Outline

Problem definition:

Can finite-Size effects in networks be represented by stochastic dynamics

Stochastic dynamics
Kramers-Moyal expansion

Approach definition …

An example – the Kuramoto network

Thermodynamic limit
Numerical simulations
Drift and diffusion coefficients in finite-size networks

Network dynamics $$\dot{x}_n = \sum_{m=1}^{N} h_m(x_m; t, \{\sigma\})$$

‘Common’ variable(s) $$\rho = \rho(t) = \mathcal{F}(x_1, x_2, \ldots, x_N;\ t, \{\sigma\})$$

‘Representative’ dynamics $$\dot{\rho} = f(\rho;\ t, \{\sigma\})$$

1. **Time-scale separation**
   - Center manifold approach / Slaving
   - Order parameter dynamics

2. **Mean-field approach**
   - Symmetries
   - **Thermodynamic limit** $N \to \infty$
   - Order parameter dynamics

Network dynamics $\dot{x}_n = \sum_{m=1}^{N} h_m(x_m; t, \{\sigma\})$

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1. **Time-scale separation**
   - Center manifold approach / Slaving
   - Order parameter dynamics

How to modify the order parameter dynamics that is valid in the thermodynamic limit to accommodate $N < \infty$?

2. **Mean-field approximation**
   - Symmetries
   - Thermodynamic limit $N < \infty$
   - Order parameter dynamics

Network dynamics

$$\dot{x}_n = \sum_{m=1}^{N} h_m(x_m;\, t, \{\sigma\})$$

‘Common’ variable(s)

$$\rho = \rho(t) = \mathcal{F}(x_1, x_2, \ldots, x_N;\ t, \{\sigma\})$$

$$N \to \infty : \quad d\rho = f(\rho;\, t, \{\sigma\})\, dt$$

Idea: Approximate the finite-size order parameter dynamics by an stochastic extension of the dynamics in the thermodynamic limit

How to modify the order parameter dynamics that is valid in the thermodynamic limit to accommodate $N < \infty$?

$$1 \ll N < \infty : \quad d\rho = f(\rho;\, t, \{\sigma\})\, dt \;+\; g(\rho;\, t, \{\sigma\})\, dw$$

Stochastic differential equations

$$d\xi(t) = f(\xi(t), t)\,dt + g(\xi(t), t)\,dw$$

$$\dot{\xi}(t) = f(\xi(t), t) + g(\xi(t), t)\,\Gamma(t)$$

Mean-centered (Gaussian) white noise

$$\mathbb{E}[\Gamma(t)] = 0 \quad \text{and} \quad \mathbb{E}\left[\Gamma(t)\Gamma(t')\right] = \delta(t - t')$$

Stochastic differential equations

$$\dot{\xi}(t) = f(\xi(t), t) + g(\xi(t), t)\,\Gamma(t)$$

Time-dependent probability densities

$$\mathbb{E}\left[\delta(x - \xi(t))\right] = p_{\xi(t)}(x) = p(x, t)$$

Markov processes

$$\forall\, t_1 > t_2 > \dots > t_n : \; p(x_1, t_1 | x_2, t_2, x_3, t_3, \dots, x_n, t_n) = p(x_1, t_1 | x_2, t_2)$$

Kramers-Moyal expansion

$$\partial_t\, p(x, t | x_0, t_0) = \sum_{k=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^k \left[D^{(k)}(x, t)\, p(x, t | x_0, t_0)\right]$$

$$\text{with} \quad D^{(k)}(x, t) = \frac{1}{k!} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\left[\left(\xi(t + \Delta t) - \xi(t)\right)^k\right]$$

System identification via the Kramers-Moyal expansion

$$\dot{\xi} = D^{(1)}(\xi, t) + \sqrt{2D^{(2)}(\xi, t)}\,\Gamma(t)$$

$$\frac{\partial}{\partial t} P(x,t) = \sum_{k}^{\infty} \frac{1}{k!} \left(-\frac{\partial}{\partial x}\right)^k D^{(k)}(x,t) P(x,t)$$

$$D^{(k)}(x,t) = \frac{1}{k!} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\left[\left(\xi(t+\Delta t) - \xi(t)\right)^k\right]$$

$$D^{(k)}(x,t) = \frac{1}{k!} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \left(x' - x\right)^k p(x', t+\Delta t | x, t)\, dx'$$

