On the Global Behavior of Generalized Characteristics of Hamilton-Jacobi Equations

Joint work of Piermarco Cannarsa and Cui Chen

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if $\mathbf{x}$ *satisfies the differential inclusion*

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3. If x_0 is singular point of u, and

$$0 \notin \operatorname{co} H_p(x_0, D^+u(x_0))\,, \tag{1.2}$$

then there exists such a singular generalized characteristic locally, see [Albano-Can, 2002], [Can-Yu, 2009] for control theory and PDE approach respectively.

Local barrier functions and inf/sup-convolution

1. Let $u \in C(\mathbb{R}^n)$ and let H be a Tonelli Hamiltonian on $\mathbb{R}^n$. Recall the *Lax-Oleinik operators* T_t^- and T_t^+, for any $u \in C(\mathbb{R}^n)$,

$$T_t^+ u(x) := \sup_{y \in \mathbb{R}^n} \{u(y) - A_t(x,y)\},$$

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3. By the regularity properties of $A_t(x, y)$ and u, ϕ_t is a locally semiconcave functions, but for $t > 0$ small enough, ψ_t is both a locally semiconcave function and a convex function. We will discuss this essential point later!

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4. In [Can-Chen-C, draft, 2015], we discussed the local propagation along this line recovering what in [Can-Yu], and more information can be obtained.
5. We will concentrate on the critical points of ψ_t in the procedure of sup-convolution, and the connection with the global propagation of singularities along generalized characteristics for general mechanical systems on $\mathbb{R}^n$ ([Can-C, draft, 2015]).

Semiconcave functions

1. Let $\Omega \subset \mathbb{R}^n$ be a convex open set, a function $u : \Omega \to \mathbb{R}$ is *semiconcave* if there exists a constant $C > 0$ such that

$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \leqslant \frac{C}{2}\lambda(1-\lambda)|x-y|^2$$

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2. Equivalently, u is semiconcave with constant C if $u(\cdot) - C|\cdot|^2/2$ is concave.
3. A function $u : \Omega \to \mathbb{R}$ is said to be *semiconvex* if $-u$ is semiconcave. A function $u : \Omega \to \mathbb{R}$ is said to be *locally semiconcave* (resp. *locally semiconvex*) if for each $x \in \Omega$, there exists an open ball $B(x, r) \subset \Omega$ such that u is a semiconcave (resp. semiconvex) function on $B(x, r)$.

Superdifferentials and Limiting differentials

1. Let $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ be a continuous function. We recall that, for any $x\in\Omega$, the closed convex sets

$$D^-u(x)=\left\{p\in\mathbb{R}^n:\liminf_{y\to x}\frac{u(y)-u(x)-\langle p,y-x\rangle}{|y-x|}\geqslant 0\right\},$$
$$D^+u(x)=\left\{p\in\mathbb{R}^n:\limsup_{y\to x}\frac{u(y)-u(x)-\langle p,y-x\rangle}{|y-x|}\leqslant 0\right\}.$$

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are called the (Dini) *subdifferential* and *superdifferential* of u at x, respectively.

2. Let $u : \Omega \to \mathbb{R}$ be locally Lipschitz. We recall that a vector $p \in \mathbb{R}^n$ is called a *limiting differential* of u at x if there exists a sequence $\{x_n\} \subset \Omega \setminus \{x\}$ such that u is differentiable at x_k for each $k \in \mathbb{N}$, and

$$\lim_{k \to \infty} x_k = x \quad \text{and} \quad \lim_{k \to \infty} Du(x_k) = p.$$

The set of all limiting differentials of u at x is denoted by $D^{*}u(x)$.

Tonelli Lagrangians

We concentrate on Tonelli systems.

A function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be a *Tonelli Lagrangian* if the following assumptions are satisfied.

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For more regularity results required, we need more conditions!

Fundamental solutions

Given $x, y \in \mathbb{R}^n$, we define

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In the literature of PDEs, $A_t(x,y)$ is called a *fundamental solution* of H-J equation

$$u_t + H\big(x, \nabla u(t,x)\big) = 0, \tag{2.2}$$

where H stands for the associated Tonelli Hamiltonian. $A_t(x,y)$ is also called *generating function* in the context of dynamical systems or symplectic geometry.

