Stochastic Closure Schemes for bi-stable Energy Harvesters Excited by Colored Noise

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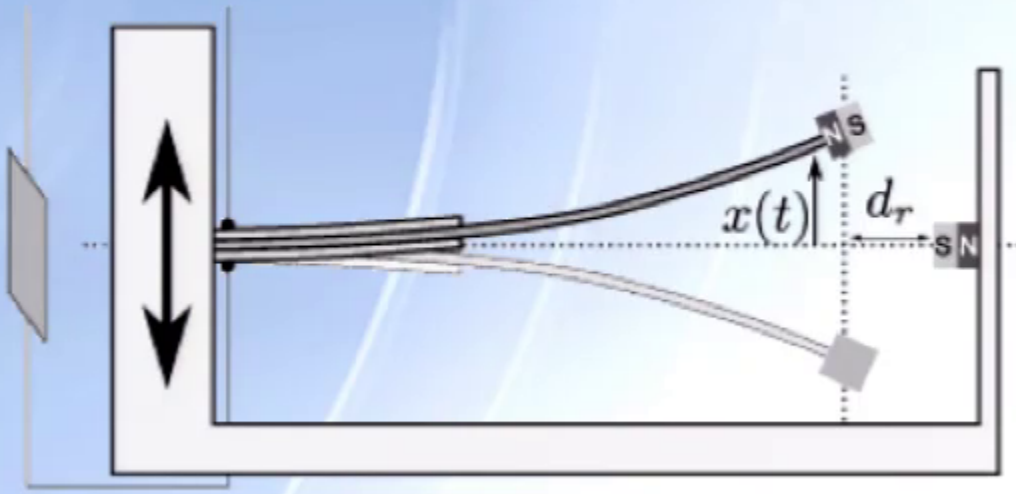
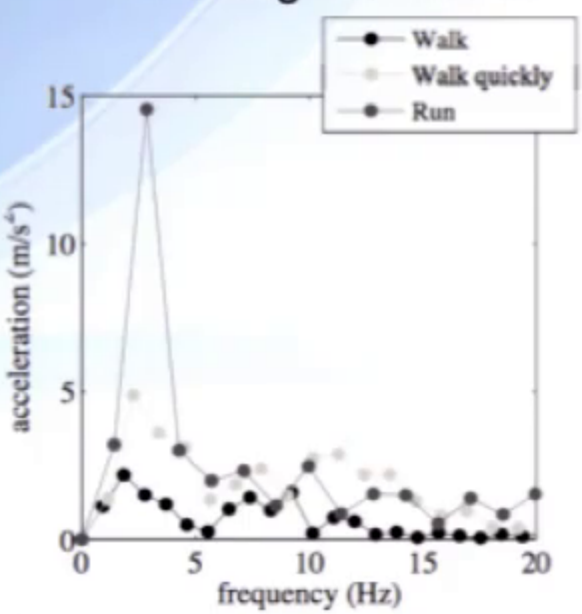
Funding provided by MIT Energy Initiative

Bi-stable energy harvester subjected to random excitations

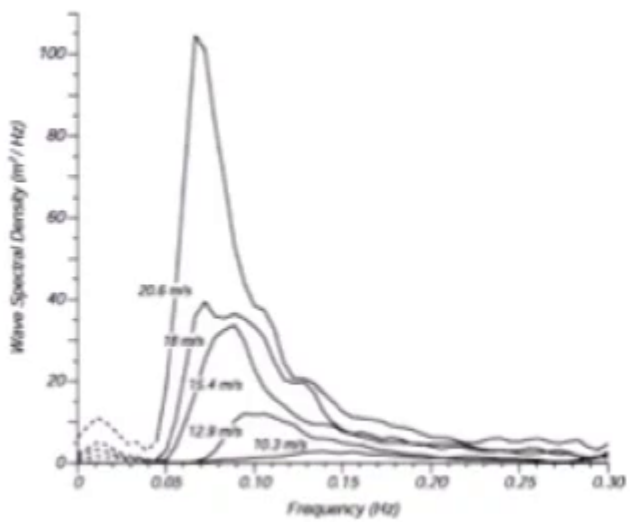


- Robustness under different loads
- Broadband operation
- Suitable for very weak loads
- Straightforward to implement

