On the well-posedness of non-convex total variation

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STRIKE!!!

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Joint work with M. Hintermüller and T. Wu

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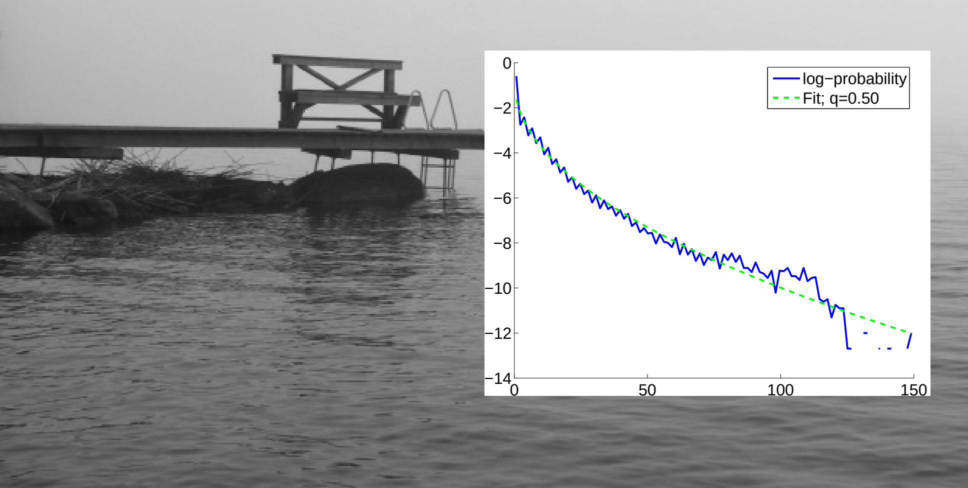
Gradient histograms

Model $t \mapsto \log P(\{x \mid \|\nabla u(x)\| = t\})$ by $-\alpha t^q$, $(\alpha, q > 0)$.



For recollection: Bayesian interpretation

The denoising problem with prior R and Gaussian noise

$$\min_u \frac{1}{2}\|u - f\|^2_{L^2(\Omega)} + \alpha R(u)$$

corresponds to the MAP estimate

$$\max_u \frac{P(f|u)P(u)}{P(f)}$$

where $P(f|u)$ is the Gaussian noise distribution, and the prior

$$P(u) = C\exp(-\alpha R(u)).$$

The TV^q model

This leads us to the image prior

$$\mathrm{TV}^q(u) := \int_\Omega \|\nabla u(x)\|^q \, dx, \quad (q \in (0,1)),$$

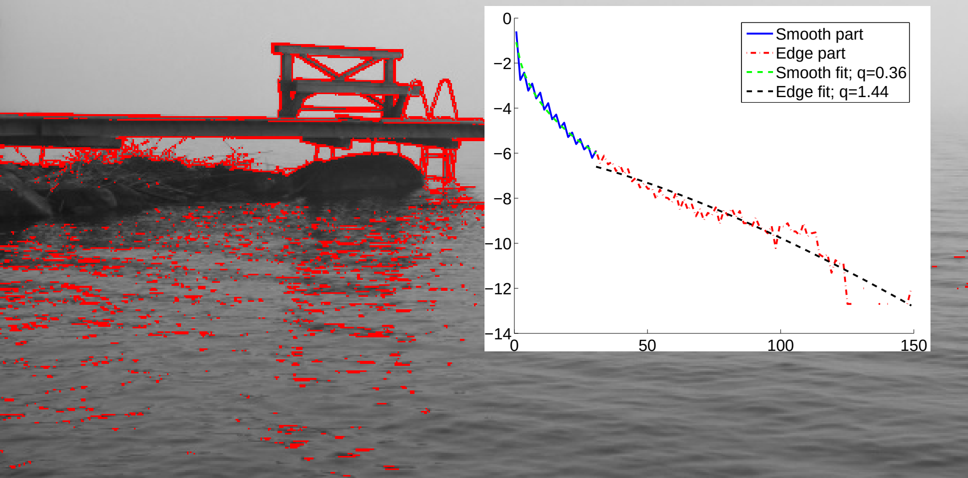
studied in (Huang and Mumford 1999; Hintermüller and Wu 2013; Hintermüller and Wu 2014; Ochs et al. 2013).

Related models for enforcing piecewise constant solutions: (Geman and Geman 1984; Nikolova 2002; Nikolova et al. 2008; Chen and Zhou 2010).