Gauss process: $\forall_{k>2} : D^{(k)} = 0$ $\quad$ => Fokker-Planck eq. (= Kolmogorov backward eq.)

$$\dot{p}\left(x, t | x_0, t_0\right) = -\frac{\partial}{\partial x}\left[D^{(1)}\left(x\right) p\left(x, t | x_0, t_0\right)\right] + \frac{\partial^2}{\partial x^2}\left[D^{(2)}\left(x\right) p\left(x, t | x_0, t_0\right)\right]$$

Approach …

1. Simulate $\dot{x}_n = \sum_{m=1}^{N} h_m(x_m; t, \{\sigma\})$ for different network sizes N

2. Compute $\rho = \rho(t) = \mathcal{F}(x_1, x_2, \ldots, x_N; t, \{\sigma\})$ for every N

3. Estimate $D^{(k)}(\rho, t) = \frac{1}{k!} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \left(\rho' - \rho\right)^k p_{\text{est}}(\rho', t + \Delta t | \rho, t)\, d\rho'$

- *Verify Markovianity via Chapman-Kolmogorov equation, statistics can be realized using either a χ^2-test or – better – via the Wasserstein distance*

Analyze the coefficients

- Test if $D^{(1)}$ resembles the thermodynamic limit
- Test if $D^{(4)}$ vanishes [Gauss process: $\forall_{k>2} : D^{(k)} = 0$]
- Test if $D^{(2)}$ is constant (additive vs. multiplicative noise)

An example – the Kuramoto network

Network dynamics

$$\dot{\phi}_n = \omega_n + \frac{K}{N}\sum_{m=1}^{N} \sin(\phi_m - \phi_n)$$

'Common' variable

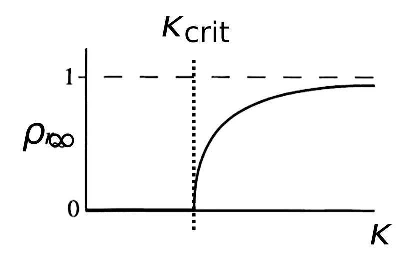
$$\rho(t) = \left| \int_0^{2\pi} e^{i\phi}\, p(\phi, t; \omega)\, d\phi \right|$$

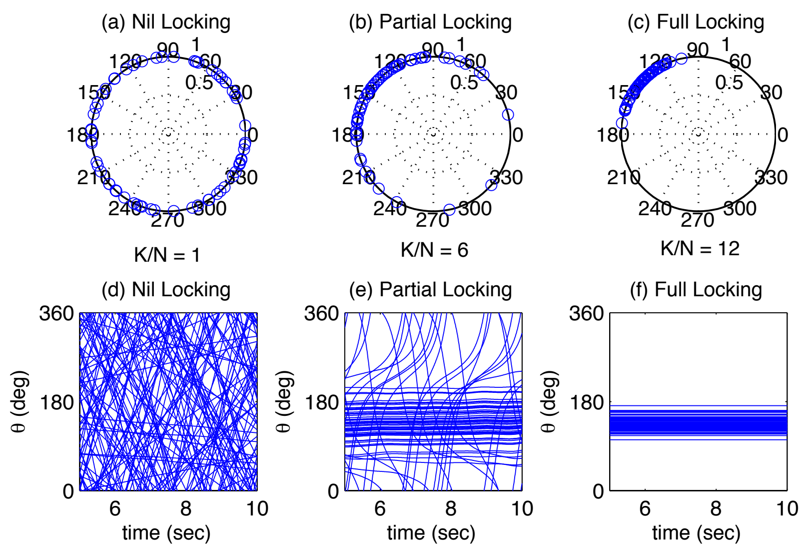
$$z = \frac{1}{N}\sum_{l=1}^{N} e^{i\phi_l}$$

'Representative' dynamics

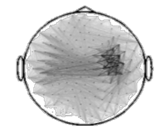
$$\dot{\rho} = \left(-\Delta + \frac{K}{2}\right)\rho - \frac{K}{2}\rho^3$$

