochastic models

Outline

1. Turbulent diffusion models with mean gradient formulation
   - Elementary models with intermittency for passive scalars
   - Rigorous intermittency in a random resonance regime
2. Two-layer baroclinic turbulence as advection flow
   - Two-layer baroclinic turbulence in ocean and atmosphere regimes
   - A reduced-order stochastic model with consistency and sensitivity
3. Predicting intermittent PDFs with low-order stochastic models
   - Model calibration in Gaussian velocity field
   - Prediction skill of low-order stochastic models

Passive tracer with a mean gradient

Passive tracer of turbulent advection, diffusion, and damping

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = -d_T T + \kappa \Delta T.$$

- The tracer field is assumed to have a background mean gradient

$$T(\mathbf{x}, t) = \alpha_1 x + \alpha_2 y + T'(x, y, t).$$

- The model for the velocity field is two dimensional and periodic in space

$$\mathbf{v}(\mathbf{x}, t) = (U(t), v(x, t)),$$

  with the *cross-sweep*, $U(t)$, and the *shear flow*, $v(x,t)$.
- $T'(x,t)$ denotes fluctuations around the mean gradient

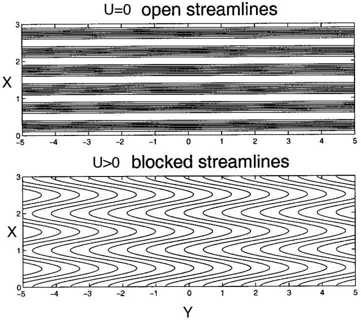
$$\frac{\partial T'}{\partial t} + U(t)\frac{\partial T'}{\partial x} = -\alpha v(x,t) - d_T T' + \kappa \Delta T'.$$

Elementary model with intermittency[2]

It is the time modulation in the transverse sweep, more precisely the fact that *the sweep crosses zero periodically*, that leads to the intermittency.

$$\frac{\partial T'}{\partial t} + \mathrm{Pe}U(t)\frac{\partial T'}{\partial x} - \frac{\partial^2 T'}{\partial x^2} = -\alpha \mathrm{Pe} v(x,t), \quad \boxed{\mathrm{Pe} = VL/\kappa}.$$

$$T = \alpha y + T', \quad U(t) = \sin \omega t, \quad v(x,t) = \sin 2\pi x.$$



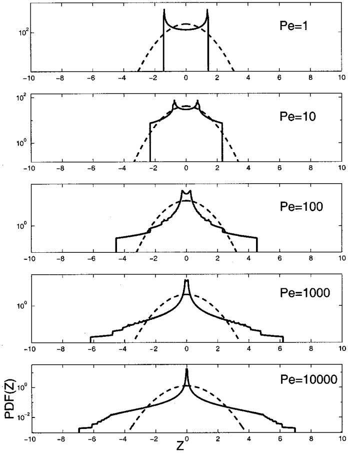
- when $U = 0$, the open streamlines in the horizontal direction, along the mean gradient αy, promote strong mixing by diffusion.
- when $U \neq 0$, the transverse sweep corresponds to blocked streamlines, little transport along the gradient, little opportunity for mixing by diffusion.

[2] *Bourlioux & Majda, Physics of Fluids, 2002*

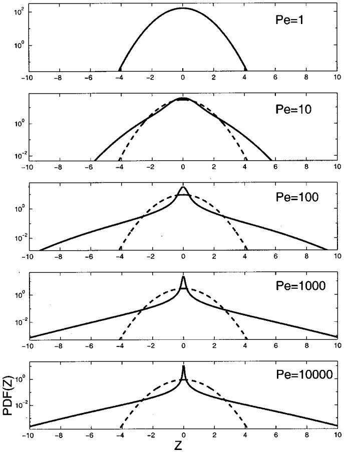
Elementary model with intermittency

$$\frac{\partial T'}{\partial t} + \mathrm{Pe}\,U(t)\,\frac{\partial T'}{\partial x} - \frac{\partial^2 T'}{\partial x^2} = -\alpha \mathrm{Pe}\, v(x,t),$$

$$i)\; v = \sin 2\pi x, \qquad ii)\; v = \xi(t),\; \langle \xi(t)\,\xi(0)\rangle = R(t).$$



