A multilevel preconditioner for data assimilation with 4D-Var

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Data assimilation

- Combine observational and background data with numerical models to obtain the best estimate of state of a system.
- Find $\mathbf{u}$ which minimises

$$
\begin{aligned}
J(\mathbf{u}) &= \frac{1}{2}(\mathbf{u}-\mathbf{u}_b)^T V_b^{-1}(\mathbf{u}-\mathbf{u}_b) \\
&+ \frac{1}{2}\sum_{i=0}^{N}(C_o(\mathbf{u}_i)-\mathbf{y}_i)^T V_o^{-1}(C_o(\mathbf{u}_i)-\mathbf{y}_i)
\end{aligned}
$$

subject to $\quad \mathbf{u}_{i+1} = \mathcal{M}_{i,i+1}(\mathbf{u}_i), \quad i = 0,\ldots,N-1.$

- Discrete nonlinear evolution operator $\mathcal{M}_{i,i+1}$.
- Incremental 4D-Var: rewrite as an unconstrained minimisation with linearised evolution operator.

Hessian matrix

- Linear system (Gauss-Newton method):

$$\mathcal{H}(\mathbf{u}_k)\delta\mathbf{u}_k = G(\mathbf{u}_k)$$

Hessian $\mathcal{H}$, gradient $G(\mathbf{u}_k)$

- PCG convergence depends on conditioning of

$$\mathcal{H} = V_b^{-1} + R^T C_o^T V_o^{-1} C_o R$$

- Discrete tangent linear operator R and its adjoint.

- $\mathcal{H}$ is usually too large to be stored in memory but all we need for PCG is $\mathcal{H}\mathbf{v}$.

- This is still very expensive to compute, so we also need a good preconditioner.

First level preconditioning

- Projected Hessian:

$$H = (V_b^{1/2})^T \mathcal{H} V_b^{1/2} = I + (V_b^{1/2})^T R^T C_o^T V_o^{-1} C_o R V_b^{1/2}$$

- Eigenvalues of H are usually clustered in a narrow band above one, with few eigenvalues distinct enough to contribute noticeably to the Hessian value.

- AIM: construct a limited-memory approximation to H^{-1} using only matrix-vector multiplication.

Limited-memory approximation

- Find n_e leading eigenvalues (by $\ln \lambda^2$) and orthonormal eigenvectors using the Lanczos method.

- Construct approximation

$$H \approx I + \sum_{i=1}^{n_e} (\lambda_i - 1)\mathbf{u}_i\mathbf{u}_i^T$$

- Easy to evaluate matrix powers:

$$H^p \approx I + \sum_{i=1}^{n_e} (\lambda_i^p - 1)\mathbf{u}_i\mathbf{u}_i^T$$

Outline of multilevel algorithm

- Represent H_0 at a given level (k, say):

$$H_{0\to k} = R_k^0(H_0 - I_0)P_0^k + I_k$$

- Precondition to improve eigenvalue spectrum:

$$\tilde{H}_{0\to k} = (B_k^{k+1})^T H_{0\to k} B_k^{k+1}$$

- Find n_k eigenvalues/eigenvectors of $\tilde{H}_{0\to k}$ using the Lanczos method.

- Approximate $\tilde{H}_{0\to k}^{-1}$:

$$\tilde{H}_{0\to k}^{-1} \approx I_k + \sum_{i=1}^{n_k} \left(\frac{1}{\lambda_i} - 1\right) \mathbf{u}_i \mathbf{u}_i^T.$$

Example

- Test using 1D Burgers' equation with initial condition

$$f(x) = 0.1 + 0.35\left[1 + \sin\left(4\pi x + \frac{3\pi}{2}\right)\right], \qquad 0 < x < 1$$

- 1D uniform grid with 7 sensors located at 0.3, 0.4, 0.45, 0.5, 0.55, 0.6, and 0.7 in $[0, 1]$.

