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Dynamics of Sweeping Processes
with
Jumps in the Driving Term

VINCENZO RECUPERO
Dipartimento di Scienze Matematiche
Politecnico di Torino (Italy)

Notations

\mathcal{H} is a real Hilbert space,

$$y_0 \in \mathcal{H},$$

$$\mathcal{C}_{\mathcal{H}} := \left\{ \mathcal{C} \subseteq \mathcal{H} : \mathcal{C} \text{ nonempty, closed, convex} \right\}$$

$$d_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) := \max \left\{ \sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d(b, \mathcal{A}) \right\}, \quad \mathcal{A}, \mathcal{B} \in \mathcal{C}_{\mathcal{H}}.$$

Lipschitz sweeping processes

Theorem (J.J. Moreau, *Proceeding CIME*, 1973).

$$\forall \mathcal{C} \in Lip_{loc}([0, \infty[; \mathcal{C}\mathcal{H}) \quad \exists ! y \in Lip_{loc}([0, \infty[; \mathcal{H}) :$$

$$\begin{cases} y(t) \in \mathcal{C}(t) & \forall t \geq 0 \\ -y'(t) \in N_{\mathcal{C}(t)}(y(t)) & \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0 \\ y(0) = \text{Proj}_{\mathcal{C}(0)}(y_0) \end{cases}$$

Proof: Moreau-Yosida regularization

$$N_{\mathcal{C}(t)}(y(t)) \rightsquigarrow (N_{\mathcal{C}(t)}(y(t)))_{\lambda} := \frac{1}{\lambda} \left(y(t) - \text{Proj}_{\mathcal{C}(t)}(y(t)) \right)$$

Solve

$$\begin{cases} y'_{\lambda}(t) + \frac{1}{\lambda} \left(y_{\lambda}(t) - \text{Proj}_{\mathcal{C}(t)}(y_{\lambda}(t)) \right) = 0 & \forall t \geq 0 \\ y_{\lambda}(0) = \text{Proj}_{\mathcal{C}(0)}(y_0) \end{cases}$$

Then

$$y(t) := \lim_{\lambda \searrow 0} y_{\lambda}(t) \text{ solves the sweeping process}$$

Rate independence

The sweeping process is “rate independent”:

if

$$\begin{cases} -y'(t) \in N_{\mathcal{C}(t)}(y(t)) & \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0 \\ y(0) = \text{Proj}_{\mathcal{C}(0)}(y_0) \end{cases}$$

$$\phi \in \text{Lip}_{\text{loc}}([0, \infty[; \mathbb{R}) \text{ increasing, } \phi(0) = 0$$

then

$$\begin{cases} -(y \circ \phi)'(t) \in N_{(\mathcal{C} \circ \phi)(t)}((y \circ \phi)(t)) & \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0 \\ (y \circ \phi)(0) = \text{Proj}_{\mathcal{C}(0)}(y_0) \end{cases}$$

Operator solution of sweeping processes

The solution operator of the sweeping processes

$$\begin{array}{ccc} \mathbf{S} : & Lip_{\text{loc}}([0, \infty[; \mathcal{C}\mathcal{H}) & \longrightarrow & Lip_{\text{loc}}([0, \infty[; \mathcal{H}) \\ & \mathcal{C} & \longmapsto & y \end{array}$$

is rate independent:

$$\mathbf{S}(\mathcal{C} \circ \phi) = \mathbf{S}(\mathcal{C}) \circ \phi$$

if $\phi \in Lip_{\text{loc}}([0, \infty[; \mathbb{R})$ is increasing, $\phi(0) = 0$.

$\mathcal{C}(t)$ constant shape: the play operator

If

$$\mathcal{Z} \in \mathcal{C}_{\mathcal{H}}, \quad z_0 \in \mathcal{Z}, \quad u \in \text{Lip}_{\text{loc}}([0, \infty[; \mathcal{H}),$$

$$\mathcal{C}(t) := u(t) - \mathcal{Z},$$

then the sweeping process reads

$$\begin{cases} u(t) - y(t) \in \mathcal{Z} & \forall t \geq 0 \\ y'(t) \in N_{\mathcal{Z}}(y(t) - u(t)) & \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0 \\ y(0) = u(0) - z_0 \end{cases}$$

$$\begin{array}{ccc} \mathbf{P} : \text{Lip}_{\text{loc}}([0, \infty[; \mathcal{H}) & \longrightarrow & \text{Lip}_{\text{loc}}([0, \infty[; \mathcal{H}) \\ u & \longmapsto & y \end{array} \quad \text{play operator.}$$

More general data $\mathcal{C}(t)$

If $\mathcal{C} \in BV_{\text{loc}}([0, \infty[; \mathcal{C}_{\mathcal{H}})$ then

$$-y'(t) \in N_{\mathcal{C}(t)}(y(t))$$

is not the proper formulation:

another notion of solution is needed.