However, are such models theoretically justified?

And is t^q the full story in terms of statistics?

Edge detection, histogram, and t^q fit

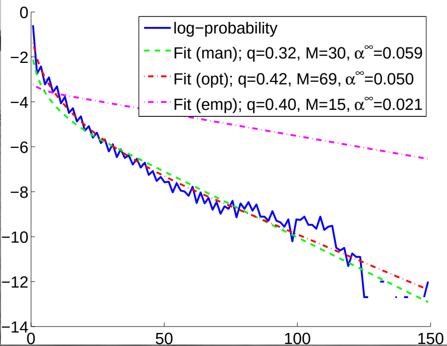


Linearised model fit

Varying M, α, q, we fit to the the function $-\alpha\varphi$ for

$$\varphi_{M,q}(t) := \begin{cases} t^q & 0 \le t \le M, \\ (1-q)M^q + qM^{q-1}t, & t > M. \end{cases}$$

We also define the *asymptotic alpha*, $\alpha^\infty := \alpha\varphi^\infty$.



Is this theoretically justified?

The TV^q model

For an energy functional φ, define

$$\widetilde{\mathrm{TV}}_{\mathrm{c}}^{\varphi}(u) = \int_{\Omega} \varphi(\|\nabla u(x)\|)\, dx, \quad (u \in C^1(\Omega)),$$

and extend this weak* lsc. to $u \in \mathrm{BV}(\Omega)$ by

$$\mathrm{TV}_{\mathrm{c}}^{\varphi}(u) := \liminf_{\substack{u^i \overset{*}{\rightharpoonup} u, \\ u^i \in C^1(\Omega)}} \widetilde{\mathrm{TV}}_{\mathrm{c}}^{\varphi}(u^i).$$

Theorem

If $\varphi(t) = t^q$ for $q \in (0,1)$, then $TV_{\mathrm{c}}^{\varphi}(u) \equiv 0$.

How about the linearised model?

Theorem

Even for the linearised model $\varphi = \varphi_{M,q}$ *with*

$$\varphi_{M,q}(t) := \begin{cases} t^q & 0 \le t \le M, \\ (1-q)M^q + qM^{q-1}t, & t > M, \end{cases}$$

we have $TV_c^{\varphi}(u) \equiv \varphi^{\infty} TV(u) = qM^{q-1} TV(u)$*.*

Difficulties

We need to replace weak* convergence – but with what?

- Weak* convergence is too weak; it demands convex integrands (cf. Bouchitté and Buttazzo 1990; Fonseca and Leoni 2007).
- Strict convergence is also not enough.
- Strong convergence in $\mathrm{BV}(\Omega)$ does not allow approximating piecewise constant functions by smooth functions, so too strong.

Area-strict convergence

Definition

Suppose $\Omega \subset \mathbb{R}^n$ with $n \geq 2$. Then $u^i \to u$ *area-strictly* in $\mathrm{BV}(\Omega)$ if

$$U^i \to U \quad \text{strictly in } \mathrm{BV}(\Omega; \mathbb{R}^{n+1})$$

with the notation $U(x) := (x/\|x\|, u(x))$.

In other words $u^i \to u$ strongly in $L^1(\Omega)$, $Du^i \overset{*}{\rightharpoonup} Du$ weakly* in $\mathcal{M}(\Omega; \mathbb{R}^n)$, and $\mathcal{A}(u^i) \to \mathcal{A}(u)$ for the *area functional*

$$\mathcal{A}(u) := \int_\Omega \sqrt{1 + \|\nabla u(x)\|^2}\, dx + |D^s u|(\Omega).$$

It can be shown that area-strict convergence is stronger than strict convergence, but weaker than norm convergence.

Area-strict continuity

Theorem (Rindler and Shaw 2013)

Let Ω *be a bounded domain with Lipschitz boundary. Let* $f \in C(\mathbb{R}^n)$ *satisfy*

$$|f(A)| \leq C(1 + |A|),$$

and suppose f^∞ *exists. Then the functional*

$$\mathcal{F}(u) := \int_\Omega f(\nabla u(x))\, dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}(x)\right) d|D^s u|(x)$$

is area-strictly continuous on $BV(\Omega)$.

