Regularity Properties of the Euler Equations in Lagrangian Variables

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joint with
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The Euler equations

- We consider the Cauchy problem for the ideal incompressible homogeneous Euler equations

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = 0 \\ \nabla \cdot u = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{E}$$

where $(x,t) \in \mathbb{R}^d \times [0,\infty)$ or $\mathbb{T}^d \times [0,\infty)$ and $d \in \{2,3\}$.

- This *Eulerian formulation* (E) is due to Euler [1757].

Well-posedness

- If $u_0 \in H^s$ with $s > d/2 + 1$, or $u_0 \in L^2 \cap C^{1,\gamma}$ for some $\gamma \in (0, 1)$, there exists a $T > 0$ and a unique solution u bounded in the same class as the datum on $[0, T)$.
- $d = 2$: Wolibner ['33], Hölder ['33]. Kato ['67].
- $d = 3$: Lichstenstein ['30]. Kato ['72].

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- $d = 2$: Wolibner ['33], Hölder ['33]. Kato ['67].
- $d = 3$: Lichstenstein ['30]. Kato ['72].
- $d = 2$: $T = \infty$, even for $\omega_0 \in L^1 \cap L^\infty$; Yudovich ['63].
- $d = 3$: the classical solution may be extended past time T iff

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < \infty$$

where $\omega = \nabla \times u$ is the vorticity; Beale-Kato-Majda ['84].
- The local existence theorems are in classes which guarantee

u is Lipschitz continuous

up to logarithms, as long as the solution exists.

Lagrangian paths

- Given a Lipschitz velocity field $u(x,t)$ the Lagrangian path starting at “label” a is given by the solution of the ODE

$$\frac{dX}{dt}(a,t) = u(X(a,t),t), \qquad X(a,0) = a.$$

- Conservation of momentum becomes

$$\partial_t^2 X(a,t) + (\nabla_x p)(X(a,t),t) = 0.$$

- Conservation of mass becomes

$$\det(\nabla_a X) = 1$$

  i.e., the map $a \mapsto X(a,t)$ is volume preserving.
- *Lagrangian description* of ideal fluids is also due to Euler [1757].
- When $u \in C^{1,\gamma} \cap L^2$ the two formulations are equivalent, and local existence and uniqueness results are “the same”.

Any difference between Lagrangian and Eulerian?

- Consider u_0 that is in $L^2 \cap C^{1,\gamma}$.
- If we view the Eulerian solution as a function of time with values in $C^{1,\gamma}$, then this function is everywhere discontinuous for generic initial data: Cheskidov-Shvydkoy ['10], Misiolek-Yoneda ['12-'14].
- See also Masmoudi-Elgindi ['14], Bourgain-Li ['14] for ill-posedness in critical spaces.

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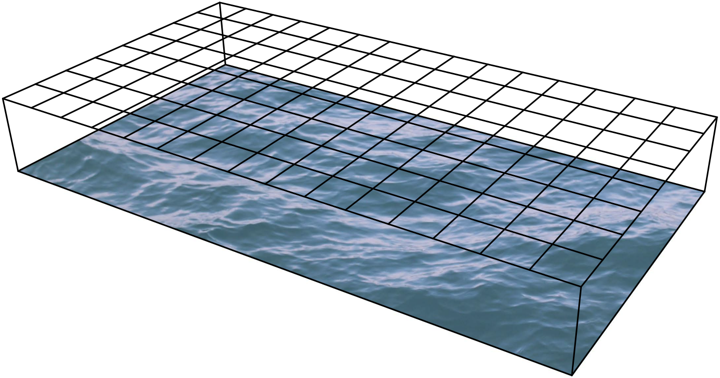
- See also Masmoudi-Elgindi ['14], Bourgain-Li ['14] for ill-posedness in critical spaces.
- On the other hand, the Lagrangian paths, viewed as functions of time with values in $C^{1,\gamma}$ are real-analytic (wrt t).

