Time Discrete Approximation for Stochastic Equations of Geophysical Fluid Dynamics

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Articles/collaborators

- N. Glatt-Holtz, R. Temam, C. Wang, Time discrete approximation of weak solutions for stochastic equations of geophysical fluid dynamics and applications, accepted.
- ——, Numerical Analysis of the Stochastic Navier-Stokes Equations: Stability and Convergence, in preparation.

Topics

- Introduction.
- Time discrete scheme.
- Existence of adapted solutions to the scheme.
- Convergence of the scheme.

The stochastic primitive equations

Let $U = (\mathbf{v}, T, S)$ be the set of prognostic variables.

The abstract form of the stochastic PEs reads

$$\begin{cases} d\,U + (AU + B(U) + E(U))\,dt = \ell\,dt + \sigma(U)dW, \\ U(0) = U_0, \end{cases}$$

where we will denote for simplicity

$$\mathcal{N}(t, U) := -(AU + B(U) + E(U)).$$

The stochastic framework

- We are given a stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$.
- The function U takes its values in a Hilbert space H.
- The cylindrical Brownian motion W takes its values in an auxiliary Hilbert space $\mathfrak{U}$ with basis $\{\psi_i\}_{i\geq 0}$:

$$W = \sum_{i=0}^{\infty} W_i \, \psi_i,$$

where $\{W_i\}_{i\geq 0}$ is a sequence of independent standard one-dimensional Brownian motions adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.
- $\sigma(U) \in L_2(\mathfrak{U}, H)$ is uniformly sublinear as a function of U.

The stochastic Primitive Equations of the oceans

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + w\partial_z \mathbf{v} + \frac{1}{\rho_0}\nabla p + f\mathbf{k}\times\mathbf{v} - \mu_{\mathbf{v}}\Delta\mathbf{v} - \nu_{\mathbf{v}}\partial_{zz}\mathbf{v} = F_{\mathbf{v}} + \sigma_{\mathbf{v}}(\mathbf{v}, T, S)\dot{W}_1,$$

$$\partial_z p = -\rho g,$$

$$\nabla\cdot\mathbf{v} + \partial_z w = 0,$$

$$\partial_t T + (\mathbf{v}\cdot\nabla)T + w\partial_z T - \mu_T\Delta T - \nu_T\partial_{zz}T = F_T + \sigma_T(\mathbf{v}, T, S)\dot{W}_2,$$

$$\partial_t S + (\mathbf{v}\cdot\nabla)S + w\partial_z S - \mu_S\Delta S - \nu_S\partial_{zz}S = F_S + \sigma_S(\mathbf{v}, T, S)\dot{W}_3,$$

$$\rho = \rho_0(1 - \beta_T(T - T_r) - \beta_S(S - S_r)).$$

- $\mathbf{u} := (\mathbf{v}, w)$ - velocity field,
- T - temperature,
- p - pressure,
- ρ - density.

The stochastic Primitive Equations of the oceans

Domain & Boundary Conditions

Physical domain: $\mathcal{M} := \left\{ \mathbf{x} := (x, y, z) \in \mathbb{R}^3 : (x, y) \in \Gamma_i, z \in (-h(x, y), 0) \right\}.$

$$
\begin{aligned}
&\partial_z \mathbf{v} + \alpha_{\mathbf{v}}(\mathbf{v} - \mathbf{v}^a) = \tau_{\mathbf{v}}, \quad w = 0, \quad \partial_z T + \alpha_T (T - T^a) = 0, \quad \partial_z S = 0, \\
&\quad \text{on } \Gamma_i \\
&\mathbf{v} = 0, \quad w = 0, \quad \partial_{\mathbf{n}} T = 0, \quad \partial_{\mathbf{n}} S = 0, \quad \text{on } \Gamma_b, \\
&\mathbf{v} = 0, \quad \partial_{\mathbf{n}} T = 0, \quad \partial_{\mathbf{n}} S = 0, \quad \text{on } \Gamma_l,
\end{aligned}
$$

where $\alpha_{\mathbf{v}}$, α_T are fixed positive constants and $\tau_{\mathbf{v}}$, $\mathbf{v}^a$, T^a are in general random and non-constant in space and time.