***BV*-solutions**

Theorem (J.J. Moreau, *JDE*, 1977).

$$\forall \mathcal{C} \in BV_{\text{loc}}^r([0, \infty[; \mathcal{C}\mathcal{H}) \quad \exists! y \in BV_{\text{loc}}^r([0, \infty[; \mathcal{H}) :$$

$$\left\{ \begin{array}{l} y(t) \in \mathcal{C}(t) \quad \forall t \geq 0 \\ Dy = w\mu \\ -w(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \geq 0 \\ y(0) = \text{Proj}_{\mathcal{C}(0)}(y_0) \end{array} \right.$$

Moreau's proof: catching up algorithm

For every $n \in \mathbb{N}$ discretize the time interval

$$0 = t_0^n < t_1^n < \cdots < t_j^n \rightarrow \infty \quad \text{as } j \rightarrow \infty.$$

Define the step function $y_n : [0, \infty[\rightarrow \mathcal{H}$ by

$$\begin{aligned} y_0^n &:= y_0, & y_j^n &:= \text{Proj}_{\mathcal{C}(t_j^n)}(y_{j-1}^n) \\ y_n(t) &:= y_j^n & \text{if } t &\in [t_{j-1}^n, t_j^n[\end{aligned}$$

Then there exists $y \in BV_{\text{loc}}^r([0, \infty[; \mathcal{H})$ such that

$$y_n \rightarrow y \quad \text{locally uniformly on } [0, \infty[,$$

and y solves the BV -sweeping process.

A “rate independent” method in $BV \cap C$

Theorem (V. Recupero, *JDE*, 2011).

If

$$\mathcal{C} \in BV_{\text{loc}}([0, \infty[; \mathcal{C}_{\mathcal{H}}) \cap C([0, \infty[; \mathcal{C}_{\mathcal{H}}),$$

$$\ell_{\mathcal{C}} : [0, \infty[\longrightarrow [0, \infty[, \quad \ell_{\mathcal{C}}(t) := V(\mathcal{C}, [0, t]),$$

then $\exists! \tilde{\mathcal{C}} \in Lip([0, \infty[; \mathcal{C}_{\mathcal{H}})$ such that $\mathcal{C} = \tilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$ and

$$\bar{S}(\mathcal{C}) := S(\tilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}$$

“easily” solves the $(BV \cap C)$ -sweeping process.

“Easily” means: simple measure theory proof

$$\text{Measure theory: } \begin{cases} D\bar{S}(\mathcal{C}) = (S(\tilde{\mathcal{C}})' \circ \ell_{\mathcal{C}}) D\ell_{\mathcal{C}}, \\ D\ell_{\mathcal{C}}(\ell_{\mathcal{C}}^{-1}(B)) = \mathcal{L}^1(B) \quad \forall B \in \mathcal{B}(\ell_{\mathcal{C}}([0, \infty[)). \end{cases}$$

If $\hat{y} := S(\tilde{\mathcal{C}}) \in \text{Lip}([0, \infty[; \mathcal{H})$ then

$$Z := \{t : -\hat{y}'(t) \notin N_{\tilde{\mathcal{C}}(t)}(\hat{y}(t))\} \implies \mathcal{L}^1(Z) = 0,$$

hence

$$\begin{aligned} & D\ell_{\mathcal{C}}(\{t : -\hat{y}'(\ell_{\mathcal{C}}(t)) \notin N_{\tilde{\mathcal{C}}(\ell_{\mathcal{C}}(t))}(\hat{y}(\ell_{\mathcal{C}}(t)))\}) \\ &= D\ell_{\mathcal{C}}(\{t : \ell_{\mathcal{C}}(t) \in Z\}) = D\ell_{\mathcal{C}}(\ell_{\mathcal{C}}^{-1}(Z)) = \mathcal{L}^1(Z) = 0, \end{aligned}$$

i.e.