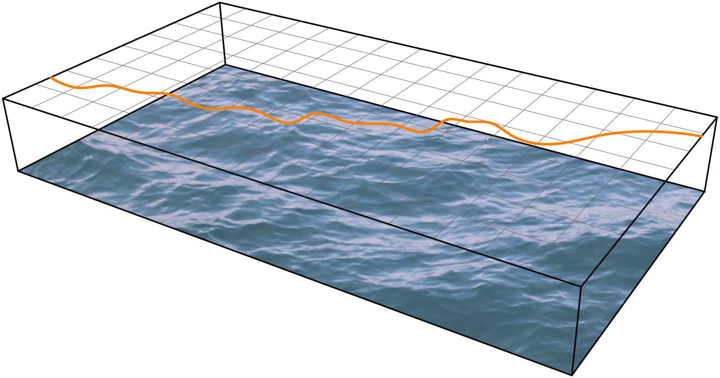
“Smooth” Sea

"Rough" Sea

Eulerian regularity



A Lagrangian path



Lagrangian analyticity for 3D Euler

- Chemin ['92], Gamblin ['94], Serfati ['95], Sueur ['11], Glass-Sueur-Takahashi ['12]: commutators, Littlewood-Paley.
- Shnirelman ['12]: Complexification of geodesic exponential map in SDiff.
- Frisch-Zheligovsky ['12-'13] "A Very Smooth Ride in a Rough Sea": Cauchy invariant gives local elliptic system in label variables. Special structure of 3D Euler.
- Nadirashvili ['13]: 2D elliptic theory yields that nondegenerate level sets of stream function in steady 2D Euler are analytic.
- Quantifying the distinct degrees of regularity for weak solutions of 3D Euler with respect to Eulerian and Lagrangian derivatives is crucial for the recent works on the Onsager conjecture: Isett ['12-'13], Buckmaster-DeLellis-Szekelyhidi ['13-'14].

Lagrangian analyticity in hydrodynamic systems

Question: is there anything robust about these results, or are all the results due to the special structure of the Euler equations?

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Theorem (Constantin-V.-Wu ('14))

Consider a well-posed hydrodynamic equation (such as 2D/3D Euler, 2D Boussinesq, 2D SQG, 2D IPM, etc...) on a time interval $[0, T)$ *when the Eulerian velocities are* $C^{1,\gamma}$*, for some* $\gamma \in (0, 1)$*.*

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Then, the Lagrangian particle trajectories $X(a, t)$ *are real-analytic functions of time.*

- The radius of analyticity in time on the interval $[0, t]$ depends on the chord-arc parameter of this interval:

$$\lambda(t) = \exp\left(\int_0^t \|\nabla u(\tau)\|_{L^\infty_x} d\tau\right).$$

- Recall: as long as $u \in L^1_t \mathrm{Lip}_x$, we have the chord-arc condition

$$\lambda(t)^{-1} \leq \frac{|a-b|}{|X(a,t) - X(b,t)|} \leq \lambda(t).$$

- Proof also applies to smooth 2D vortex patches. In contrast, for generic vorticity in the Yudovich class, only Gevrey-3 regularity in time appears to be known: Gamblin ['94], Sueur ['11].

- Reformulation as closed Lagrangian system. The Lagrangian path, X, obeys

$$\frac{dX}{dt}(a,t) = \frac{1}{4\pi}\int \frac{X(a,t) - X(b,t)}{|X(a,t) - X(b,t)|^3} \times (\nabla_b X(b,t)\omega_0(b))db.$$

and

$$\frac{d(\nabla_a X)}{dt}(a,t) = (\nabla_a X)(a,t)\int K(X(a,t) - X(b,t))\,(\nabla_b X(b,t)\omega_0(b))\,db$$
$$+\frac{1}{2}(\nabla_a X(a,t)\omega_0(a)) \times (\nabla_a X)(a,t).$$

with kernel K is given by

$$(K(x)y)_{ij} = \frac{3}{8\pi}\frac{(x \times y)_i\, x_j + (x \times y)_j\, x_i}{|x|^5}.$$

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- Key observations: initial datum just appears as a parameter; and the equations are closed ODEs with values in $C^{1,\gamma}$.