Application of area-strict continuity

Corollary

Suppose $\varphi \in C(\mathbb{R}^{0,+})$, φ^∞ *exists, and* $\varphi(t) \leq C(1+t)$, $(t \in \mathbb{R}^{0,+})$. *Then the functional*

$$TV_{\mathrm{as}}^{\varphi}(u) := \int_\Omega \varphi(\|\nabla u(x)\|)\,dx + \varphi^\infty |D^s u|(\Omega), \quad (u \in BV(\Omega)),$$

is area-strictly continuous on $BV(\Omega)$.

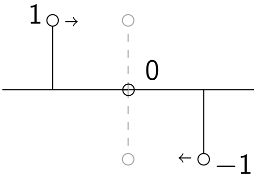
The problem now is: how do we obtain area-strict convergence of a minimising sequence to the denoising problem

$$\min_{u \in \mathrm{BV}(\Omega)} \frac{1}{2}\|f-u\|^2_{L^2(\Omega)} + \mathrm{TV}^{\varphi}_{\mathrm{as}}(u)?$$

Annihilation

Question: What strict convergence lacks that weak* can exhibit?

Answer: Annihilation effects.



How to avoid them?

Multiscale analysis

With $\eta_0 > 0$ and $\{\rho_\epsilon\}_{\epsilon>0}$ a family of mollifiers satisfying the semigroup property $\rho_{\epsilon+\delta} = \rho_\epsilon * \rho_\delta$, we define

$$\eta(\mu) := \eta_0 \sum_{\ell=1}^{\infty} \int_{\mathbb{R}^n} (|\mu| * \rho_{2^{-i}})(x) - |\mu * \rho_{2^{-i}}|(x)\, dx, \quad (\mu \in \mathcal{M}(\Omega; \mathbb{R}^n)).$$

Theorem ($T.V.$ 2011; $T.V.$ 2012)

If $\sup_i \eta(\mu) < \infty$ *and* $\mu^i \overset{*}{\rightharpoonup} \mu$, *then* $|\mu^i|(\Omega) \to |\mu|(\Omega)$.

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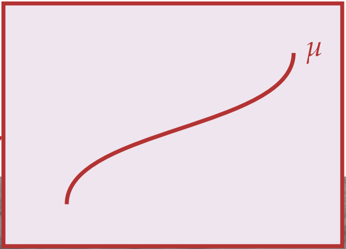

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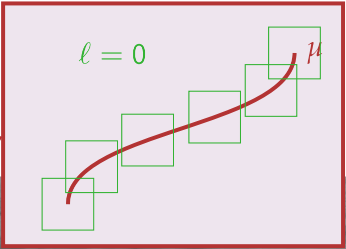

Theorem (*T.V.* 2011; *T.V.* 2012)

If $\sup_i \eta(\mu) < \infty$ *and* $\mu^i \overset{*}{\rightharpoonup} \mu$, *then* $|\mu^i|(\Omega$

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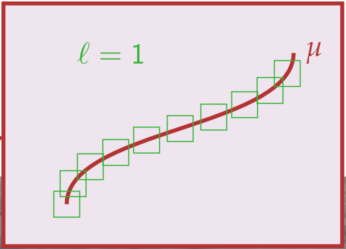

Theorem (*T.V.* 2011; *T.V.* 2012)

If $\sup_i \eta(\mu) < \infty$ *and* $\mu^i \overset{*}{\rightharpoonup} \mu$*, then* $|\mu^i|(\Omega)$

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Theorem (*T.V.* 2011; *T.V.* 2012)

If $\sup_i \eta(\mu) < \infty$ *and* $\mu^i \overset{*}{\rightharpoonup} \mu$, *then* $|\mu^i|(\Omega) \to |\mu|(\Omega)$.

The fixed TV^q model

Theorem

Let $\varphi = \varphi_{M,q}$ be the linearised t^q integrand. Suppose $\Omega \subset \mathbb{R}^n$ is bounded with Lipschitz boundary. Then the functional

$$G(u) := \frac{1}{2}\|f - u\|^2_{L^2(\Omega)} + \alpha TV^{\varphi}_{\mathrm{as}}(u) + \eta(DU), \quad (u \in BV(\Omega)),$$

admits a minimiser $u \in BV(\Omega)$.

Remark (convergence of minimising sequences)

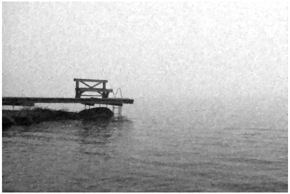
The linearisation of φ is needed for a bound in $\mathrm{BV}(\Omega)$ and weak* convergence. Then a bound on η gives area-strict convergence.