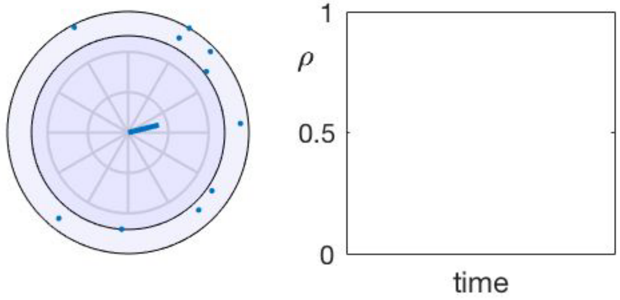
$$\dot{z} = \left(-\Delta + \frac{K}{2}\right) z - \frac{K}{2}|z|^2 z$$



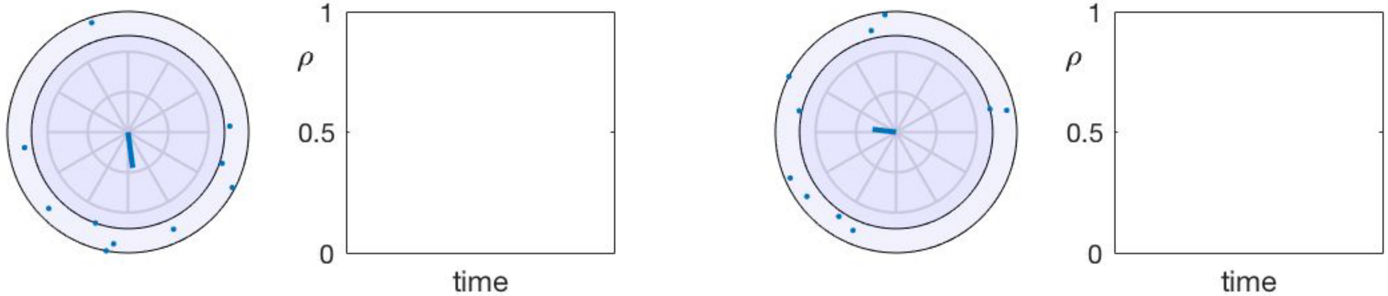


The Kuramoto network $\dot{\phi}_n = \omega_n + \frac{\kappa}{N}\sum_{m=1}^{N}\sin(\phi_m - \phi_n)$





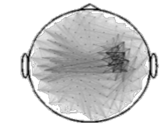
$$\rho = \left|\frac{1}{N}\sum_{l=1}^{N} e^{i\phi_l}\right|$$

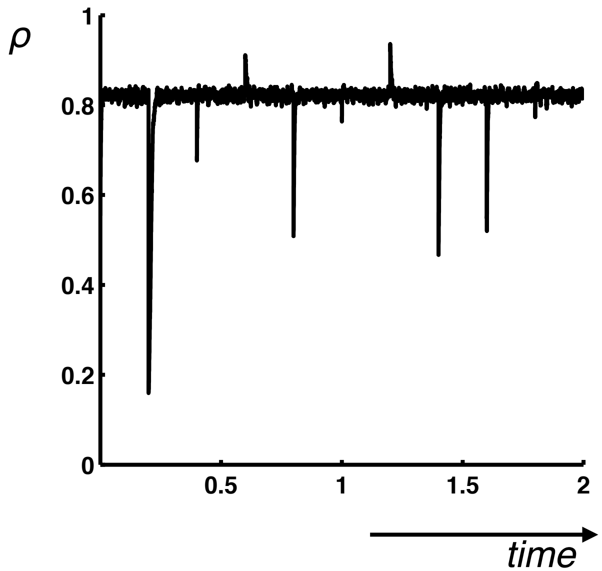


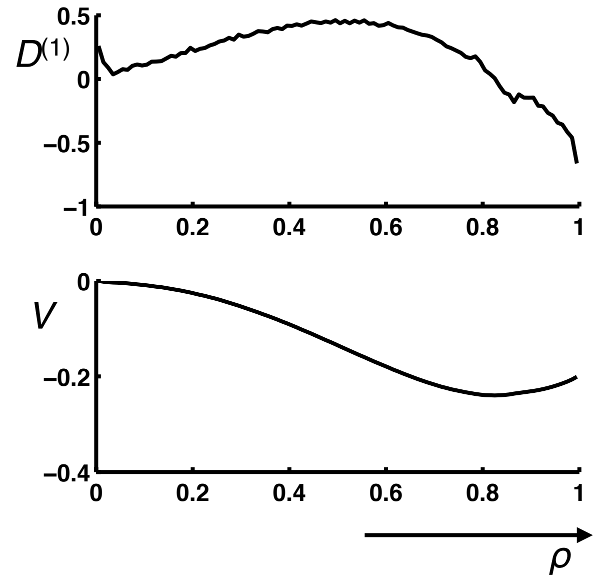
Does the order parameter evolve like $\dot{\rho} = D^{(1)}(\rho,t) + \sqrt{2D^{(2)}(\rho,t)}\,\Gamma(t)$?