- Recall: If K is real-analytic and $X \in C^0$ is a solution of

$$\frac{dX}{dt} = K(X)$$

then in fact X is real-analytic with respect to t.

- Proof: keep track of *proper Cauchy inequalities*

$$|\partial_t^n X| \leq (-1)^{n-1} \binom{1/2}{n} \frac{(2C)^n}{R^{n-1}} n!$$

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- Instead, for a large class of inviscid hydrodynamical models:

$$\begin{aligned}&\frac{d}{dt}[X, \nabla X](a)\\&= \mathcal{P}_1(X(a), \nabla X(a))\\&\quad \times p.v. \int \mathcal{K}(X(a) - X(b)) \mathcal{P}_2(X(b), \nabla X(b)) \mathcal{P}_3(u_0(b), \nabla u_0(b)) db\end{aligned}$$

where $\mathcal{P}_i$ are polynomials, and $\mathcal{K}$ are Calderon-Zygmund kernels.

- C-Z operators remain OK after composition with $C^{1,\gamma}$ maps.

Fully Lagrangian formulation of the Euler equations

- The Lagrangian velocity v and the pressure q are obtained by composing with X

$$v(a,t) = u(X(a,t),t), \quad q(a,t) = p(X(a,t),t).$$

- Denote the matrix inverse of the Jacobian of the particle map as

$$Y(a,t) = (\nabla_a X(a,t))^{-1}.$$

- The Lagrangian formulation of Euler is given in components by

$$\begin{cases} \partial_t v^i + Y_i^j \partial_j q = 0 \\ Y_i^k \partial_k v^i = 0 \\ \partial_t Y_i^k = -Y_l^k (\nabla v)_j^l Y_i^j \end{cases} \tag{L}$$

used summation convention on repeated indices, and $\partial_k = \partial_{a_k}$.

- The evolution of Y follows from $\det(\nabla X) = 1$ and $\nabla_a(\partial_t X = v)$.
- The closed system for (v, q, Y) is supplemented with initial datum

$$v(a,0) = v_0(a) = u_0(a), \quad Y(a,0) = I.$$

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- Then as long as the smooth solution exists (i.e. does not blow up in $L^1_t \mathrm{Lip}_x$), it remains real-analytic wrt x.

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- Bardos-Benachour-Zerner ['76], Bardos-Benachour ['77], Alinhac-Metivier ['86], Levermore-Oliver ['97], Kukavica-V. ['09-'11], Zheligovsky ['11], Sueur ['11], Glass-Sueur-Takahashi ['12], Sawada ['13].
- Best lower bounds on the uniform spatial analyticity radius $\tau(t)$ are given explicitly in terms of the chord-arc parameter

$$\tau(t) \geq \frac{\tau_0}{\lambda(t)} = \tau_0 \exp\left(- \int_0^t \|\nabla u(s)\|_{L^\infty} ds \right)$$

- Analyticity with respect to label a follows (with possibly different convergence radius) due to composition of real-analytic functions, and Cauchy-Kowalevski.
- Analyticity in time in this case follows directly from the equations: $\partial_t u =$ real-analytic function.

Constantin-Kukavica-V. ('15)

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- In the Lagrangian formulation, one may solve the equations locally in time, at fixed analyticity radius.
- In Eulerian variables, it may deteriorate instantaneously.
- The Lagrangian formulation allows solvability in highly anisotropic classes, e.g. functions which have analyticity in one variable, but are not analytic in the others.
- In the Eulerian formulation, the equations are ill-posed in such functions spaces.

Norms for real-analytic and Gevrey functions

- Fix $r > d/2$, so that $H^r(\mathbb{R}^d)$ is an algebra.
- For a Gevrey-index $s \geq 1$ and Gevrey-radius $\delta > 0$, we denote the isotropic Gevrey norm by

$$\|f\|_{G_{s,\delta}} = \sum_{\beta \geq 0} \frac{\delta^{|\beta|}}{|\beta|!^s} \|\partial^\beta f\|_{H^r} = \sum_{m \geq 0} \frac{\delta^m}{m!^s} \left(\sum_{|\beta|=m} \|\partial^\beta f\|_{H^r} \right)$$

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- When $s = 1$ this norm corresponds to the space of real-analytic functions, and δ represents the uniform radius of analyticity of f.
- The ℓ^1 norm in m is essential $\rightarrow$ Wiener algebra.
- See Oliver-Titi ['01] for an equivalent Fourier description.

