Witness Complexes for Time Series Analysis

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Outline

Goal: Online regime-shift detection from time series using TDA.

Pipeline:

1) Receive time series

2) Delay coordinate reconstruction

3) Compute persistent homology

4) Statistics on persistence diagrams

Outline

Goal: Online regime-shift detection from time series using TDA.

Pipeline:

1) Receive time series - storage, incorporation of new data

2) Delay coordinate reconstruction - choice of delay and dimension parameters

3) Compute persistent homology - witness complexes

4) Statistics on persistence diagrams - metrics, stability

2. delay reconstruction

Delay Coordinate Reconstruction

$$\mathbf{x}(t) = (x(t), x(t-\tau), x(t-2\cdot\tau), \ldots, x(t-m\cdot\tau))$$

Two parameters: m dimension;
τ delay

Theoretical bounds: $m \geq 2(\text{box-dim}) + 1$;
τ generic

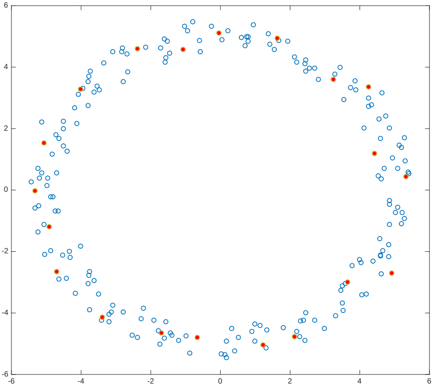
Heuristics: m estimations (False Nearest Neighbor);
τ estimations (1st Minimum of Average Mutual Information)

Packard et. al, Physical Review Letters ('80), Takens, Springer Dynam. Sys. and Turbulence ('81) , Sauer et. al, Journal of Statistical Physics ('91)

A. Fraser et. al., Phys. Rev. A ('86) , M. Kennel et. al., Phys. Rev. A ('92) , L. Pecora et. al., Chaos ('07)

3. witness complexes

Witness Complexes for Time Series Analysis

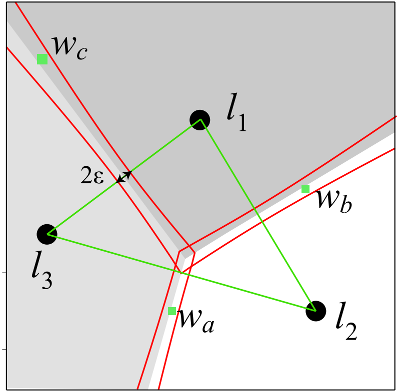


$\Gamma = \{w_1, \ldots, x_N\}$, where $w_{i+1} = \hat{\mathcal{F}}(w_i, \Delta t_i)$; *called witnesses*

$L = \{l_1, \ldots, l_M\}$, some subset of the witnesses; *called landmarks*

V. de Silva, G. Carlsson, Eurographics Symposium on Point-Based Graphics ('04)

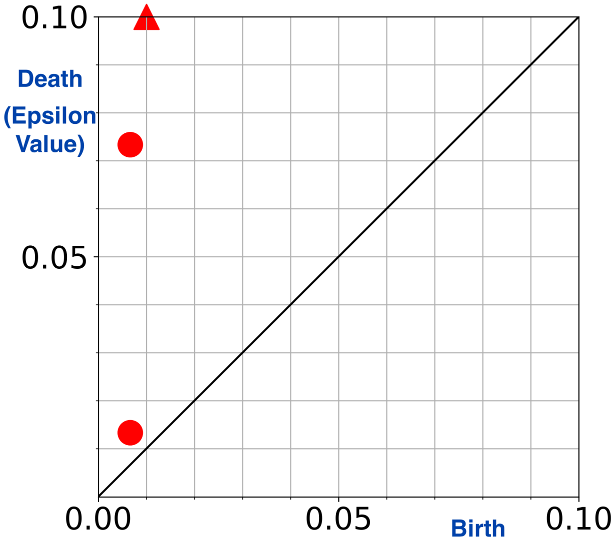
Witness Complexes for Time Series Analysis



"is a witness of landmark": $w_t \in W^{\epsilon}(l_i)$ if $d(w_t, l_i) \leq d(w_t, L) + \epsilon$

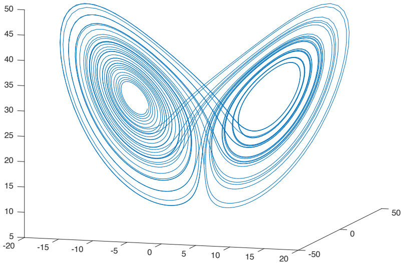
"is a simplex in the witness complex": $\sigma = \langle l_{i_1}, \ldots, l_{i_k} \rangle \in \mathcal{W}^{\epsilon}(\Gamma, L)$ if $\exists\, w_t \in \bigcap_{j=1}^{k} W^{\epsilon}(l_{i_j})$.