Walking vibrations



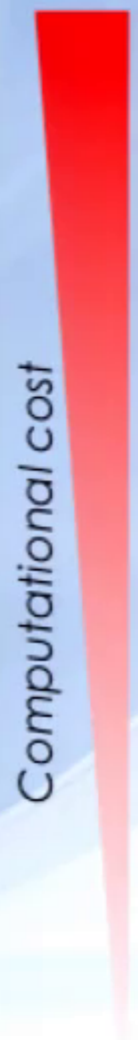
Ocean waves



Challenge: Most sources of energy are neither broadband nor monochromatic

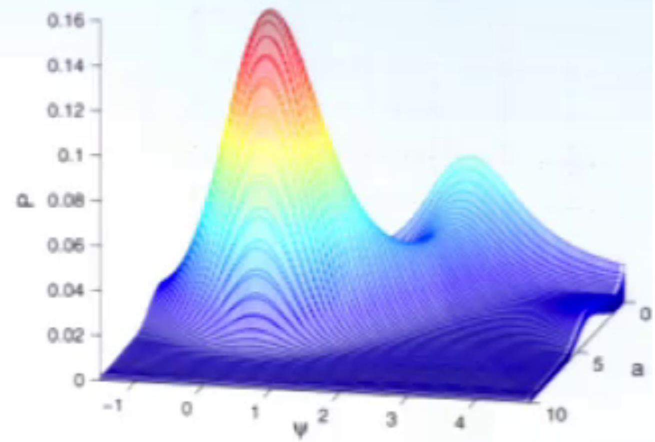
Goal: Model the stochastic dynamics of a strongly nonlinear system involving multiple time scales and correlated excitations

Methods to analyze systems subjected to correlated noise



Fokker Planck Equation + Filters

- Model the excitation as filtered white-noise
- Solve the coupled system+filter FP equation
- Very expensive and often unrealistic



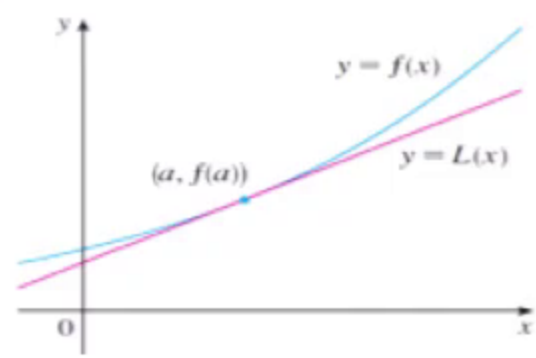
Polynomial Chaos (Wiener, …)

- Expand excitation and solution in a PC series
- Slow convergence for non-Gaussian responses…
- Little information about non-Gaussian statistics

$$u = \sum_{J=0}^{N_{PC}} \hat{u}_J \Psi_J(\xi)$$

Statistical (non) linearization (Booton, Caughey, …)

- Approximate dynamics by the closest linear system
- Very powerful method for vibrational systems
- Fails for bi-stable (bi-modal) systems



Plan of the presentation – overview of the method

- **Overview of statistical linearization methods and their limitations**
- **The moment-equation-closure minimization method**
  → *Moment equations expressing two-times statistics*
  → *Two-times pdf representations and induced closure schemes*
  → *Simultaneous error minimization for both the moments and the closure*
- **Representation of the full probability density function**
- **Application to bistable systems**
  → *Application to Duffing oscillator excited by correlated noise*
  → *Application to a bistable electromechanical energy harvester*
  → *Comparison with Gaussian closure methods*

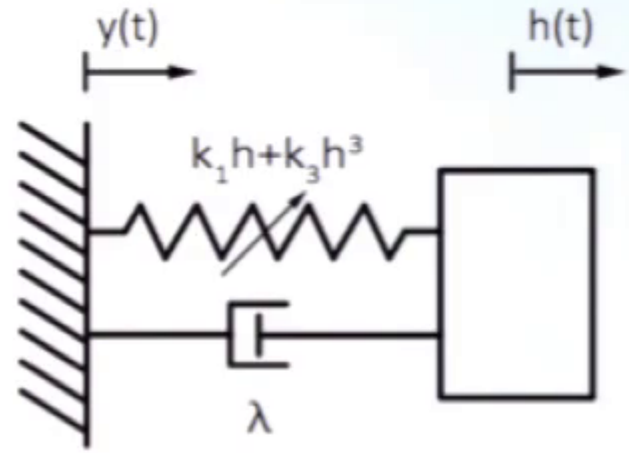
Overview of statistical linearization methods

Consider a nonlinear (SDOF) oscillator of the form:

$$\ddot{x} + \lambda \dot{x} + g(x) = \ddot{y}$$

$$g(x) = k_1 x + k_3 x^3$$

$$y \rightarrow \text{Correlated random excitation}$$



Statistical linearization: *substitute the non-linear system by the "closest" linear*

$$\ddot{x} + \lambda \dot{x} + k_0 x = \ddot{y}$$

How to choose k_0?