Time discrete scheme

Fix $T > 0$, and, for any integer N, let $\Delta t = T/N$,

$$t^n = t^n_N = n\Delta t, \;\; \eta^n = \eta^n_N = W(t^n) - W(t^{n-1}), \; n = 1, \ldots, N.$$

We introduce the following implicit Euler scheme:

$$\frac{U^n_N - U^{n-1}_N}{\Delta t} + AU^n_N + B(U^n_N) + EU^n_N = \ell^n_N + \xi(t^n, U^n_N) + \sigma_N(t^{n-1}, U^{n-1}_N)\frac{\eta^n_N}{\Delta t},$$

where

$$\ell^n_N(U^\sharp) = \frac{1}{\Delta t}\int_{(n-1)\Delta t}^{n\Delta t} \ell(t, U^\sharp)dt \quad \text{for } n = 1, 2, \ldots, N,$$

$$\lim_{N\to\infty} \sigma_N(t, U_N) = \sigma(t, U), \quad \text{whenever } U_N \to U \text{ in } H.$$

Existence of weak martingale solutions

Theorem (GHTW, '14)

We only specify U^0 and ℓ as measures (denoted as μ_{U^0}, μ_ℓ) such that

$$\mu_{U^0}(|\cdot|_H^2) < \infty \quad and \quad \mu_\ell(\|\cdot\|^2_{L^2_{loc}(0,\infty;V')}) < \infty.$$

Then there exists a ***martingale solution*** $(\widetilde{\mathcal{S}}, \widetilde{U}, \widetilde{\ell})$ *of the abstract stochastic equation satisfying*

$$\widetilde{U} \in L^2(\widetilde{\Omega}; L^\infty_{loc}(0,\infty;H) \cap L^2_{loc}(0,\infty;V)),$$

and $\widetilde{U}$ a.s. weakly continuous in H.

Challenges due to the stochasticity

- A measurable selection theorem is not enough to derive the existence of adapted solutions to the scheme.

- Piecewise linear approximation introduces troublesome error terms due to the necessity of ensuring that the processes are adapted.

- Compactness arguments are made more difficult due to the complicated error terms and lack of higher order moments for the estimates of U_N^n.

A measurable selection theorem

Lemma (Bensoussna & Temam, '73)

Let $\Lambda(t, F)$ *be a multivalued mapping from* $(0, T) \times V'$ *into the subsets of* V *with the values being all the admissible solutions of the Euler scheme. Then there exists a map* $\Gamma : (0, T) \times V' \to V$ *which is universally Radon measurable such that for every* $t \in (0, T)$ *and every* $F \in V'$*,*
$U := \Gamma(t, F) \in \Lambda(t, F)$*,*

The Wiener space and a specific stochastic basis

Let $\Omega = \mathcal{C}([0,T];\mathfrak{U})$ equipped with its Borel σ-algebra denoted as $\mathcal{G}$ and the Wiener measure $\mathbb{P}$.

Then $W(\omega,t) := \omega(t)$, $\omega \in \Omega$, $t \in [0,T]$, is a cylindrical Wiener process in $\mathfrak{U}$, with the filtration $\mathcal{G}_t$ defined as the completion of $\sigma\{W(s); s \in (0,t)\}$ with respect to $\mathbb{P}$.

Let $\mathcal{S}_{\mathcal{G}} = (\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0}, \mathbb{P}, W)$, then it has the following key feature

$$\sigma\{W(s); s \in [0,t])\} = \phi_t^{-1}(\mathcal{B}(\mathcal{C}([0,t];\mathfrak{U})).$$

where $(\phi_t^{-1}\omega)(s) = \omega(t \wedge s)$; $0 \leq s \leq T$

Existence of the U_N^n's adapted to $\mathcal{G}_{t^n}$

Proposition (GHTW, '14)

Consider the stochastic basis $\mathcal{S}_{\mathcal{G}} = (\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\geq 0}, \mathbb{P}, W)$

Then there exists a sequence $\{U_N^n\}_{n=0}^N$ *as an admissible solution of the Euler scheme (satisfying the scheme and certain energy bounds) where* U_N^n *is adapted to* $\mathcal{G}_{t^n}$.