$$-\hat{y}'(\ell_{\mathcal{C}}(t)) \in N_{\tilde{\mathcal{C}}(\ell_{\mathcal{C}}(t))}(\hat{y}(\ell_{\mathcal{C}}(t))) \quad \text{for } D\ell_{\mathcal{C}}\text{-a.e. } t \geq 0. \quad \blacksquare$$

The discontinuous BV case

If

$$\mathcal{C} \in BV_{\text{loc}}^r([0, \infty[; \mathcal{C}_{\mathcal{H}})$$

$$\ell_{\mathcal{C}} : [0, \infty[\longrightarrow [0, \infty[, \quad \ell_{\mathcal{C}}(t) := V(\mathcal{C}, [0, t]),$$

then $\exists! \tilde{\mathcal{C}} \in Lip(\ell_{\mathcal{C}}([0, \infty[); \mathcal{C}_{\mathcal{H}})$ such that $\mathcal{C} = \tilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$.

We need to extend $\tilde{\mathcal{C}}$ to $[0, \infty[$,

i.e.

we need to define $\tilde{\mathcal{C}}$ on $[\ell_{\mathcal{C}}(t-), \ell_{\mathcal{C}}(t+)]$ for $t \in \text{Discont}(\mathcal{C})$

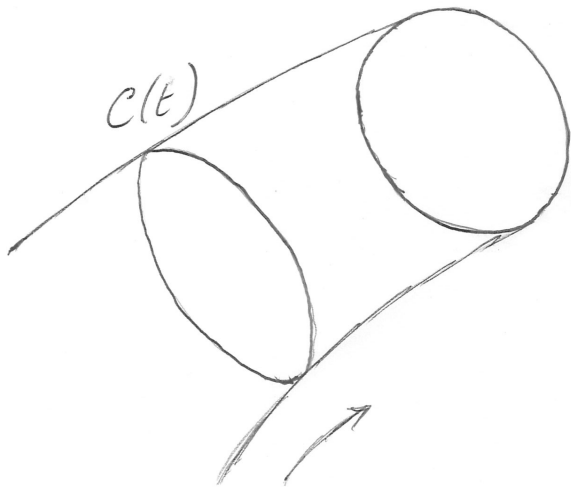
i.e.

we need to fill in the jumps of \mathcal{C} at every $t \in \text{Discont}(\mathcal{C})$.

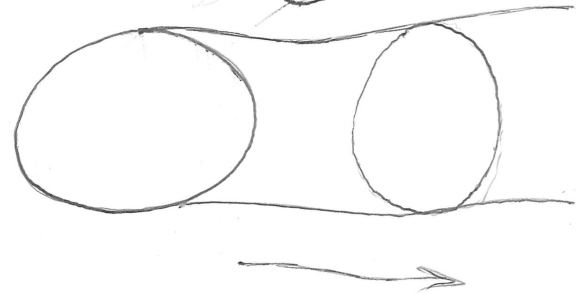
Jump

t_0 jump point

$C(t_0^-) = A$



$C(t_0^+) = B$



The particular case $S = P$

A natural choice:

fill in the jumps with segments.

If $u \in BV_{\text{loc}}^r([0, \infty[; \mathcal{H})$ and $\ell_u(t) := V(u, [0, t])$
then $\exists! \tilde{u} \in Lip([0, \infty[; \mathcal{H})$ such that