Does the deterministic part look like $\dot{\rho} = -\left(\Delta - \frac{\kappa}{2}\right)\rho - \frac{\kappa}{2}\rho^3$?

The Kuramoto network $\dot{\phi}_n = \omega_n + \frac{\kappa}{N}\sum_{m=1}^{N}\sin(\phi_m - \phi_n)$



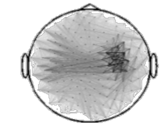




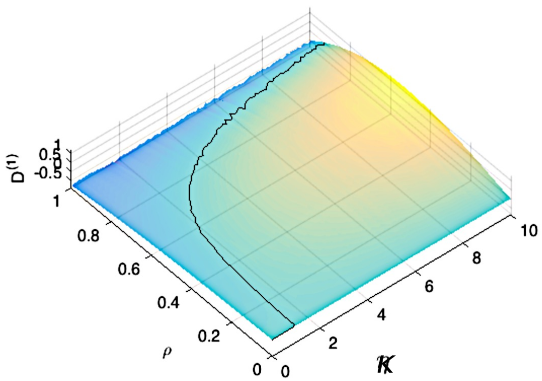
- ***N*** **oscillators, integration via Runge-Kutta 4/5; initial step-size 10^{-3}; duration 10s**
- **100 initial conditions $\phi_n(0)$ drawn from a Von Mises distrib. specifying $\rho(0)$**

The Kuramoto network

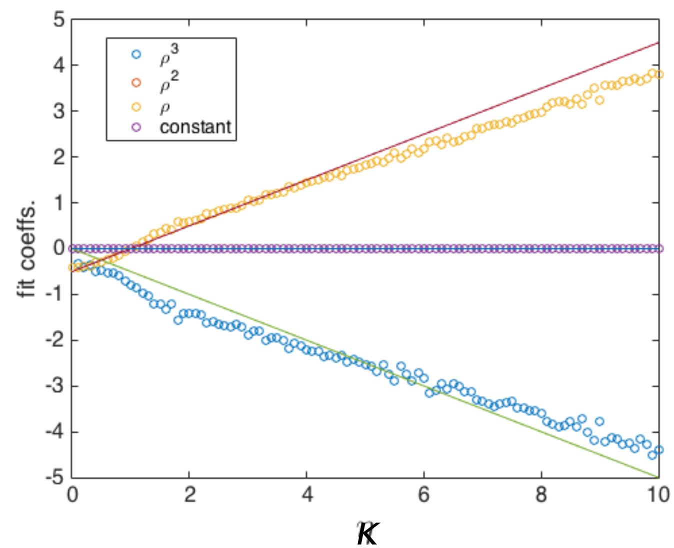
$$\dot{\phi}_n = \omega_n + \frac{\kappa}{N} \sum_{m=1}^{N} \sin(\phi_m - \phi_n)$$