Viscosity solutions

We recall that a continuous function u is called a *viscosity subsolution* of equation (2.2) if, for any $x \in \mathbb{R}^n$,

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Finally, u is called a *viscosity solution* of equation (2.2), if it is both a viscosity subsolution and a supersolution.

Two basic facts on viscosity solutions

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$\operatorname{Ext} D^+u(x) = D^*u(x)$ *for any viscosity solution* u *of* (2.2) *and any* $x \in \mathbb{R}^n$.

Here we denote by $\operatorname{Ext} C$ the set of extremal points of C.

One important observation

Proposition

Let $x \in \mathbb{R}^n$ and $u : \mathbb{R}^n \to \mathbb{R}$ be a viscosity solution of the Hamilton-Jacobi equation (2.2)*. Then $p \in D^*u(x)$ if and only if there exists a unique C^2 curve $\gamma : (-\infty, 0] \to \mathbb{R}^n$ with $\gamma(0) = x$ which is a backward calibrated curve, and $p = L_v(x, \dot{\gamma}(0))$.*

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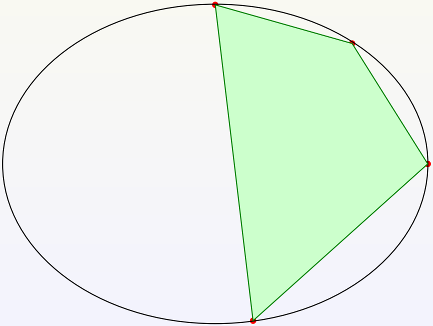

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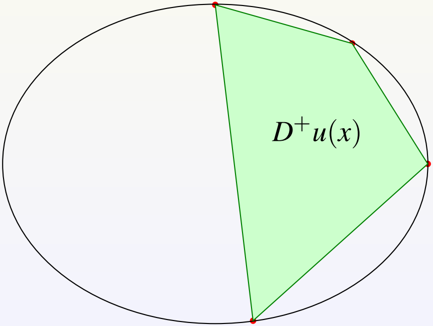

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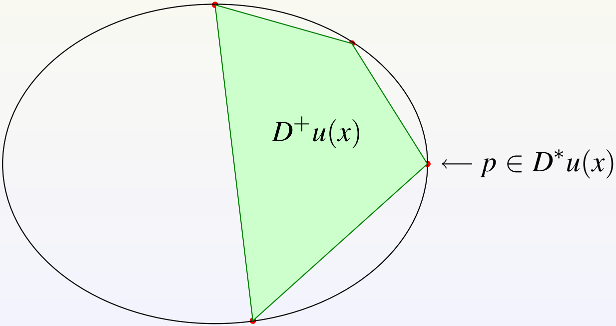

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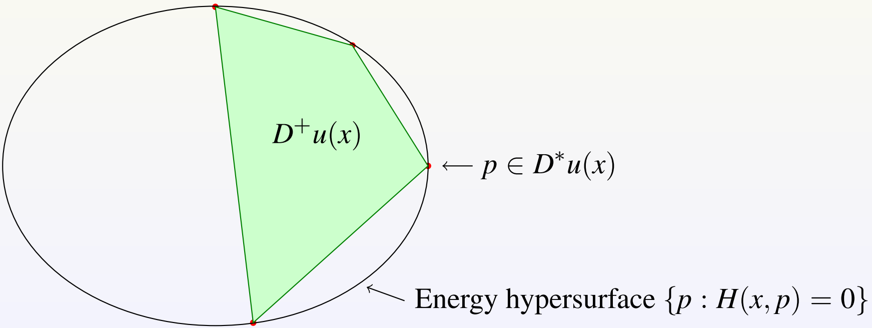

Singular set and cut loci

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2. When talking about the viscosity solution, $\overline{\Sigma_u}$ is called the *cut loci* of u. (Under a certain regularity condition, $\overline{\Sigma_u} = \Gamma_u \cup \Sigma_u$, where Γ_u is the conjugate loci.)

Maximizers of ψ_t: local case

1. We begin with a local argument. Let x_0 be a singular point of u, and the local barrier function ψ_t is defined w.r.t x_0 for $t > 0$. We want to find the maximizer, say y_t, of ψ_t in $\bar{B}(x_0, R)$.