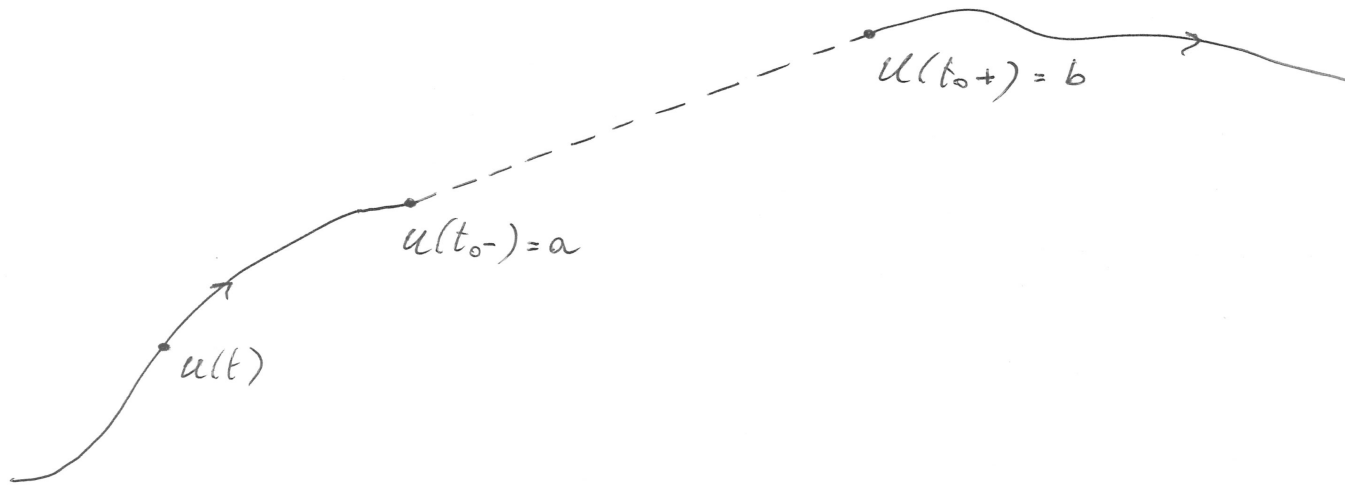
$$u = \tilde{u} \circ \ell_u$$

\tilde{u} is a geodesic segment on $[\ell_u(t-), \ell_u(t+)]$.

Is $\bar{P}(u) := P(\tilde{u}) \circ \ell_u$ the BV -solution?

Geodesic segment in \mathcal{H}

$$s(t) = (1-t)a + tb$$



No!

Theorem (V. Recupero, *Ann. SNS Pisa*, 2011).

$$\bar{\mathbf{P}}(u) := \mathbf{P}(\tilde{u}) \circ \ell_u, \quad u \in BV_{\text{loc}}^r([0, \infty[; \mathcal{H}),$$

is the unique continuous extension of

$\mathbf{P} : Lip_{\text{loc}}([0, \infty[; \mathcal{H}) \longrightarrow Lip_{\text{loc}}([0, \infty[; \mathcal{H})$ *when*

the domain has the BV-strict topology,

the codomain has the L^1 -topology.

But $\bar{\mathbf{P}}(u)$ is not the BV-solution.

Theorem (P. Krejčí, V. Recupero, *JCA*, 2014).

If $\dim(\mathcal{H}) < \infty$,

$\bar{\mathbf{P}}(u)$ is the BV-solution $\iff \mathcal{Z}$ is a non-obtuse polyhedron

Convex-valued “segments”

Segments correspond to convex-valued geodesics of the form

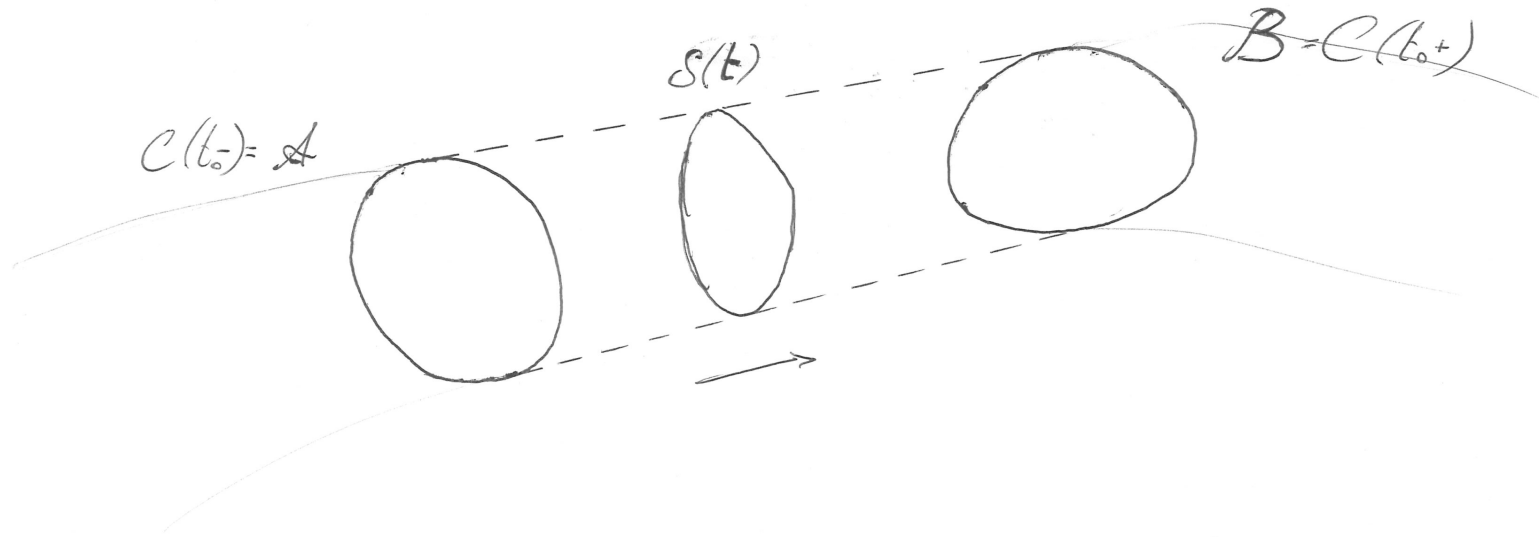
$$\mathcal{S}(t) := (1 - t)\mathcal{A} + t\mathcal{B}, \quad t \in [0, 1].$$