$$E\left[\Delta^2\right] = E\left[\left(k_0 x - g(x)\right)^2\right] = \min$$

$$\Rightarrow k_0 = \frac{E\left[x g(x)\right]}{E\left[x^2\right]}$$

Overview of statistical linearization methods

Step 1: Adoption of a pdf representation (single-time statistics)

e.g. if Gaussian statistics for the response is assumed:

$$k_0 = \frac{E[xg(x)]}{E[x^2]} = \frac{E[k_1 x^2 + k_3 x^4]}{E[x^2]} = k_1 + 3k_3 \boxed{\sigma_X^2}$$ *Isserlis' Theorem*

unknown

Non-Gaussian pdf may also be utilized (statistical non-linearization)

Step 2: Two-times moment equation for the linear system

$$\ddot{x} + \lambda \dot{x} + k_0 x = \ddot{y}$$

WK Theorem

$$S_{XX}(\omega) = \frac{-\omega^2}{\left|-\omega^2 + \lambda i \omega + k_0\right|^2} S_{YY}(\omega)$$ *Two-time statistics*

$$\sigma_X^2 = \int_0^\infty \frac{-\omega^2}{\left|-\omega^2 + \lambda i \omega + k_1 + 3k_3 \sigma_X^2\right|^2} S_{YY}(\omega)$$

Algebraic equation for σ_X^2

Limitations of statistical linearization methods

1. Moment equations express two-times statistics but adopted pdf representation is for a single-time statistics.

$$k_0 = \frac{E[xg(x)]}{E[x^2]}$$

Closure relies on single-time statistics…

Important for bi-stable system where we have rich correlation structure

2. Closure has to be exactly satisfied and all the mismatch is handled by the equation.

$$\sigma_X^2 = \int_0^\infty \frac{-\omega^2}{\left|-\omega^2 + \lambda i\omega + k_1 + 3k_3\sigma_X^2\right|^2} S_{YY}(\omega)$$

Information obtained by the equation under the condition

$$k_0 = \frac{E[xg(x)]}{E[x^2]}$$

What if the closure condition is not exactly satisfied?

bi-stable systems have non-trivial pdf structure

The moment-equation-closure minimization method

Step 1: Develop a pdf representation for two-times statistics

We want this representation to:

i. incorporate specific properties or information about the response pdf (single time statistics) in the statistical steady state

ii. incorporate a given correlation structure between the statistics of the response and the excitation, e.g. Gaussian

iii. have a consistent marginal with the excitation pdf (for the case of the joint response-excitation pdf),

iv. induce a non-Gaussian closure scheme that will be consistent with all the above properties.

The moment-equation-closure minimization method

Step 1: Develop a pdf representation for two-times statistics

Single-time statistics:

$$f(x;\gamma) = \frac{1}{\mathcal{F}} \exp\left\{ -\frac{1}{\gamma}\left(\frac{1}{2}k_1 x^2 + \frac{1}{4}k_3 x^4\right)\right\}$$

Shape that is consistent with the exact solution of the FP equation but with a free parameter

Two-times statistics:

response-excitation pdf
x(t)y(s)

$$q(x,y) = \frac{1}{\mathcal{M}} f(x) g(y) e^{cxy}$$

response-response pdf
x(t)x(s)

$$p(x,z) = \frac{1}{\mathcal{N}} f(x) f(z) e^{cxz}$$

$f(x)$: marginal for x(t) or x(s)

$g(y)$: marginal for x(t)

c : depends on *t-s & expresses degree of correlation*

Generic non-Gaussian marginals - Gaussian correlation structure

The moment-equation-closure minimization method

Two-times statistics:

response-excitation pdf
x(t)y(s)