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2. By the regularity properties of $A_t(x, y)$, when $t > 0$ small enough, we have that $A_t(x_0, \cdot)$ is convex locally (with the constant of convexity large if t small) and $C^{1,1}_{loc}$ consequently. Thus ψ_t is concave in $\bar{B}(x_0, R)$. So there exists a unique $y_t \in \bar{B}(x_0, R)$.

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3. The essential difficulty is if the maximizer y_t can be attained in the interior of $\bar{B}(x_0, R)$. If so, we can prove that there exists $t_0 > 0$ dependent on the initial point x_0, such that the arc $\mathbf{y} : [0, t_0] \to \mathbb{R}^n$ defined by

$$\mathbf{y}(t) = \begin{cases} x_0, & t = 0, \\ y_t, & t \in (0, t_0], \end{cases}$$

satisfying that $\mathbf{y}(t)$ is a singular point of u for all $t \in [0, t_0]$.

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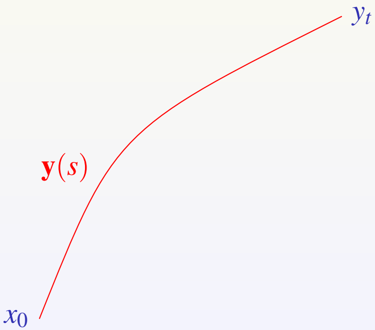
2. Suppose these two conditions are satisfied. $\xi_t : [0, t] \to \mathbb{R}^n$ is the unique minimizer in the definition of $A_t(x_0, y_t)$.



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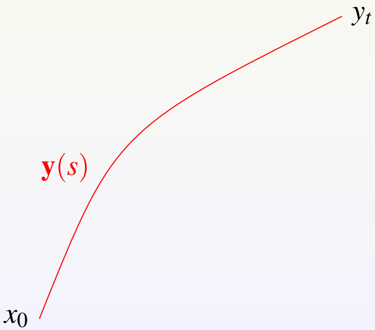

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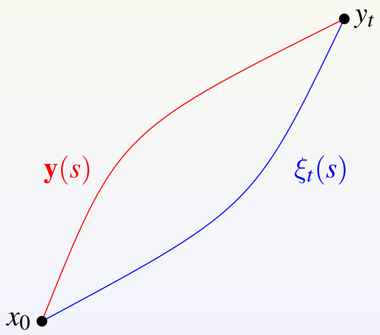

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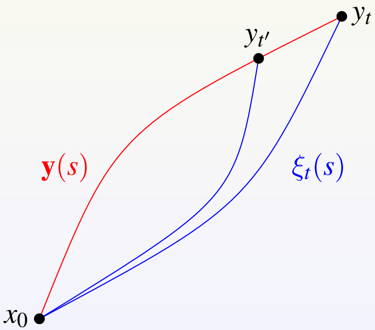

Maximizers of ψ_t: global case

1. We need more conditions in global case.
   1. There exists $t_0 > 0$ such that for any $t \in (0.t_0]$, there exists $R = R(t) > 0$ such that, $A_t(x, \cdot)$ is convex in $\bar{B}(x, R)$ uniformly;
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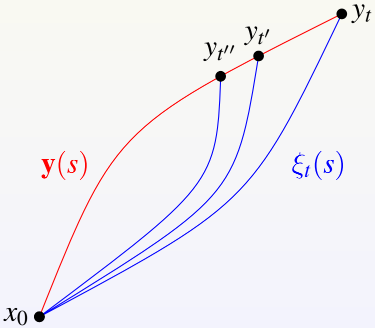

The velocity of $\mathbf{y}(\cdot)$ at x_0

1. Since $\{\dot{\xi}_t(\cdot)\}_{t\in(0,t_0]}$ are equi-Lipschitz, then for any sequence $t_k \to 0^+$ such that $v_{t_k} := \frac{\xi_{t_k}(t_k)-x_0}{t_k}$ converges, it is not hard to have that $v_0 := \lim_{k\to\infty} v_{t_k} = \lim_{k\to\infty} \dot{\xi}_{t_k}(t_k)$ exists.