connecting \mathcal{A} to \mathcal{B} . Therefore \mathcal{S} is not a good choice.

Minkowski sum-type geodesic

$$S(t) = (1-t)A + tB$$

No!



Another convex-valued curve is needed

We look for a curve $\mathcal{G}(t)$ connecting \mathcal{A} and \mathcal{B} such that

any point $a \in \mathcal{A}$ is swept by $\mathcal{G}(t)$ to its projection on \mathcal{B} ,

i.e. for *any* initial datum $a \in \mathcal{A}$, we want $\text{Proj}_{\mathcal{B}}(a)$ to be the final point of trajectory of the solution of the sweeping process driven by $\mathcal{G}(t)$.

This is consistent with the catching-up algorithm.

A convex-valued geodesic which works

If $\rho := d_{\mathcal{H}}(\mathcal{A}, \mathcal{B})$

$$\mathcal{G}_{\mathcal{A}, \mathcal{B}}(t) := (\mathcal{A} + B_{t\rho}(0)) \cap (\mathcal{B} + B_{(1-t)\rho}(0)), \quad t \in [0, 1],$$

is a geodesic connecting \mathcal{A} to \mathcal{B} .

For any $a \in \mathcal{A}$, the solution $y \in Lip([0, 1]; \mathcal{H})$ of

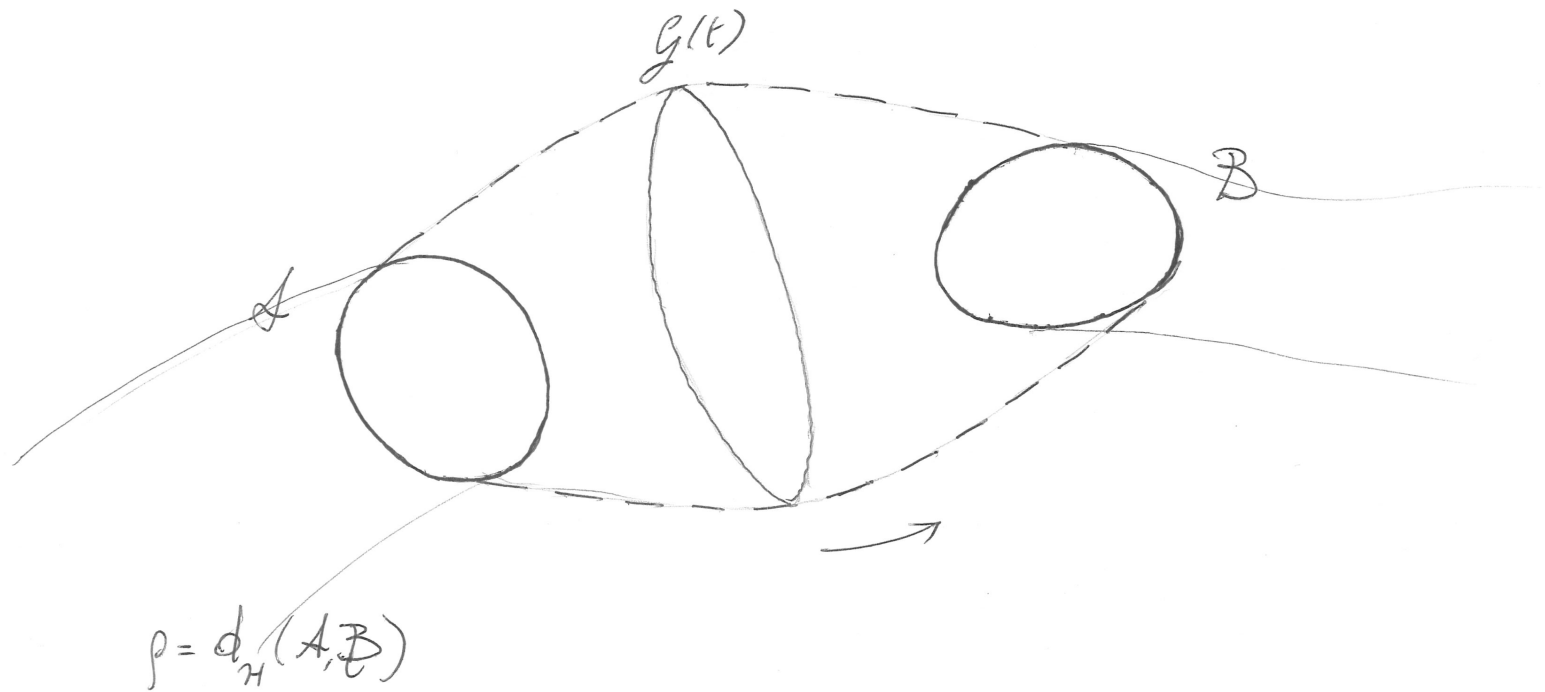
$$\begin{cases} y(t) \in \mathcal{G}(t) & \forall t \in [0, 1], \\ -y'(t) \in N_{\mathcal{G}(t)}(y(t)) & \text{for } \mathcal{L}^1\text{-a.e. } t \in [0, 1], \\ y(0) = a \end{cases}$$

satisfies

$$y(1) = \text{Proj}_{\mathcal{B}}(a).$$

The geodesic $\mathcal{G}_{A,B}$

$$\mathcal{G}(t) = (A + B_{t\rho}^{(0)}) \cap (\mathcal{B} + B_{(1-t)\rho}^{(0)}) \quad \underline{\text{YES}}$$