- Multilevel preconditioning with four grid levels:

k	0	1	2	3
grid points	401	201	101	51

Assessing approximation accuracy

- Riemannian distance:

$$\delta(A,B) = \left\| ln(B^{-1}A) \right\|_F = \left(\sum_{i=1}^{n} ln^2 \lambda_i \right)^{1/2}$$

- Compare eigenvalues of H^{-1} and $\tilde{H}^{-1}$ on the finest grid level $k = 0$ using

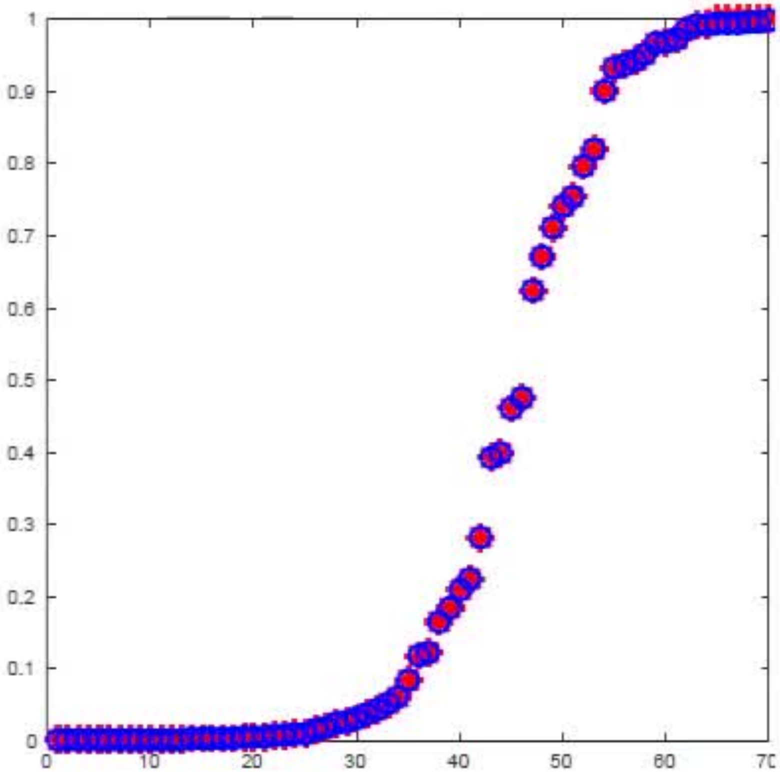
$$D = \frac{\delta(H^{-1}, \tilde{H}^{-1})}{\delta(H^{-1}, I)}$$

- Vary number of eigenvalues chosen on each grid level

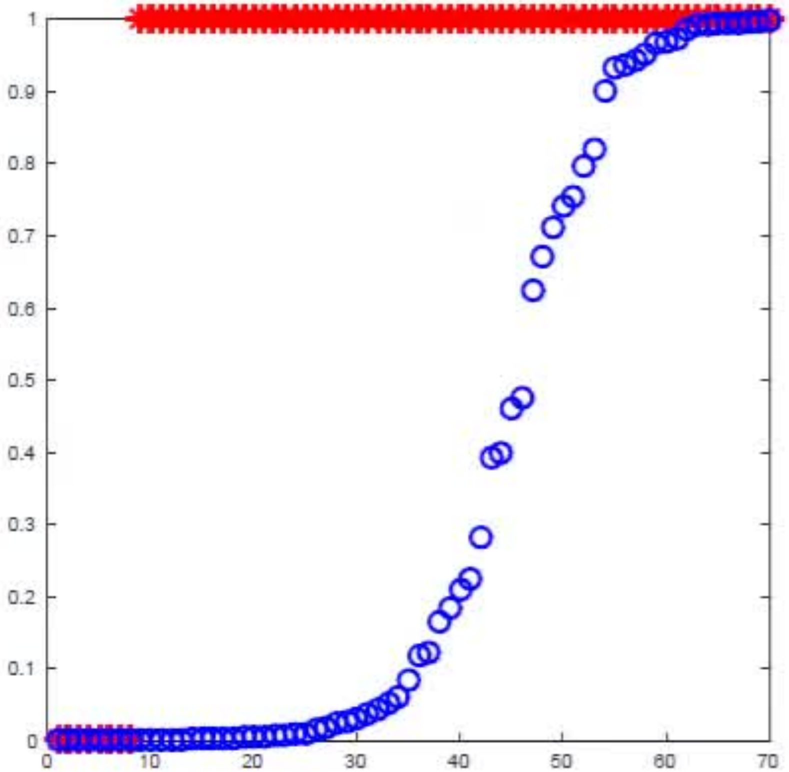
$$N_e = (n_0, n_1, n_2, n_3)$$

Eigenvalues of the inverse Hessian

- Exact (blue circles), approximated (red stars)