$$q(x,y) = \frac{1}{\mathcal{M}} f(x) g(y) e^{cxy}$$

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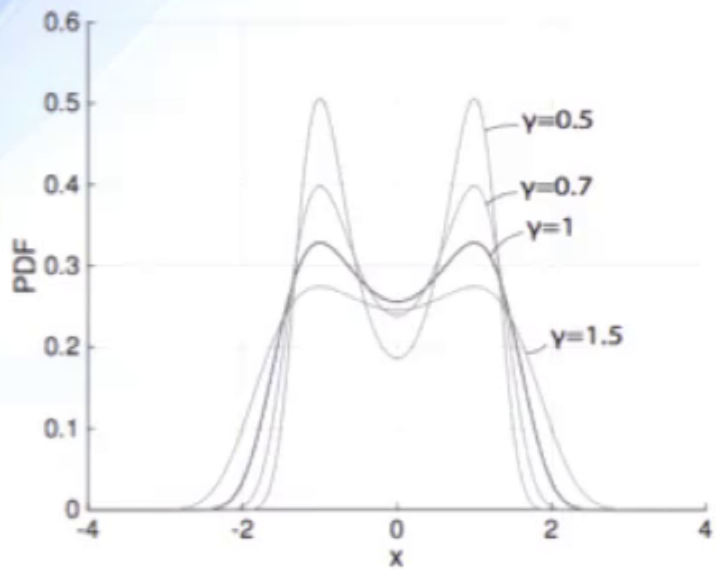
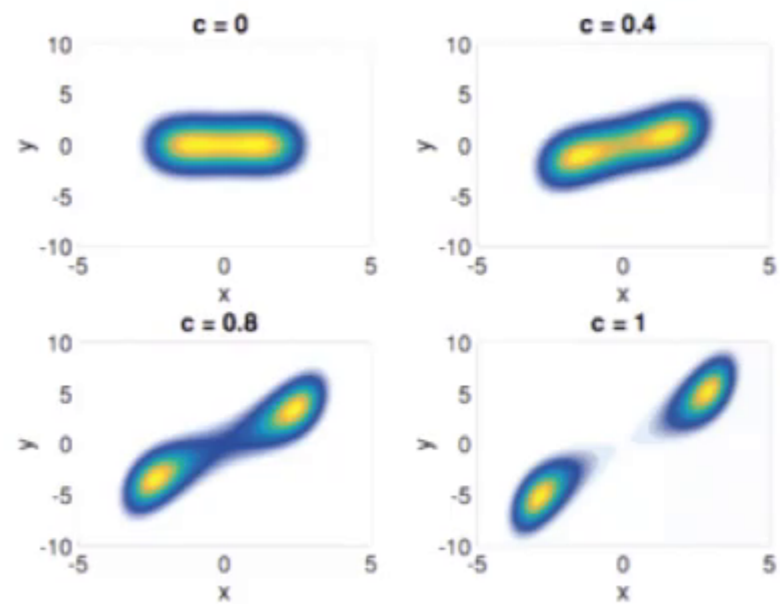




The moment-equation-closure minimization method

Step 2: Formulation of moment equations for the original system

$$\overline{\ddot{x}(t)y(s)} + \lambda\overline{\dot{x}(t)y(s)} + k_1\overline{x(t)y(s)} + k_3\overline{x(t)^3y(s)} = \overline{\ddot{y}(t)y(s)}$$
$$\overline{\ddot{x}(t)x(s)} + \lambda\overline{\dot{x}(t)x(s)} + k_1\overline{x(t)x(s)} + k_3\overline{x(t)^3x(s)} = \overline{\ddot{y}(t)x(s)}$$

Assuming statistical stationarity: $\tau = t - s$

$$\frac{\partial^2}{\partial\tau^2}C_{xy}(\tau) + \lambda\frac{\partial}{\partial\tau}C_{xy}(\tau) + k_1C_{xy}(\tau) + k_3\boxed{\overline{x(t)^3y(s)}} = \frac{\partial^2}{\partial\tau^2}C_{yy}(\tau),$$
$$\frac{\partial^2}{\partial\tau^2}C_{xx}(\tau) + \lambda\frac{\partial}{\partial\tau}C_{xx}(\tau) + k_1C_{xx}(\tau) + k_3\boxed{\overline{x(t)^3x(s)}} = \frac{\partial^2}{\partial\tau^2}C_{xy}(-\tau)$$

Different equations for $C_{xx}(\tau)$ *and* $C_{xy}(\tau)$

In statistical (non-) linearization closure is applied directly to the governing eq.

Here we apply closure to the exact two-times moment equations instead…

The moment-equation-closure minimization method

Step 3*: Induced two-times closures for the terms* $\overline{x(t)^3y(s)}$ ***and*** $\overline{x(t)^3x(s)}$

With some explicit computations using the two-times pdf representations we obtain

Closure constraint

$$\overline{x(t)^3x(s)} = \rho_{x,x}\,\overline{x(t)x(s)} \qquad \rho_{x,x} = \frac{\overline{x^4}}{\overline{x^2}}$$

function of γ

Using similar arguments we obtain a closure for $\overline{x(t)^3y(s)}$

$$\rho_{x,y} = \frac{\overline{x^3y}}{\overline{xy}} = \frac{\bar{x^4}\bar{y^2}c + \frac{1}{6}\left(\bar{x^6}\bar{y^4} - 3\bar{x^4}\bar{x^2}(\bar{y^2})^2\right)c^3}{\bar{x^2}\bar{y^2}c + \frac{1}{6}\left(\bar{x^4}\bar{y^4} - 3(\bar{x^2})^2(\bar{y^2})^2\right)c^3}$$