Reduction from *BV* to *Lip*

Theorem (V. Recupero, 2015).

If

$$\mathcal{C} \in BV_{\text{loc}}^r([0, \infty[; \mathcal{C}_{\mathcal{H}}), \quad \ell_{\mathcal{C}}(t) := V(\mathcal{C}, [0, t]),$$

then $\exists! \tilde{\mathcal{C}} \in \text{Lip}([0, \infty[; \mathcal{C}_{\mathcal{H}})$ *such that*

$$\mathcal{C} = \tilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$$

$\tilde{\mathcal{C}}$ *is the geodesic* $\mathcal{G}_{\mathcal{C}(t-), \mathcal{C}(t+)}$ *on* $[\ell_{\mathcal{C}}(t-), \ell_{\mathcal{C}}(t+)]$.

and

$\bar{S}(\mathcal{C}) := S(\tilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}$ “*easily*” *solves the BV-sweeping process:*

*existence, continuous dependence, convergence
of the catching-up algorithm are deduced from the Lip-case.*

Basic (“new”) vector measure theory tools

If

$$f \in Lip_{\text{loc}}([0, \infty[; \mathcal{H}), \quad h : [0, \infty[\longrightarrow [0, \infty[\text{ increasing ,}$$

then

$$\begin{cases} Dh(h^{-1}(B)) = \mathcal{L}^1(B) & \forall B \in \mathcal{B}(h(\text{Cont}(h))). \\ D(f \circ h) = g Dh \end{cases}$$

where

$$g(t) := \begin{cases} f'(h(t)) & \text{if } t \in \text{Cont}(h) \\ \frac{f(h(t+)) - f(h(t-))}{h(t+) - h(t-)} & \text{if } t \in \text{Discont}(h) \end{cases}$$

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