(c) deterministic



(d) random Gaussian

Outline

1. Turbulent diffusion models with mean gradient formulation
   - Elementary models with intermittency for passive scalars
   - Rigorous intermittency in a random resonance regime
2. Two-layer baroclinic turbulence as advection flow
   - Two-layer baroclinic turbulence in ocean and atmosphere regimes
   - A reduced-order stochastic model with consistency and sensitivity
3. Predicting intermittent PDFs with low-order stochastic models
   - Model calibration in Gaussian velocity field
   - Prediction skill of low-order stochastic models

The two-layer flow as advection flow

The **two-layer quasi-geostrophic model** is one simple but fully nonlinear fluid model capable in capturing the essential physics.

Two-layer model

$$\frac{\partial q_\psi}{\partial t}+J\left(\psi,q_\psi\right)+J\left(\tau,q_\tau\right)+\beta\frac{\partial\psi}{\partial x}+U\frac{\partial\Delta\tau}{\partial x}=-\frac{\kappa}{2}\Delta\left(\psi-\tau\right)-\nu\Delta^s q_\psi,$$

$$\frac{\partial q_\tau}{\partial t}+J\left(\psi,q_\tau\right)+J\left(\tau,q_\psi\right)+\beta\frac{\partial\tau}{\partial x}+U\frac{\partial}{\partial x}\left(\Delta\psi+k_d^2\psi\right)=-\frac{\kappa}{2}\Delta\left(\tau-\psi\right)-\nu\Delta^s q_\tau.$$

barotropic and baroclinic modes:

$$q_\psi=\Delta\psi,\ \psi=\frac{1}{2}\left(\psi_1+\psi_2\right),$$

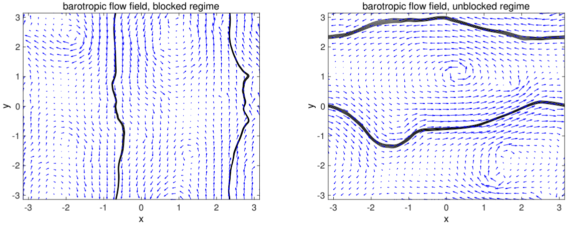
$$q_\tau=\Delta\tau-k_d^2\tau,\ \tau=\frac{1}{2}\left(\psi_1-\psi_2\right).$$

energy and heat flux:

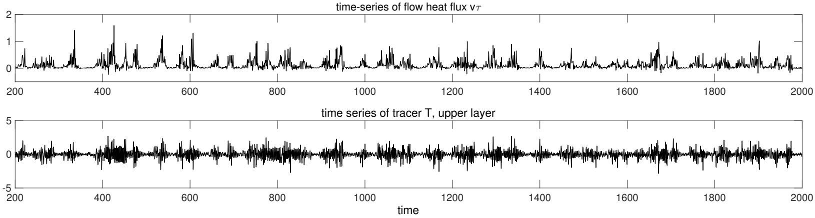
$$E=\frac{1}{2}\int\left|\nabla\psi\right|^2+\left|\nabla\tau\right|^2+k_d^2\tau^2,$$

$$H_f=k_d^2U\int v\tau.$$

Illustration about tracer intermittency with heat flux



(a) blocked regime (b) unblocked regime



(c) time-series of heat flux and scalar tracer

Stochastic formulations for the passive tracer and turbulent flow field

- **reduced-order advection flow equations** $\vec{q}_{\mathbf{k}} = \left(\hat{q}_{1,\mathbf{k}}, \hat{q}_{2,\mathbf{k}}\right)^T$

$$d\vec{q}_{M,\mathbf{k}} = -\left(\gamma_{q,\mathbf{k}} + i\omega_{q,\mathbf{k}}\right)\vec{q}_{M,\mathbf{k}}dt - D_{q,\mathbf{k}}^{M}\vec{q}_{M,\mathbf{k}}dt + \Sigma_{q,\mathbf{k}}^{M}d\vec{W}_{q,\mathbf{k}}, \quad 1 \leq |\mathbf{k}| \leq M.$$