The moment-equation-closure minimization method

Substitute the induced two-times closures to the moment equations

$$\frac{\partial^2}{\partial\tau^2}C_{xy}(\tau)+\lambda\frac{\partial}{\partial\tau}C_{xy}(\tau)+(k_1+\rho_{x,y}k_3)C_{xy}(\tau)=\frac{\partial^2}{\partial\tau^2}C_{yy}(\tau),$$
$$\frac{\partial^2}{\partial\tau^2}C_{xx}(\tau)+\lambda\frac{\partial}{\partial\tau}C_{xx}(\tau)+(k_1+\rho_{x,x}k_3)C_{xx}(\tau)=\frac{\partial^2}{\partial\tau^2}C_{xy}(-\tau)$$

We transform the two time equations to spectrum equations

$$S_{xx}(\omega)=\left|\frac{\omega^4}{\{k_1+\rho_{x,y}k_3-\omega^2+j(\lambda\omega)\}\{k_1+\rho_{x,x}k_3-\omega^2-j(\lambda\omega)\}}\right|S_{yy}(\omega)$$

From which we obtain the following constraint:

Dynamic constraint

$$\overline{x^2}=\int_0^\infty\left|\frac{\omega^4}{\{k_1+\rho_{x,y}k_3-\omega^2+j(\lambda\omega)\}\{k_1+\rho_{x,x}k_3-\omega^2-j(\lambda\omega)\}}\right|S_{yy}(\omega)d\omega$$

The moment-equation-closure minimization method

Step 4*: Simultaneous minimization of the two constraints*

$$\mathcal{J}(\gamma, \rho_{x,x}) = \underbrace{\left(\overline{x^2} - \int_0^{\infty} \left| \frac{\omega^4 S_{yy}(\omega)}{\{k_1 + \rho_{x,y}k_3 - \omega^2 + j(\lambda\omega)\}\{k_1 + \rho_{x,x}k_3 - \omega^2 - j(\lambda\omega)\}} \right| d\omega\right)^2}_{\text{Dynamics constraint}} + \underbrace{\left(\overline{x^2} - \frac{\overline{x^4}}{\rho_{x,x}}\right)^2}_{\text{Closure constraint}}$$

Notes:

- For the case where the closure constraint is exactly satisfied we recover the statistical (non-) linearization method.
- After we obtain the two unknowns we can go back and recover the correlation functions $C_{xx}(\tau)$ and $C_{xy}(\tau)$

The moment-equation-closure minimization method

- Using the values of the correlation functions we can find the constant *c* to obtain the full joint (two-times) pdf:

Full pdf representation

$$f_{x(t)x(t+\tau)y(t+\tau)}(x,z,y) = \frac{1}{\mathcal{R}} f(x;\gamma) f(z;\gamma) g(y) exp(c_1 xz + c_2 xy + c_3 yz)$$

Correlation functions from the pdf…

$$C_{xx}(\tau) = \iiint xz f_{x(t)x(t+\tau)y(t+\tau)}(x,z,y) dxdydz = c_1 \left(\bar{x^2}\right)^2 + \mathcal{O}\left(c_1^2\right)$$

$$C_{xy}(\tau) = \iiint xy f_{x(t)x(t+\tau)y(t+\tau)}(x,z,y) dxdydz = c_2 \bar{x^2}\bar{y^2} + \mathcal{O}\left(c_2^2\right)$$