Continuous time approximations

We define

$$U_N(t) = \begin{cases} U_N^0, & \text{for } t \in [0, t^1], \\ U_N^n, & \text{for } t \in (t^n, t^{n+1}],\ \ n \geq 1. \end{cases}$$

$$\bar{U}_N(t) = \begin{cases} U_N^0, & \text{for } t \in [0, t^1], \\ U_N^{n-1} + \frac{U_N^n - U_N^{n-1}}{\Delta t}(t - t^n), & \text{for } t \in (t^n, t^{n+1}],\ \ n \geq 1. \end{cases}$$

Remark

The processes U_N and $\bar{U}_N$ are their deterministic analogues evaluated at time t by their value at time $t - \Delta t$, so that they are adapted to $\mathcal{G}_t$.

Derivation of the error terms

$$\bar{U}_N(t) = U_N^0 + \int_0^t (\mathcal{N}_N(U_N) + \ell_N)ds + \int_0^t \sigma_N(U_N)dW + \mathcal{E}_N^D(t) + \mathcal{E}_N^S(t),$$

where

$$\mathcal{E}_N^D(t) := -\mathcal{N}(U_N^0)\Delta t \wedge t - \left(\int_{t^{N_*^t-1}}^t \ell_N ds + \ell_N^{N_*^t-1}(t^{N_*^t} - t)\chi_{t>t^1} \right),$$

$$\mathcal{E}_N^S(t) := -\sigma_N(U_N^{N_*^t-2})\frac{\eta_N^{N_*^t-1}}{\Delta t}(t^{N_*^t} - t)\chi_{t>t^1} - \int_{t^{N_*^t-1}}^t \sigma_N(U_N)dW,$$

$$N_*^t = \inf\{n : t^n \geq t\}.$$

Convergence of the error terms

Proposition

$\mathcal{E}_N^S$ and $\mathcal{E}_N^D$ converge to 0 in $L^2(\Omega; L^2(0,T;H))$ strongly, and they remain bounded in $L^2(\Omega; L^2(0,T;V))$.

Problem: With only L^2 estimates on U_N, it is complicated to derive estimates on the fractional derivatives in time on these error terms.

An auxiliary sequence

We define

$$V_N(t) := \bar{U}_N(t) - \mathcal{E}_N^D(t) - \mathcal{E}_N^S(t)$$
$$= U_N^0 - \int_0^t (\mathcal{N}_N(U_N) + \ell_N) ds + \int_0^t \sigma_N(U_N) dW,$$

Then whenever we can verify the following

$$V_N \rightharpoonup U \text{ in distribution,}$$
$$\mathcal{E}_N^D, \quad \mathcal{E}_N^S \to 0 \text{ a.s.,}$$

we obtain

$$\bar{U}_N \rightharpoonup U \text{ in distribution.}$$

Summary

- As a first step towards the extension of the numerical schemes from the deterministic to the stochastic case, we have explored time discretization scheme for the stochastic Primitive Equations.
- As a second step, we are exploring the time and space discretization scheme for the stochastic Navier-Stokes equation, where the stability conditions need to be developed.

Thank you

A measurable selection theorem

Lemma (Bensoussna & Temam, '73)

Let $\Lambda(t, F)$ *be a multivalued mapping from* $(0, T) \times V'$ *into the subsets of* V *with the values being all the admissible solutions of the Euler scheme. Then there exists a map* $\Gamma : (0, T) \times V' \to V$ *which is universally Radon measurable (Radon measurable for every Radon measure on* $(0, T) \times V'$*), such that for every* $t \in (0, T)$ *and every* $F \in V'$*,* $U := \Gamma(t, F) \in \Lambda(t, F)$*,*

Construction of an adapted solution

We can build the desired sequence $\{U_N^n\}_{n=0}^N$ inductively as follows:

$$U_N^n = f_N^n(W\big|_{[0,t^n]}),$$

with $f_N^n : \mathcal{C}([0,t^n];\mathfrak{U}_0) \to V$ measurable for V equipped with $\mathcal{B}(V)$ and $\mathcal{C}([0,t^n];\mathfrak{U}_0)$ equipped with $\mathcal{G}_{t^n}$.
Suppose that we have obtained U_N^{n-1} for some $n \geq 2$. Then by the measurable selection theorem, we have

$$\begin{aligned} U_N^n &= \Gamma(t^n, f_N^{n-1}(W\big|_{[0,t^{n-1}]}), L_N^n(W\big|_{[0,t^n]}), \eta_N^n) \\ &:= f_N^n(W\big|_{[0,t^n]}). \end{aligned}$$

where Γ is universally Radon measurable and hence f_N^n is measurable with respect to $\mathcal{G}_{t^n}$.