$$\mathbf{v}_M = \nabla^{\perp}\vec{\psi}_M, \quad \vec{q}_{M,\mathbf{k}} = H_{\mathbf{k}}\vec{\psi}_{M,\mathbf{k}},$$

$$\boxed{-\frac{1}{2}\sum_{\mathbf{m}+\mathbf{n}=\mathbf{k}}\left(A_{\mathbf{km}}\vec{q}_{\mathbf{m}} \circ \vec{q}_{\mathbf{n}} + A_{\mathbf{kn}}\vec{q}_{\mathbf{n}} \circ \vec{q}_{\mathbf{m}}\right) \rightarrow -D_{q,\mathbf{k}}^{M}\vec{q}_{M,\mathbf{k}}dt + \Sigma_{q,\mathbf{k}}^{M}d\vec{W}_{q,\mathbf{k}}.}$$

- **reduced-order passive tracer equations** $\vec{T}_{\mathbf{k}} = \left(\hat{T}_{1,\mathbf{k}}, \hat{T}_{2,\mathbf{k}}\right)^T$

$$d\vec{T}_{M,\mathbf{k}} + \left(\mathbf{v}_M \cdot \nabla\vec{T}_M\right)_{\mathbf{k}} d\tilde{t} = \alpha\Gamma_{\mathbf{k}}\vec{\psi}_{M,\mathbf{k}}d\tilde{t} - \left(\gamma_{T,\mathbf{k}} + i\omega_{T,\mathbf{k}}\right)\vec{T}_{M,\mathbf{k}}d\tilde{t}, \quad 1 \leq |\mathbf{k}| \leq M.$$

$$\mathbf{v}_M = \sum_{|\mathbf{k}| \leq M_1} i\mathbf{k}^{\perp}\vec{\psi}_{M,\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Model calibration in the advection flow field

- The imperfect model calibration parameters $\left(D_q^M, \Sigma_q^M\right)$ need properly reflect the true nonlinear energy mechanism.
- The statistical equations for the fluctuation covariance matrix $R_{\mathbf{k}}^q = \left\langle \vec{q}_{\mathbf{k}} \vec{q}_{\mathbf{k}}^* \right\rangle$ become[4]

$$\frac{dR_{\mathbf{k}}^q}{dt} + Q_F^q = \left(\mathscr{L}_{\mathbf{k}}^q + \mathscr{D}_{\mathbf{k}}^q\right) R_{\mathbf{k}}^q + R_{\mathbf{k}}^q \left(\mathscr{L}_{\mathbf{k}}^q + \mathscr{D}_{\mathbf{k}}^q\right)^*, \quad |\mathbf{k}| \leq N,$$
$$Q_F^q(\vec{q}_{\mathbf{k}}) = \frac{1}{2} \sum_{\mathbf{m}+\mathbf{n}=\mathbf{k}} \left\langle \left(A_{\mathbf{km}} \vec{q}_{\mathbf{m}} \circ \vec{q}_{\mathbf{n}} + A_{\mathbf{kn}} \vec{q}_{\mathbf{n}} \circ \vec{q}_{\mathbf{m}}\right) \vec{q}_{\mathbf{k}}^* \right\rangle.$$

- In the statistical steady state, $dR_{\mathbf{k}}^q/dt = 0$, then the nonlinear fluxes can be calculated at equilibrium as $t \to \infty$

$$Q_{F,\mathrm{eq}}^q = \left(\mathscr{L}_{\mathbf{k}}^q + \mathscr{D}_{\mathbf{k}}^q\right) R_{\mathbf{k},\mathrm{eq}}^q + R_{\mathbf{k},\mathrm{eq}}^q \left(\mathscr{L}_{\mathbf{k}}^q + \mathscr{D}_{\mathbf{k}}^q\right)^*.$$