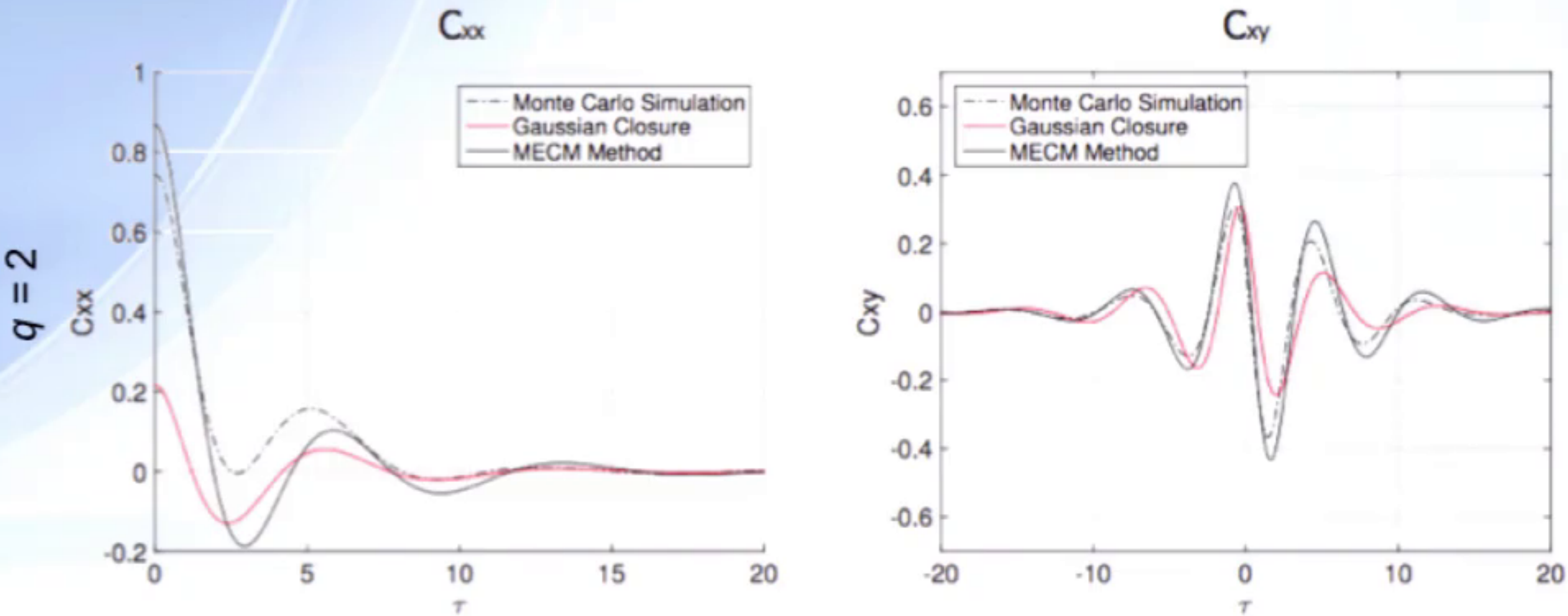
$$C_{xy}(0) = \iiint yz f_{x(t)x(t+\tau)y(t+\tau)}(x,z,y) dxdydz = c_3 \bar{x^2}\bar{y^2} + \mathcal{O}\left(c_3^2\right)$$