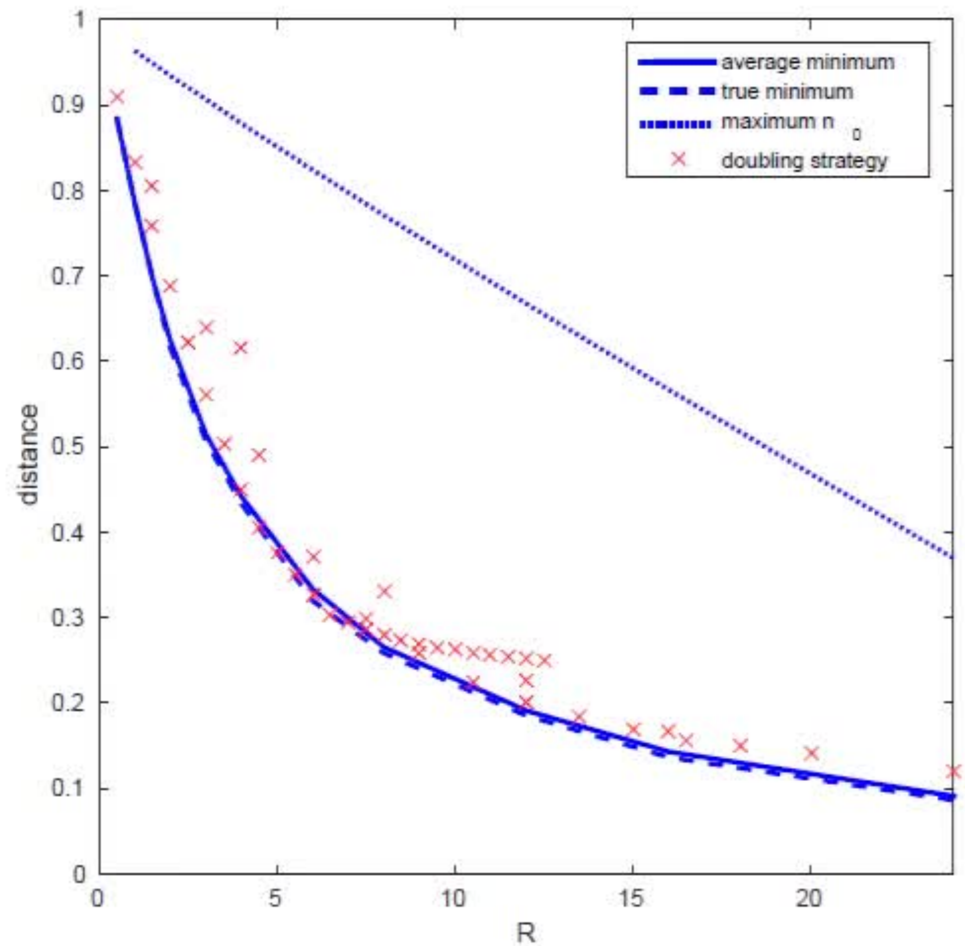
$$N_e = (64, 0, 0, 0)$$
$$D = 2.98e - 4$$