[4] *Sapsis & Majda, Physica D, 2013*

Imperfect model correction from equilibrium statistics and additional terms

- the first proposal for the linear damping and Gaussian random noise correction can be introduced as

$$D_{q,\mathbf{k}}^{\mathrm{eq}} = -\frac{1}{2} Q_{F,\mathrm{eq},\mathbf{k}}^{q,-} \left(R_{\mathbf{k},\mathrm{eq}}^{q} \right)^{-1}, \quad \Sigma_{q,\mathbf{k}}^{\mathrm{eq}} = \left(Q_{F,\mathrm{eq},\mathbf{k}}^{q,+} \right)^{1/2};$$

- a further correction for the noise and damping with a simple constant damping and noise

$$Q_{M,\mathbf{k}}^{\mathrm{add}} = -D_{M}^{\mathrm{add}} R_{M,\mathbf{k}} + \left(\Sigma_{M,\mathbf{k}}^{\mathrm{add}} \right)^{2}, \quad D_{M}^{\mathrm{add}} = \begin{bmatrix} d_M + i\omega_M & \\ & d_M - i\omega_M \end{bmatrix}.$$

- Combining the ideas, we propose the additional damping and noise corrections for the reduced-order flow vorticity mode

$$D_{q,\mathbf{k}}^{M} = -\frac{1}{2} Q_{F,\mathrm{eq},\mathbf{k}}^{q,-} \left(R_{\mathbf{k},\mathrm{eq}}^{q} \right)^{-1} - D_{M}^{\mathrm{add}}, \quad \Sigma_{q,\mathbf{k}}^{M} = \left(Q_{F,\mathrm{eq},\mathbf{k}}^{q,+} + \left(\Sigma_{M,\mathbf{k}}^{\mathrm{add}} \right)^{2} \right)^{1/2}.$$

Outline

1. Turbulent diffusion models with mean gradient formulation
   - Elementary models with intermittency for passive scalars
   - Rigorous intermittency in a random resonance regime
2. Two-layer baroclinic turbulence as advection flow
   - Two-layer baroclinic turbulence in ocean and atmosphere regimes
   - A reduced-order stochastic model with consistency and sensitivity
3. Predicting intermittent PDFs with low-order stochastic models
   - Model calibration in Gaussian velocity field
   - Prediction skill of low-order stochastic models

Gaussian velocity stochastic models for passive tracer statistics

- Apply the Galerkin truncation strategy by resolving only the first M leading modes in spectral space

$$\frac{d\hat{u}_k^M}{dt} = \left(-\gamma_{u_k} + i\omega_{u_k}\right)\hat{u}_k^M + \sigma_{u_k}\dot{W}_k, \quad |k| \le M.$$

- One of the simplest and most direction way to estimate the undetermined coefficients $\gamma_{u_k}, \omega_{u_k}, \sigma_{u_k}$ is through the mean stochastic model (MSM):

$$E_k \equiv \mathrm{var}\left(\hat{u}_k(t)\right) = \left\langle \left|\hat{u}_k(t) - \langle\hat{u}_k\rangle\right|^2 \right\rangle,$$
$$R_k = \int \mathscr{R}_k(t) \equiv \int \frac{\left\langle \left(\hat{u}_k(\tau) - \langle\hat{u}_k\rangle\right)\left(\hat{u}_k(\tau+t) - \langle\hat{u}_k\rangle\right)^* \right\rangle}{\mathrm{var}\left(\hat{u}_k(\tau)\right)}.$$

Spectral Information criterion for improving imperfect model prediction skill

A natural way of measuring the lack of information is the relative entropy

$$\mathscr{P}\left(\pi,\pi^{M}\right)=\int\pi\log\frac{\pi}{\pi^{M}}.$$

Khinchin's formula: if the autocorrelation function $\mathscr{R}(t)$ is smooth and rapid-decay,

$$u(t)=\int_{-\infty}^{\infty}e^{i\lambda t}\hat{Z}(d\lambda),\quad \mathscr{R}(t)=\int_{-\infty}^{\infty}e^{i\lambda t}dF(\lambda)=\int_{-\infty}^{\infty}e^{i\lambda t}E(\lambda)d\lambda,$$