Application to the Duffing equation under correlated excitation

$$\ddot{x} + \lambda\dot{x} + k_1 x + k_3 x^3 = \ddot{y}$$

excitation is a Gaussian with spectrum

$$S(\omega) = q\,\frac{1}{\omega^5}\exp(-\frac{1}{\omega^4})$$



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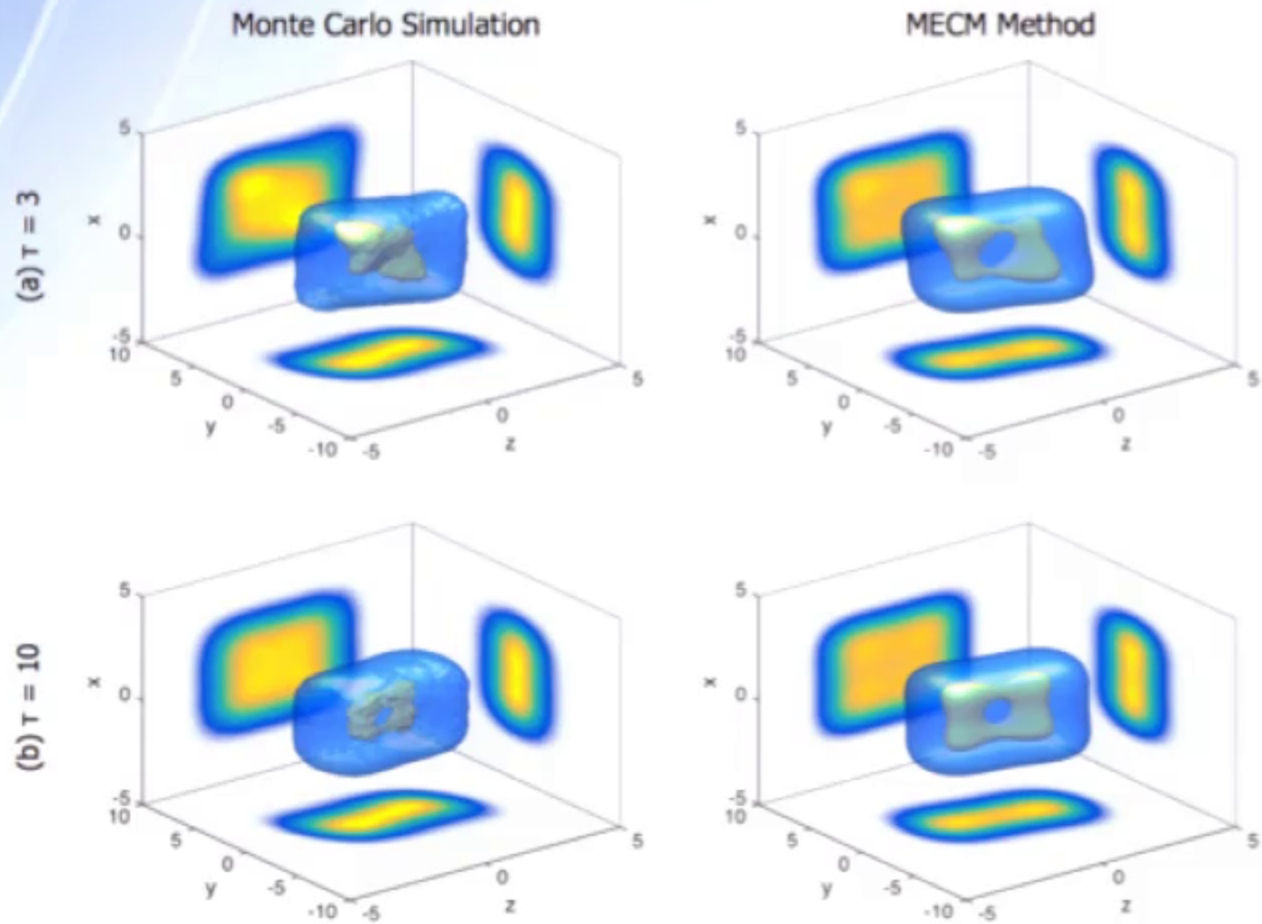
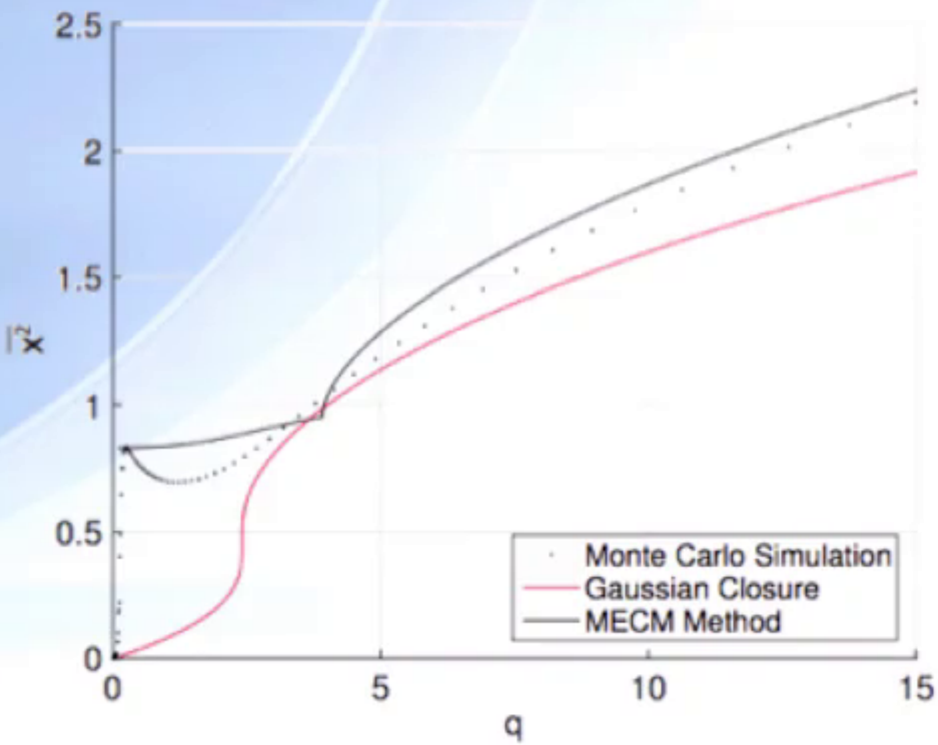
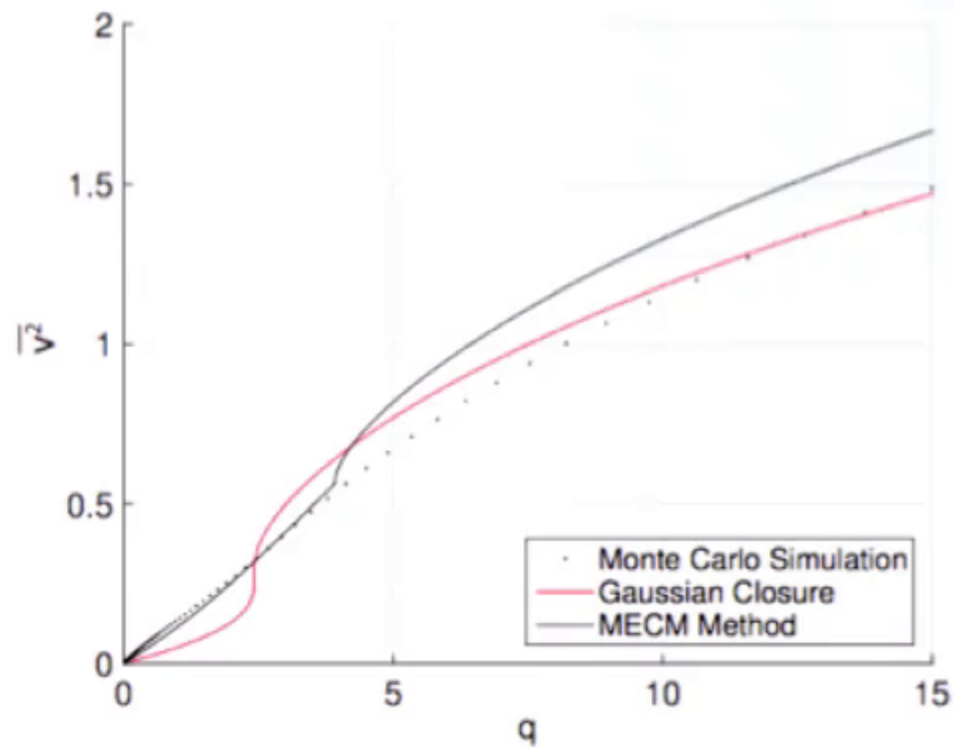