Drift coefficient $D^{(1)}$

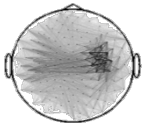


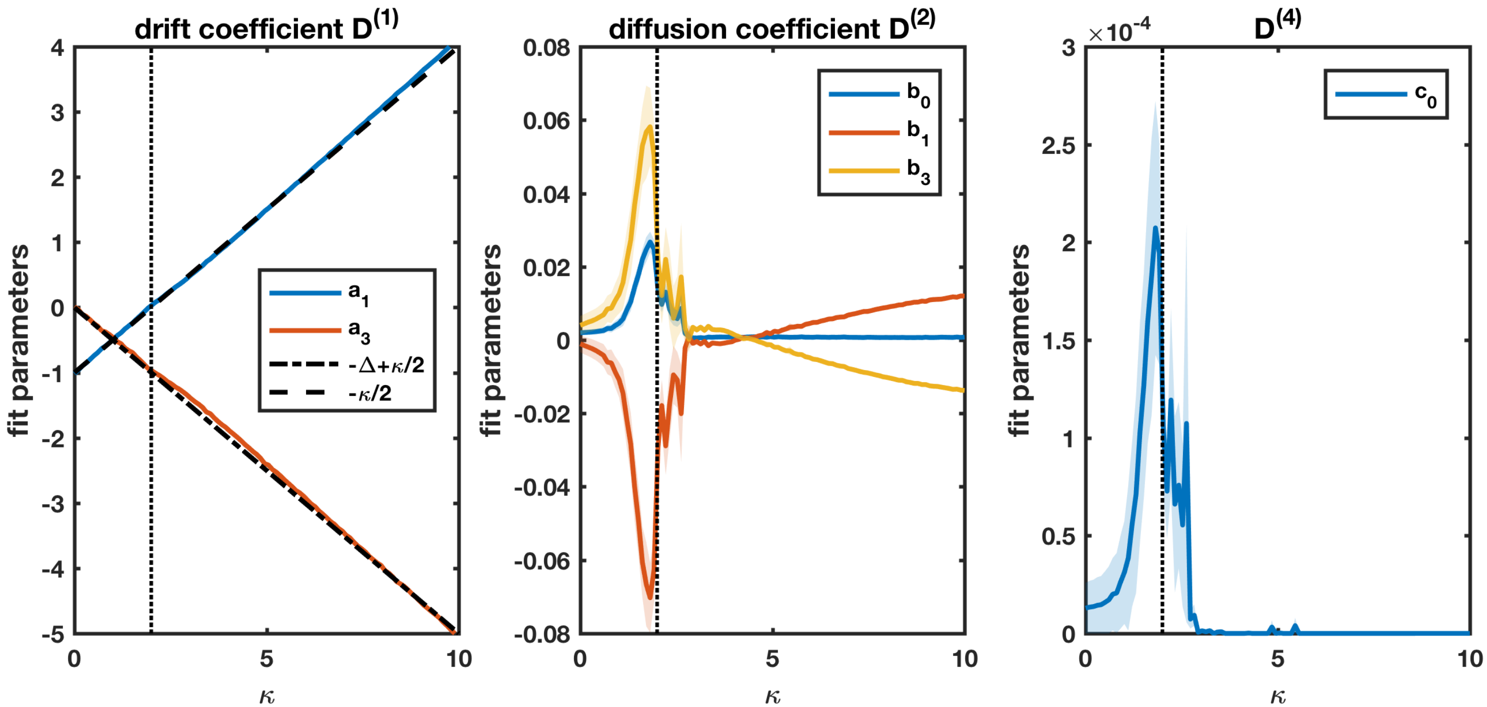
$D^{(1)}(\rho) \approx a_0 + a_1\rho + a_2\rho^2 + a_3\rho^3$



The Kuramoto network

$$\dot{\phi}_n = \omega_n + \frac{\kappa}{N}\sum_{m=1}^{N} \sin(\phi_m - \phi_n)$$