Energy spectrum can be represented by $E(\lambda)$ or $dF(\lambda)$

$$dF(\lambda)=E(\lambda)d\lambda=\mathbb{E}\left|\hat{Z}(d\lambda)\right|^{2}.$$

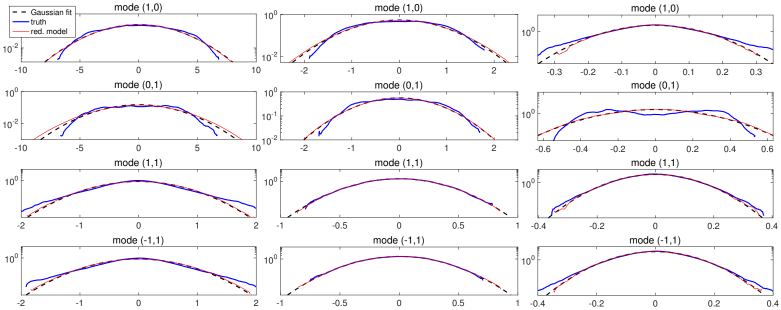
Spectral information criterion

A) $\max\left|\mathscr{R}(t)-\mathscr{R}^{M}(t)\right|\leq2\sqrt{3}\left(\int\left(E^{2}(\lambda)+E_{M}^{2}(\lambda)\right)d\lambda\right)^{1/2}\mathscr{P}\left(E(\lambda),E^{M}(\lambda)\right)^{1/2};$

B) $\int\left|\mathscr{R}(t)-\mathscr{R}^{M}(t)\right|^{2}dt\leq12\max\left|E^{2}(\lambda)+E_{M}^{2}(\lambda)\right|\mathscr{P}\left(E(\lambda),E^{M}(\lambda)\right).$

PDFs in stream functions of the advection flow

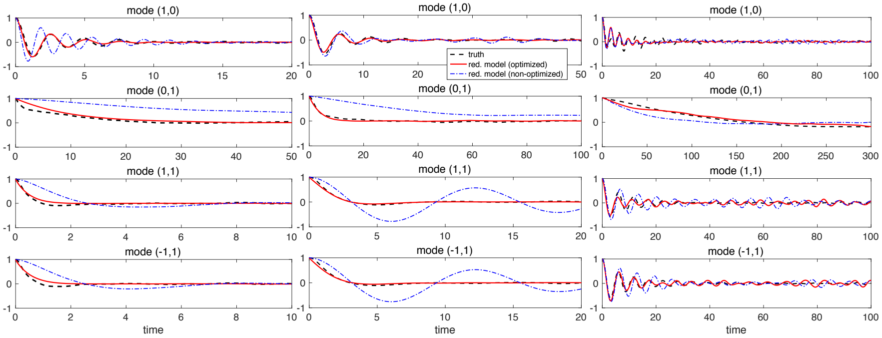
regime	N	β	k_d	U	r	d_T	κ	α
ocean, high lat.	128	2	40	0.1	0.1	0.5	0.001	1
atmosphere, high lat.	128	2	4	0.2	0.2	0.1	0.001	1
atmosphere, mid lat.	128	1	4	0.2	0.1	0.1	0.001	1



(a) high lat. ocean (b) high lat. atmosphere (c) mid lat. atmosphere

Autocorrelation functions for representative modes

$$\mathscr{R}_k(t) = \frac{\left\langle (\hat{u}_k(\tau) - \langle \hat{u}_k \rangle)(\hat{u}_k(\tau+t) - \langle \hat{u}_k \rangle)^* \right\rangle}{\mathrm{var}(\hat{u}_k(\tau))}.$$