Bistable oscillator coupled to an electromechanical harvester

$$\ddot{x} + \lambda\dot{x} + k_1 x + k_3 x^3 + \alpha v = \ddot{y},$$
$$\dot{v} + \beta v = \delta\dot{x}$$

excitation is a Gaussian with spectrum $S(\omega) = q\,\frac{1}{\omega^5}\exp(-\frac{1}{\omega^4})$





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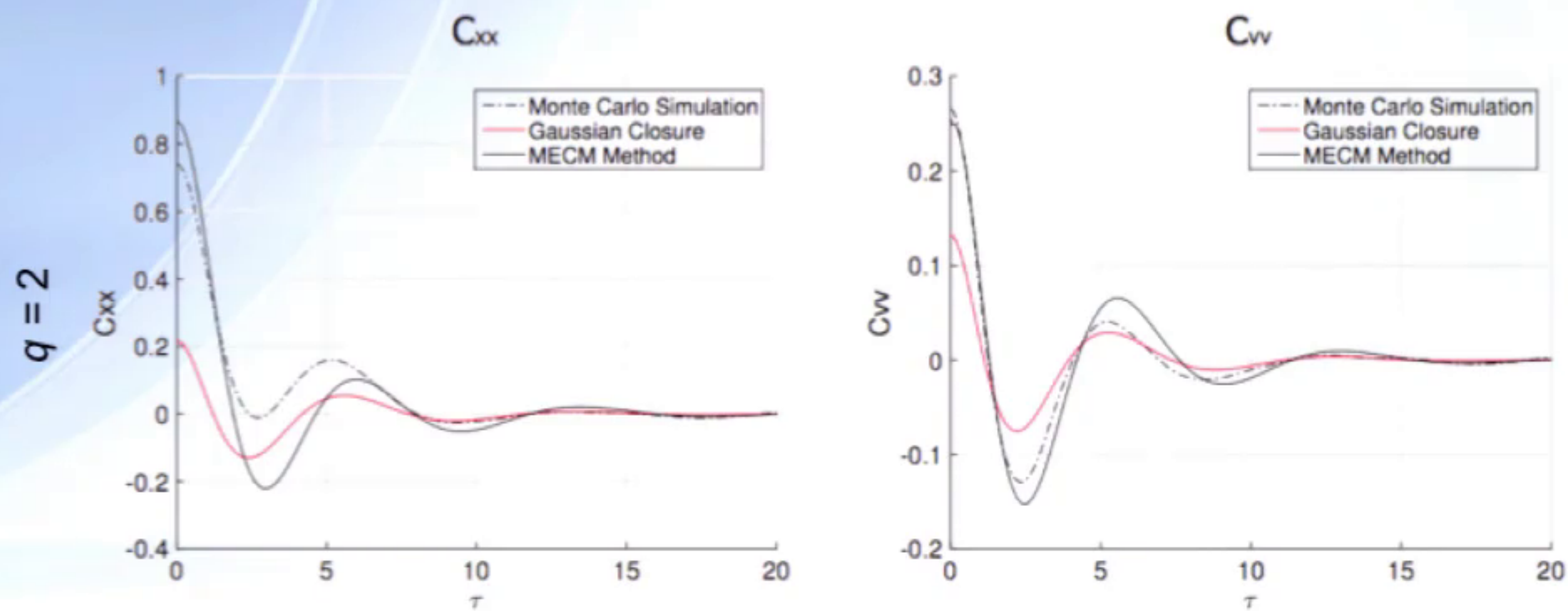

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