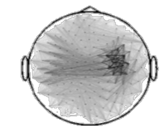


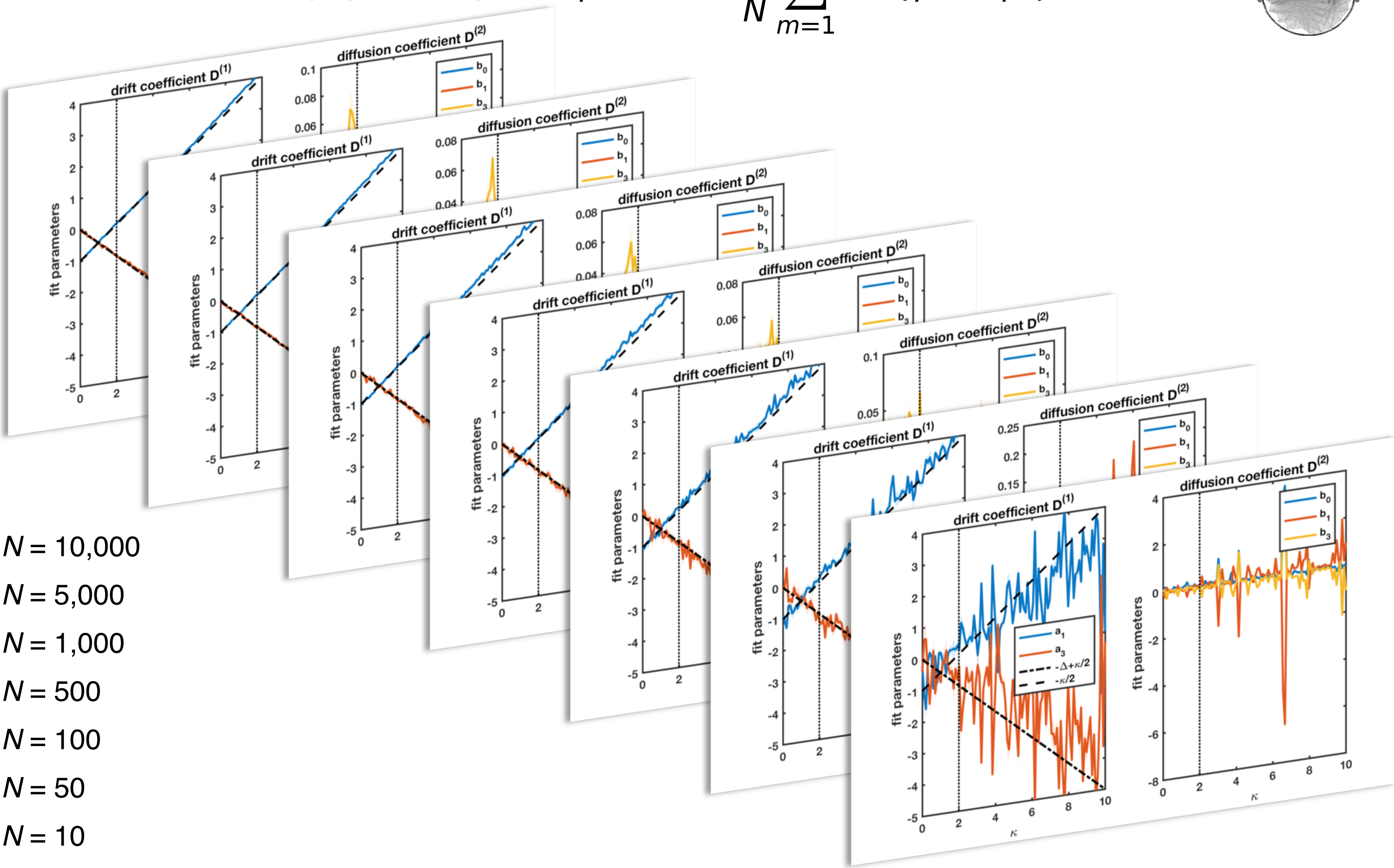


Number of nodes $N = 100{,}000$

The Kuramoto network

$$\dot{\phi}_n = \omega_n + \frac{\kappa}{N}\sum_{m=1}^{N}\sin(\phi_m - \phi_n)$$