(a) high lat. ocean (b) high lat. atmosphere (c) mid lat. atmosphere

Skill of stochastic models in two-layer QG

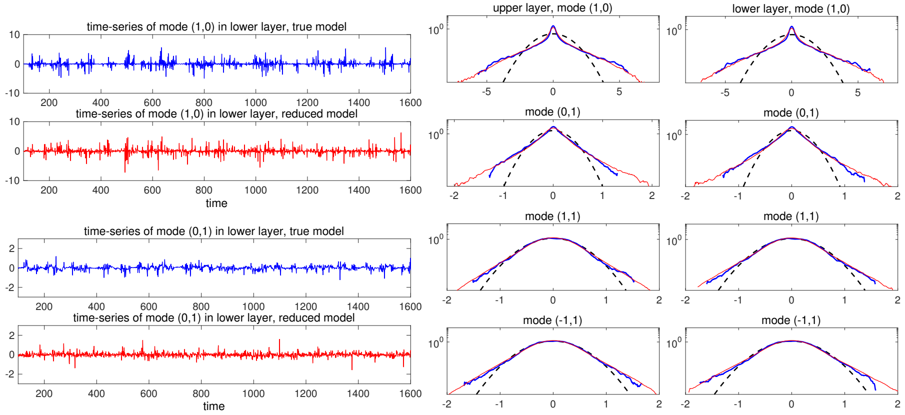


Figure: Prediction about tracer intermittency in high latitude ocean regime.

Skill of stochastic models in two-layer QG

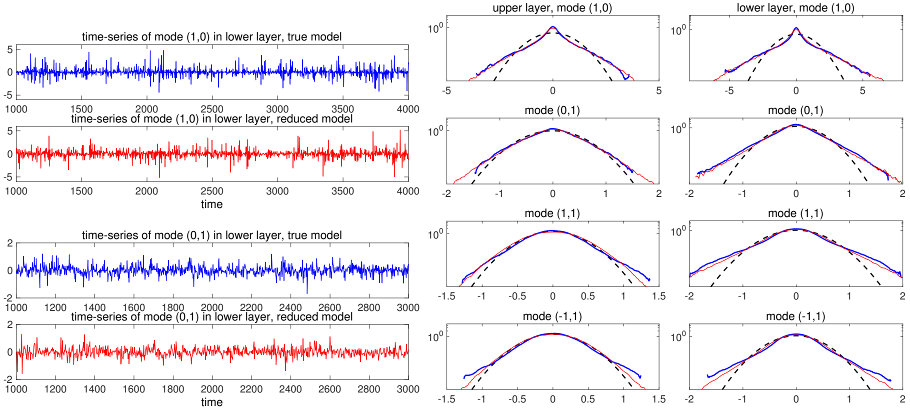


Figure: Prediction about tracer intermittency in high latitude atmosphere regime.

Skill of stochastic models in two-layer QG

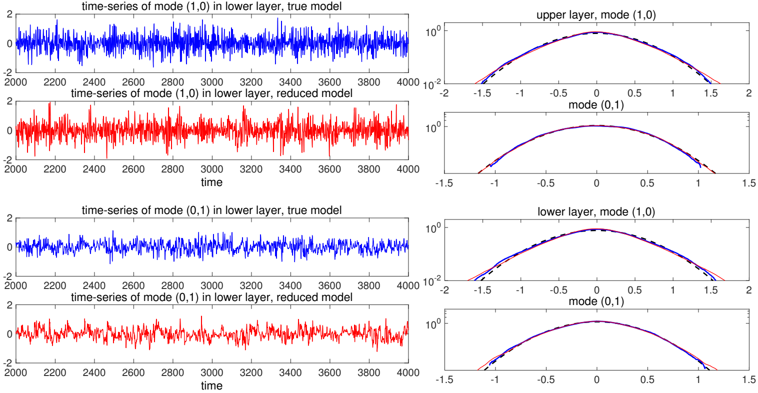


Figure: Prediction about tracer intermittency in mid latitude atmosphere regime with parameters.

Summary

- The single-point probability distribution function of a passive scalar has been the focus of much interest and requires better understanding.
- The generation of tracer intermittency is related with the competition between the blocked and unblocked modes in advection flow.
- The complex strongly turbulent dynamical system requires proper reduced-order modeling strategy of adopting simple advection flow models.
- Reduced-order stochastic models in passive scalar turbulence are useful in predicting extreme events and intermittency.

- D. Qi and A. Majda, *Predicting fat-tailed Intermittent probability distributions in passive scalar turbulence with imperfect models through empirical information theory*, CMS, 2015.
- D. Qi and A. Majda, *Predicting extreme events for passive scalar turbulence in two-layer baroclinic flows through reduced-order stochastic models*, submitted, 2017.

Thank you for your attention!