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- The ℓ^1 norm in m is essential $\to$ Wiener algebra.
- See Oliver-Titi ['01] for an equivalent Fourier description.
- Similarly, given a coordinate $j \in \{1, \ldots, d\}$, we define the anisotropic s-Gevrey norm with radius $\delta > 0$ by

$$\|f\|_{G^{(j)}_{s,\delta}} = \sum_{m \geq 0} \frac{\delta^m}{m!^s} \|\partial_j^m f\|_{H^r}.$$

Persistence of Lagrangian analyticity radius

Theorem (Constantin-Kukavica-V. ('15))

Assume that $v_0 \in L^2$ *and*

$$\nabla v_0 \in G_{s,\delta}$$

for some Gevrey-index $s \geq 1$ *and a Gevrey-radius* $\delta > 0$*.*
Then there exists $T > 0$ *and a unique solution* $v \in C([0, T]; H^{r+1})$*,* $Y \in C([0, T], H^r)$ *of the Lagrangian Euler system* (L)*, which moreover satisfies*

$$\nabla v, Y \in L^\infty([0, T], G_{s,\delta}).$$

Instantaneous decay of Eulerian analyticity radius

Theorem (Constantin-Kukavica-V. ('15))

There exist smooth periodic functions f, g *such that*

$$\|u_0\|_{G_{1,1}} < \infty$$

and such that the unique solution u *of the Euler equations* (E) *measured in the Eulerian variables obeys*

$$\|u(t)\|_{G_{1,1}} = \infty$$

as soon as $t > 0$.

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- Let f, g be two 2π-periodic functions. The function

$$u(x_1, x_2, x_3, t) = (f(x_2), 0, g(x_1 - tf(x_2)))$$

is an exact solution of the Euler equations (E) on $\mathbb{T}^3$, with datum

$$u_0(x_1, x_2, x_3) = (f(x_2), 0, g(x_1)).$$

and vanishing pressure. Di Perna-Majda ['87]; Bardos-Titi ['10].

Proof

- Simply letting

$$f(y) = \sin(y) \quad \text{and} \quad g(y) = \frac{1}{\sinh(1)^2 + \sin(y)^2}$$

does not work, since then $u_0 \notin G_{1,1}$ (ℓ^1 vs ℓ^∞ in derivative order).

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- As soon as we turn on time, the holomorphic extension of the function

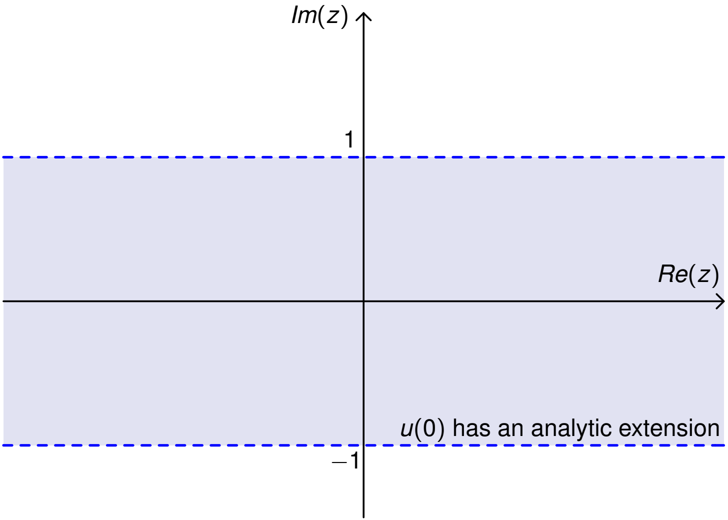
$$\partial_{x_1}^3 u_3(x_1, x_2, x_3, t) = \partial_{x_1}^3 (g(x_1 - tf(x_2)))$$

  has a singularity in the complex plane at

$$z_1 = 0 - (1-t)i$$
$$z_2 = 0 + i \log 2.$$

- Thus, $u(t) \notin G_{1,\delta(t)}$, for any $\delta(t) > 1 - t$.