$N = 10{,}000$

$N = 5{,}000$

$N = 1{,}000$

$N = 500$

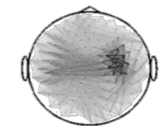
$N = 100$

$N = 50$

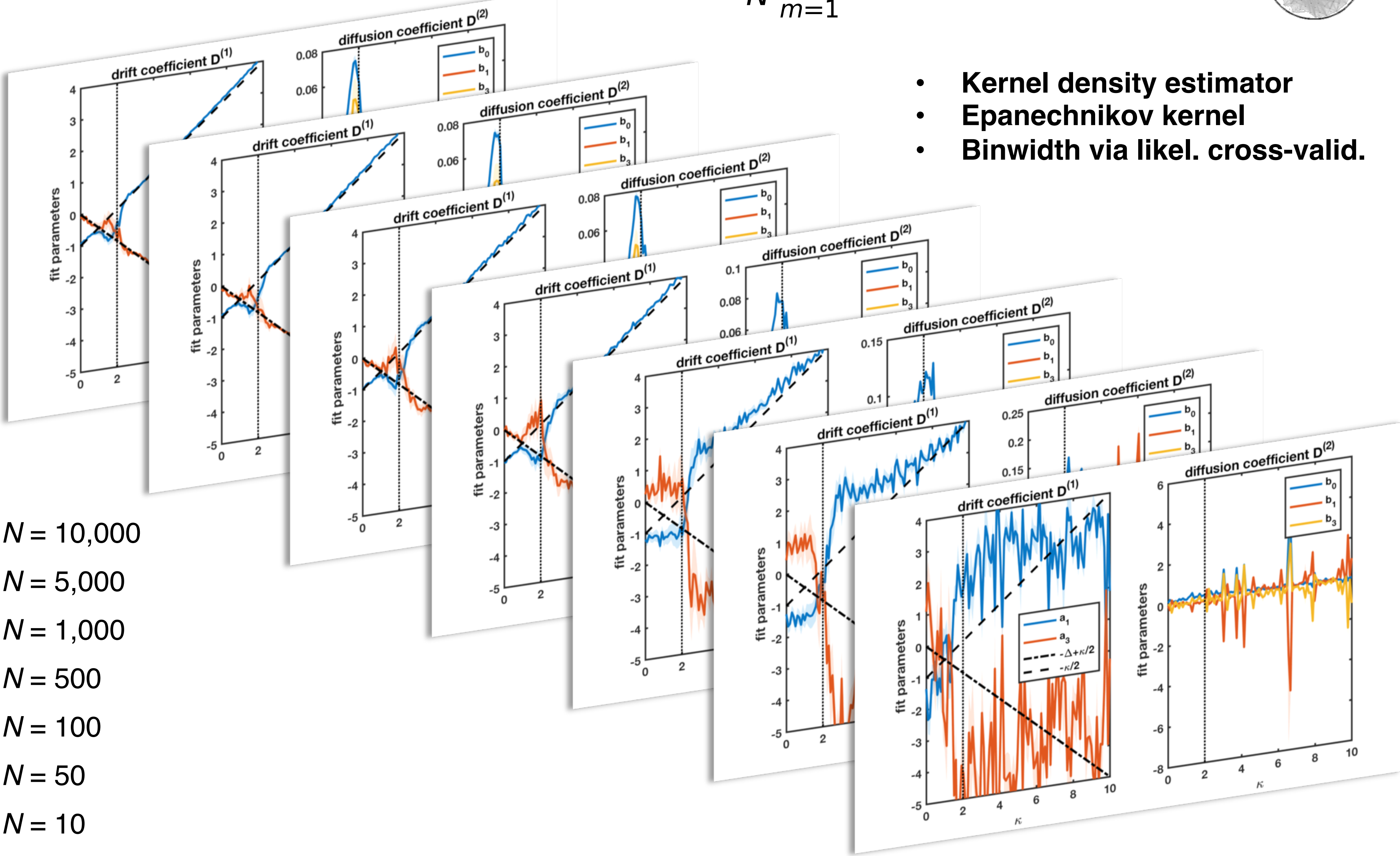
$N = 10$

The Kuramoto network

$$\dot{\phi}_n = \omega_n + \frac{\kappa}{N}\sum_{m=1}^{N} \sin(\phi_m - \phi_n)$$



- **Kernel density estimator**
- **Epanechnikov kernel**
- **Binwidth via likel. cross-valid.**



$N = 10{,}000$

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Conclusions

Does the order parameter evolve like $\dot{\rho} = D^{(1)}(\rho, t) + \sqrt{2D^{(2)}(\rho, t)}\Gamma(t)$?

Yes it does, but only if…

Does the deterministic part look like $\dot{\rho} = -\left(\Delta - \frac{\kappa}{2}\right)\rho - \frac{\kappa}{2}\rho^3$?

Yes it does, but only if …

… the network size is sufficiently large

In order to identify this …

- **the observed data span (a good portion of) the state space**
- **transient and stationary (steady) solutions should be present in the data, the latter at least to good approximation**

Thanks to…

- **COSMOS consortium**
- Nicolás Deschle
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- Till Frank
- Anke van Mourik
- Rudolf Friedrich
- Hermann Haken
- Klaus Lehnertz
- Hu Gang
- Joachim Peinke

I try to keep an open mind, but not so open that my brains fall out.

-- Judge Harold T. Stone