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2. By the semiconcavity of $u_0(\cdot)$, we have

$$\langle p - L_v(y_{t_k}, \dot{\xi}_{t_k}(t_k)), v_{t_k}\rangle + t_k C|v_{t_k}|^2 \geqslant 0, \quad \forall p \in D^+u(x_0).$$

Taking limit, then $\langle p, v_0\rangle \geqslant \langle L_v(x_0, v_0), v_0\rangle$, for all $p \in D^+u(x_0)$. In other words,

$$H(x_0, p) \geqslant H(x_0, p_0), \quad \forall p \in D^+u(x_0),$$

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3. This leads to the assertion that
$$\dot{\mathbf{y}}^+(0) = \lim_{t\to 0^+} \frac{\xi_t(t) - x_0}{t} = \lim_{t\to 0^+} \dot{\xi}_t(t) = v_0,$$
with $v_0 = H_p(x_0, p_0)$.

The singularities along $\mathbf{y}(\cdot)$ on $[0, t_0]$

1. We define $p_t(s) := L_v(\xi_t(s), \dot{\xi}_t(s))$ for all $s \in [0, t]$, and since y_t is maximizer of ψ_t in $B(x_0, R)$, we have

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4. It is worth noting t_0 is independent on x_0 since our uniformness assumption on L are satisfied.

$\mathbf{y}(\cdot)$ as a generalized characteristic on $[0, t_0]$

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Moreover,

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4. For mechanical systems, a uniqueness result on generalized characteristic have been obtained in [Albano-Can, 2002] or [Can-Yu, 2009].

Extend $\mathbf{y}(\cdot)$ to $[0, +\infty)$

1. We construct the global propagation by induction. First, we obtain a singular generalized characteristic $\mathbf{x}^0(s)$ on $[0, t_0]$ satisfying

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3. The juxtaposition of $\{\mathbf{x}^k\}$ gives the desired singular g. c..

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Strong conditions All the second order partial derivatives of L and H are bounded uniformly, see, also, [Bernard, 2012].

Boundedness results

Proposition

Let $t > 0$, $R > 0$ and suppose L satisfies condition (L1)-(L3). Let $\xi \in \Gamma^t_{x,y}$ be a minimizer for $A_t(x, y)$, $x \in \mathbb{R}^n$, $y \in \bar{B}(x, R)$, and let $p(s)$ be the dual arc of $\xi(s)$. Then we have

$$\sup_{s\in[0,t]} |\dot{\xi}(s)| \leqslant \Delta(t, R), \qquad \sup_{s\in[0,t]} |p(s)| \leqslant \Delta(t, R),$$

where $\Delta(t, R)$ is strictly increasing in R and strictly decreasing in t.

Moreover, if $0 < t \leqslant 1$, then $\Delta(t, R) = \kappa(R/t)$ where the function $\kappa : (0, \infty) \to (0, \infty)$ is continuous and strictly increasing. We also have $\sup_{s\in[0,t]} |\xi(s) - x| \leqslant \Delta(t, R)$.

Note that under our assumptions, the constant Δ is independent on x. But for a local result we can have x-dependence of such Δ.

Compactness

1. Suppose $R > 0$ and L satisfies (L1) and (L2). For any $0 < t \leqslant 1$, and $y \in B(x, R)$, let $\xi_{t,y} \in \Gamma^t_{x,y}$ be a minimizer in $A_t(x, y)$, and $p_{t,y}$ its dual arc, then we have

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$$\sup_{s\in[0,t]} |\dot{\xi}_{t,y}(s)| \leqslant \kappa(R/t), \qquad \sup_{s\in[0,t]} |p_{t,y}(s)| \leqslant \kappa(R/t).$$

2. Now, take $0 < t \leqslant 1$ and $R(t) = \frac{1}{2}t$, we denote

$$\mathbf{K}_x := \bar{B}(x,\kappa(1/2)) \times \bar{B}(0,\kappa(1/2)), \tag{4.1}$$

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3. For the estimate in the proof of the regularity property of $A_t(x, y)$, all the partial derivatives of L or H involved are bounded by certain constants on $\mathbf{K}_x$ or $\mathbf{K}^*_x$ respectively.