$$N_e = (8, 0, 0, 0)$$
$$D = 7.71e - 1$$

Fixed memory ratio

- Fixed memory ratio $R = \sum_{k=0}^{k_c} \frac{n_k}{2^k}$



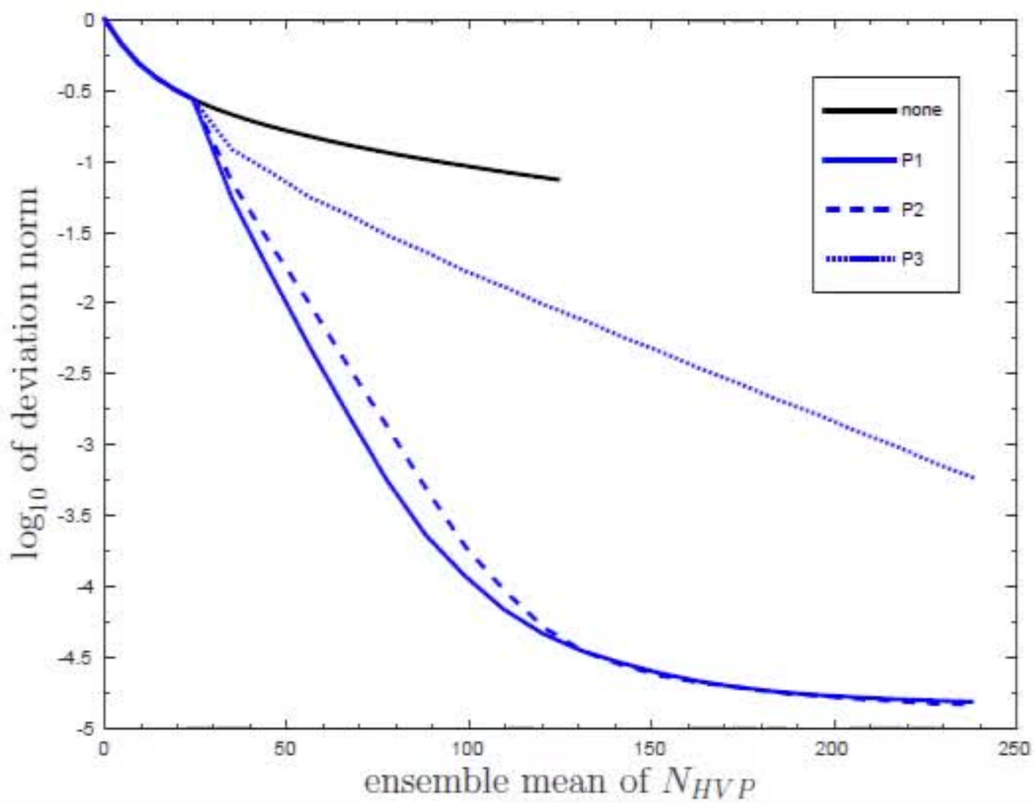
Practical approach: version 1

- Assemble local Hessians for each sensor to form H_a, then apply mlpre to H_a.
- Local Hessians cheaper to compute:
  - Potentially smaller area of influence.
  - Could run local rather than global model.
  - Compute local Hessians at level l.
  - Use limited-memory form with n_l eigenpairs.
  - Can be computed in parallel.
- More memory required:
  - Need to store additional local Hessians.

Iteration counts

Preconditioner	N_e	l	n_l
P1	(200,0,0,0)	1	8
P2	(0,8,16,32)	1	8
P3	(0,4,8,16)	1	8

log(error) vs number of HVP



Practical approach: version 2

- Can reduce memory requirements further.

- Approximate local Hessians by applying mlpre to local inverse Hessians using N_e^l.

- Construct a reduced-memory assembled Hessian H_a^{rm}.

- Use mlpre again on H_a^{rm}.

Conclusions and next steps

- Similar results with other configurations (e.g. moving sensors, different initial conditions).
- Multilevel preconditioning looks promising for constructing a good limited-memory approximation to H^{-1}.
- The balance between restrictions on memory/cost limitations may vary between particular applications.
- Identifying globally appropriate values for (n_0, n_1, n_2, n_3) is tricky.

- Now ready for two dimensions!

Iteration counts

Preconditioner	N_e	l	n_l	N_e^l
P1	(200,0,0,0)	1	8	-
P2	(0,8,16,32)	1	8	-
P3	(0,4,8,16)	1	8	-
P4	(0,8,16,32)	1	8	(0,0,8,0)
P5	(0,8,16,32)	2	8	(0,0,0,8)

log(error) vs number of HVP