For numerical experiments

- We use a modification of the method of (Hintermüller and Wu 2013; Hintermüller and Wu 2014).
- We vary the cut-off M while keeping the asymptotic α fixed.
  - Defined by $\alpha^\infty := \alpha\varphi^\infty$.
  - Justification: for TV, $\alpha^\infty = \alpha$, so same edge regularisation.
- Empirically optimal q discovered by trial and error.

(a) Original

(b) Noisy image



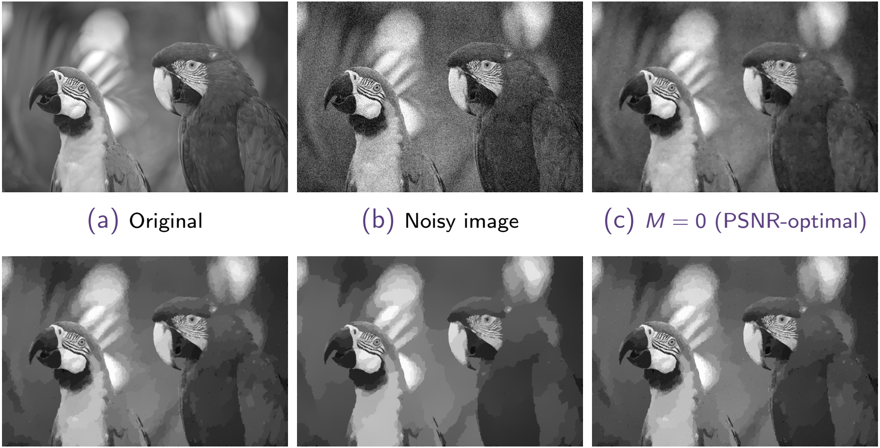
(c) $M = 0$

(d) $M = 10$ (PSNR-optimal)

(e) $M = 40$ (SSIM-optimal)

(f) $M = \infty$

Figure: Pier photo denoising results with noise level $\sigma = 30$ (Gaussian), for varying cut-off M, fixed $q = 0.4$ and fixed $\alpha^\infty = 0.0207$.



(a) Original

(b) Noisy image

(c) $M = 0$ (PSNR-optimal)

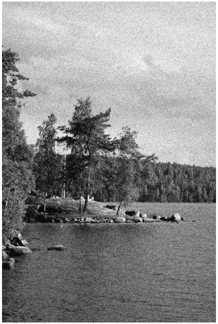
(d) $M = 15$ (SSIM-optimal)

(e) $M = 40$

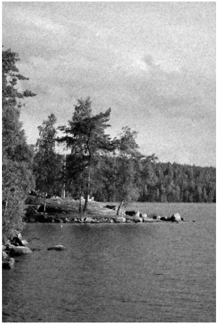
(f) $M = \infty$

Figure: Parrot photo denoising results with noise level $\sigma = 30$ (Gaussian), for varying cut-off M, fixed $q = 0.5$ and fixed $\alpha^{\infty} = 0.0253$.

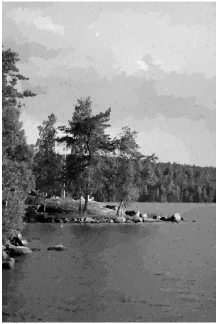
(a) Original



(b) Noisy image



(c) $M = 0$



(d) $M = 20$ (PSNR-optimal)

Figure: Summer photo denoising results with noise level $\sigma = 60$ (Gaussian), for varying cut-off M, fixed $q = 0.3$ and fixed $\alpha^{\infty} = 0.00430$.

Effect of the η term

(a) $\eta_0 = 7\text{e}^{-03}$, $\epsilon_1 = 1$

(b) $\eta_0 = 7\text{e}^{-03}$, $\epsilon_1 = 2$

(c) $\eta_0 = 7\text{e}^{-04}$, $\epsilon_1 = 1$

(d) $\eta_0 = 7\text{e}^{-04}$, $\epsilon_1 = 2$

(e) $\eta_0 = 7\text{e}^{-05}$, $\epsilon_1 = 1$

(f) $\eta_0 = 7\text{e}^{-05}$, $\epsilon_1 = 2$

Conclusion

We may conclude:

- The cut-off M for linearising t^q
  - Is required theoretically
  - Can be seen in image gradient statistics
  - Improves results in practise
- The multiscale regularisation η is a “theoretical artefact” that has yet to be justified in practise.

Thank you for your attention!