Convexity result of $A_t(x, y)$

Proposition (Convexity of fundamental solutions)

Suppose L is a Tonelli Lagrangian satisfying (L1)-(L3). Fix any $x \in \mathbb{R}^n$, then there exists $t_0 > 0$, such that for $0 < t \leqslant t_0$, $(t, y) \mapsto A_t(x, y)$ is locally convex in

$$S(x, t_0) = \{(t, y) \in \mathbb{R} \times \mathbb{R}^n : 0 < t \leqslant t_0, |y - x| \leqslant R(t)\},$$

with $R(t)$ defined above.
More precisely, there exists constants $C_1, C_2 > 0$ such that, if $y \in B(x, R(t))$, then, for $|h| \ll 1$ and $|z| \ll 1$, we have

$$A_{t+h}(x, y + z) + A_{t-h}(x, y - z) - 2A_t(x, y) \geqslant \frac{C_1}{t^3}|h|^2 + \frac{C_2}{t}|z|^2. \tag{4.3}$$

$C^{1,1}_{loc}$ result of $A_t(x, y)$

Proposition

Suppose L is a Tonelli Lagrangian satisfying (L1)-(L3). For any $x \in \mathbb{R}^n$*, there exists* $t_0 > 0$*, such that the functions* $w : (t, y) \mapsto A_t(x, y)$ *and* $(t, y) \mapsto A_t(y, x)$ *are both of class* $C^{1,1}_{loc}$ *in*

$$S(x, t_0) = \{(t, y) \in \mathbb{R} \times \mathbb{R}^n : 0 < t \leqslant t_0, |y - x| \leqslant R(t)\},$$

with $R(t)$ *defined above, for* $0 < t \leqslant t_0$*. In Particular, for any* $t \in (0, t_0]$*,*

$$D_yA_t(x, y) = L_v(\xi(t), \dot{\xi}(t)), \tag{4.4}$$

$$D_xA_t(x, y) = -L_v(\xi(0), \dot{\xi}(0)), \tag{4.5}$$

$$D_tA_t(x, y) = -E_{t,x,y}, \tag{4.6}$$

where $\xi \in \Gamma^t_{x,y}$ *is the unique minimizer for* $A_t(x, y)$ *and* $E_{t,x,y}$ *is the energy of the Hamiltonian trajectory* $(\xi(s), p(s))$ *with* $p(s) = L_v(\xi(s), \dot{\xi}(s))$*.*

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2. More precise condition of (L3): let C_2 be the constant of convexity of the function $(t,y) \mapsto A_t(x,y)$ in $B(x,R(t))$, and let C_1 be the constant of semiconcavity of u, then for any $p \in D^+u(x)$, $p' = D_yA_t(x,x)$ and $0 < t \leqslant t_0$, we have
$$\frac{2|p-p'|}{C_2/t - C_1} < R(t),$$
where t_0 is determined such that $\psi_t(\cdot)$ is strictly concave by Theorem 4.3.

Global propagation

Theorem ([Can-C, 2015])

Let H be a C^2 Tonelli Hamiltonian with the associated Lagrangian satisfies conditions (L1)-(L3), and u be a viscosity solution of the Hamilton-Jacobi equation

$$H(x, Du(x)) = 0, \quad x \in \mathbb{R}^n.$$

If x belongs to Σ_u, the singular set of u, then there exists a generalized characteristic $\mathbf{x} : [0, +\infty) \to \mathbb{R}^n$ such that $\mathbf{x}(s) \in \Sigma_u$ for all $s \in [0, +\infty)$.

An example

Under our conditions, an example satisfies the conditions is the mechanical systems in the following form:

$$H(x,p) = \frac{1}{2}\langle A^{-1}(x)p, p\rangle + V(x) - E, \quad x \in \mathbb{R}^n, p \in \mathbb{R}^n,$$

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3. $V \in C^2(\mathbb{R}^n, R)$, $V \leqslant 0$ and $\sup_{x\in\mathbb{R}^n} V(x) = 0$.

An example

Under our conditions, an example satisfies the conditions is the mechanical systems in the following form:

$$H(x,p) = \frac{1}{2}\langle A^{-1}(x)p, p\rangle + V(x) - E, \quad x \in \mathbb{R}^n, p \in \mathbb{R}^n,$$

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Rescaling

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$$\dot{\mathbf{x}}^\varepsilon(s) \in A(\mathbf{x}^\varepsilon(s))D^+u^\varepsilon(\mathbf{x}^\varepsilon(s)), \quad s \in [0, +\infty)$$

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   is what we want, since u and u^{ε} share the singularities.

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$$L^{\varepsilon}(x, v) = \frac{1}{2}\langle A(x)v, v\rangle - \varepsilon^2 l(x, v), \quad (x, p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

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Thanks for your attention!