Proof



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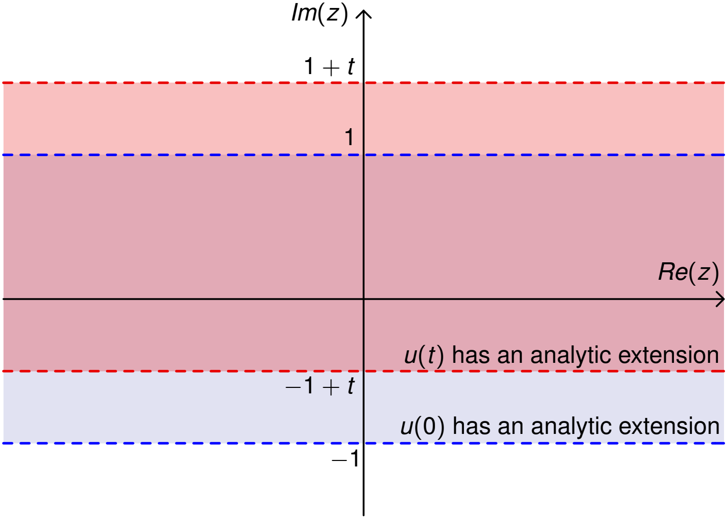

$Im(z)$
$1+t$
1
$Re(z)$
$u(t)$ has an analytic extension
$-1+t$
$u(0)$ has an analytic extension
-1

Solvability in anisotropic Lagrangian Gevrey classes

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for some index $s \geq 1$ *and radius* $\delta > 0$. *Then there exists* $T > 0$ *and a unique solution* $v \in C([0, T], H^{r+1})$, $Y \in C([0, T], H^r)$ *of the Lagrangian Euler system* (L), *which moreover satisfies*

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- At low regularity, i.e. Hölder classes, the equivalent question is the propagation of smoothness along vector fields transported by the Euler flow: "striated regularity". Bae-Kelliher ['15], following earlier works of Chemin ['93], Gamblin-Saint Raymond ['95], Danchin ['99], in spaces with negative degrees of smoothness.

Ill-posedness for anisotropic Eulerian real-analyticity

Theorem (Constantin-Kukavica-V. ('15))

There exists $T > 0$ and an initial datum $u_0 \in C^\infty(\mathbb{R}^2)$ for which u_0 and ω_0 are real-analytic in x_1, uniformly with respect to x_2, such that the unique $C([0, T]; H^r)$ solution $\omega(t)$ of the Cauchy problem for the Euler equations (E) *is not real-analytic in x_1, for any $t \in (0, T]$.*

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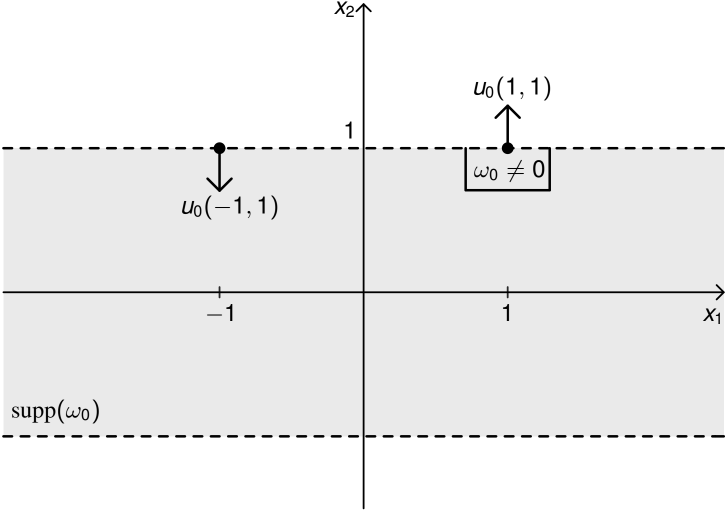
- The fact that the Eulerian version of the theorem does not hold should not be so surprising: isotropy and time-reversibility of the Euler equations.
- By contrast, the fact that the Lagrangian formulation keeps the memory of initial anisotropy is a bit more puzzling.
- Navier-Stokes $\approx$ Euler + Prandtl?

Proof



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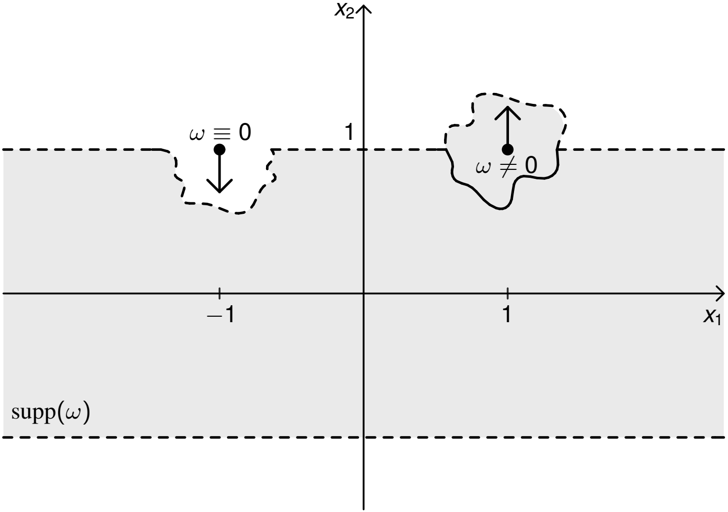

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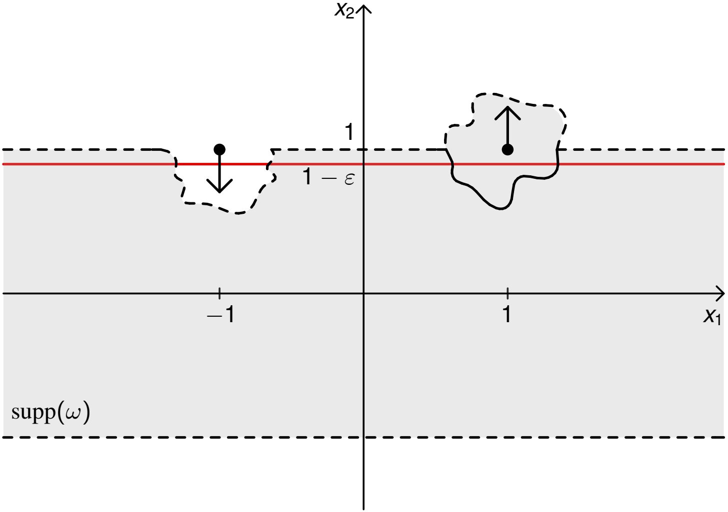

The Lagrangian vorticity in 2D

- For $d = 2$ the Lagrangian scalar vorticity

  $$\zeta(a, t) = \omega(X(a, t), t)$$

  is conserved in time

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  for $t \geq 0$.

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- The Lagrangian velocity v can then be computed from the Lagrangian vorticity ζ using the elliptic curl-div system

$$\begin{aligned} \varepsilon_{ij} Y_i^k \partial_k v^j &= Y_1^k \partial_k v^2 - Y_2^k \partial_k v^1 = \zeta = \omega_0 \\ Y_i^k \partial_k v^i &= Y_1^k \partial_k v^1 + Y_2^k \partial_k v^2 = 0 \end{aligned}$$

  where ε_{ij} is the sign of the permutation $(1,2) \mapsto (i,j)$.

The Cauchy identities for Lagrangian vorticity in 3D

- For $d = 3$ the vorticity vector is not conserved along particle trajectories, and instead we have the vorticity transport formula

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- Thus, in three dimensions, the elliptic curl-div system becomes

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$$Y^k_i \partial_k v^i = 0$$

- In order to make use of the above identity, we need to reformulate it so that the right side is time-independent, as in 2D.
- Multiplying the equation for the Lagrangian curl with Y^m_i and summing in i, we get

$$\varepsilon_{ijk} Y^m_i Y^l_j \partial_l v^k = \omega_0^m$$

which is the form of the Cauchy [1827] identity containing only Y.

Proof of Lagrangian persistence

- Fix $s \geq 1$ and $\delta > 0$ so that $\|\nabla v_0\|_{G_{s,\delta}} \leq M$, that is

$$\Omega_m := \sum_{|\alpha|=m} \|\partial^\alpha \nabla v_0\|_{H^r}$$

obeys

$$\sum_{m\geq 0} \Omega_m \frac{\delta^m}{m!^s} \leq M$$

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$$\sum_{m \geq 0} \Omega_m \frac{\delta^m}{m!^s} \leq M$$

- Fix $T > 0$, to be chosen later sufficiently small in terms of M and s, and for $m \geq 0$ define

$$V_m = V_m(T) = \sup_{t \in [0,T]} \sum_{|\alpha|=m} \|\partial^\alpha \nabla v(t)\|_{H^r},$$

$$Z_m = Z_m(T) = \sup_{t \in [0,T]} t^{-1/2} \sum_{|\alpha|=m} \|\partial^\alpha (Y(t) - I)\|_{H^r}.$$

Velocity estimates

- In order to estimate ∇v and its derivatives, we use the three-dimensional div-curl system we conclude that for $\alpha \in \mathbb{N}_0^3$:

$$\begin{aligned}\|\partial^\alpha \nabla v\|_{H^r} \leq{}& C\|\partial^\alpha \omega_0^m\|_{H^r} + C\|\partial^\alpha(\varepsilon_{ijk}(\delta_{im} - Y_i^m)(\delta_{jl} - Y_j^l)\partial_l v^k)\|_{H^r} \\ &+ C\|\partial^\alpha(\varepsilon_{mjk}(\delta_{jl} - Y_j^l)\partial_l v^k)\|_{H^r} + C\|\partial^\alpha(\varepsilon_{ijk}(\delta_{im} - Y_i^m)\partial_j v^k)\|_{H^r} \\ &+ C\|\partial^\alpha((\delta_{ik} - Y_i^k)\partial_k v^i)\|_{H^r}.\end{aligned}$$

- Summing the above inequality over all multi-indices with $|\alpha| = m$ and taking a supremum over $t \in [0, T]$ we arrive at

$$\begin{aligned}V_m \leq{}& C\Omega_m + CTZ_m Z_0 V_0 + CTZ_0^2 V_m + CT^{1/2} Z_0 V_m + CT^{1/2} Z_m V_0 \\ &+ CT^{1/2} \sum_{0<j<m} \binom{m}{j} Z_j V_{m-j} + CT \sum_{0<(j,k)<m} \binom{m}{j\ k} Z_j Z_k V_{m-j-k}\end{aligned}$$

for all $m \geq 0$.

Flow map estimates

- In order to bound Z_m we appeal to the evolution for $Y(t) - I$:

$$I - Y(t) = \int_0^t (Y - I) : \nabla v : (Y - I)\, d\tau + \int_0^t (Y - I) : \nabla v\, d\tau + \int_0^t \nabla v : (Y - I)\, d\tau + \int_0^t \nabla v\, d\tau$$

- We obtain

$$Z_m \le CT^{1/2}(TZ_0^2 V_m + TZ_m Z_0 V_0 + T^{1/2} Z_0 V_m + T^{1/2} Z_m V_0 + V_m) + CT^{3/2} \sum_{0<|(j,k)|<m} \binom{m}{j\ k} Z_j Z_k V_{m-j-k} + CT \sum_{j=1}^{m-1} \binom{m}{j} Z_j V_{m-j}$$

for all $m \ge 0$.

- Summing over m completes the proof.

Thank you!