Persistence Diagrams for Witness Complexes

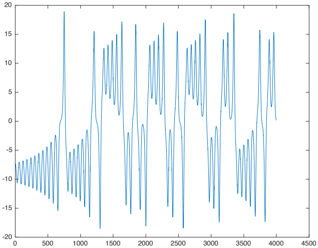


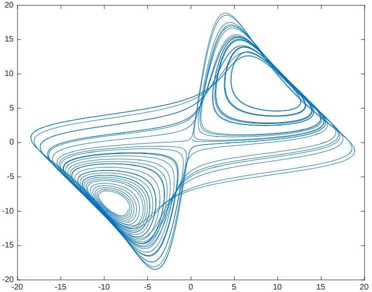
4. into the pipeline

Test Case: Lorenz 63

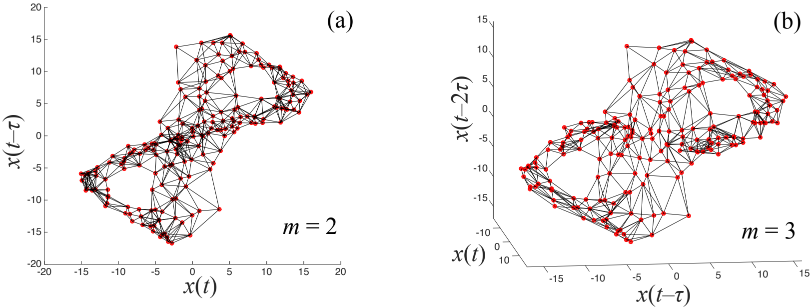


- $m = 2, \tau = 20$
- 5000 W, 100 L
- coarse-grain topology





Setting $m = 2$.

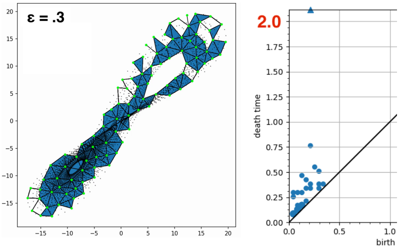


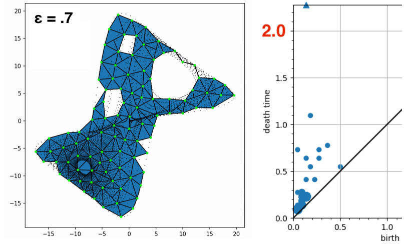
Increasing the embedding dimension and tracking edge formation/destruction between landmarks showed that $m = 2$ is often sufficient to correctly capture the H_1 homology with a witness complex.

Next: See what we can squeeze out of varying delay parameter τ.

Garland et. al., Physica D ('14)

Some problems with standard witness complexes



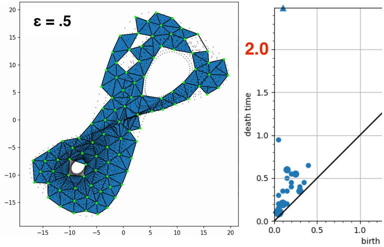


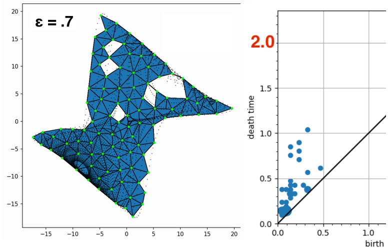
* $\tau = 6$, small reach

* $\tau = 18$, high speed/low density

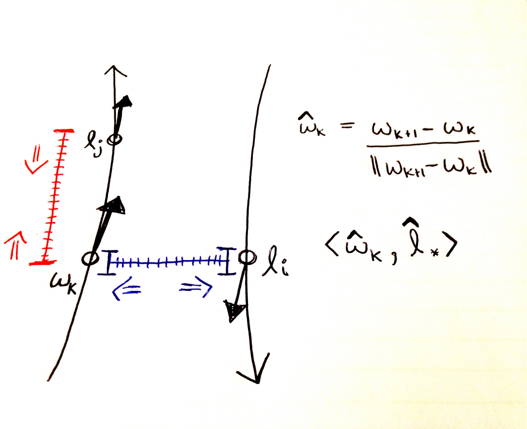
* $\tau = 12$, luck

* $\tau = 24$, folding/projection





Some observations about delay reconstruction

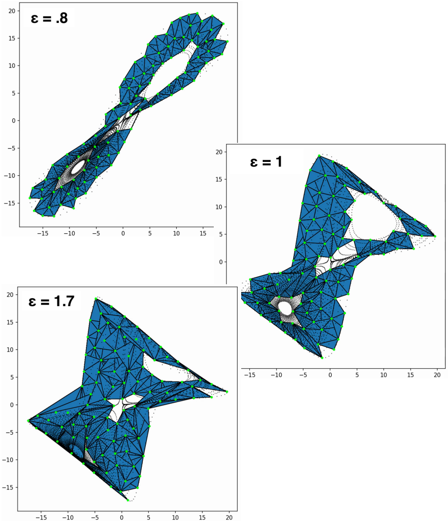


Observation: Holes that "matter" have tangent vectors across the hole that point in opposite directions.

Novel Witness Complexes: Additive Penalty

$$d_A(w, l) = d_E + k_A \cdot (1 - \langle \hat{w}, \hat{l} \rangle)$$

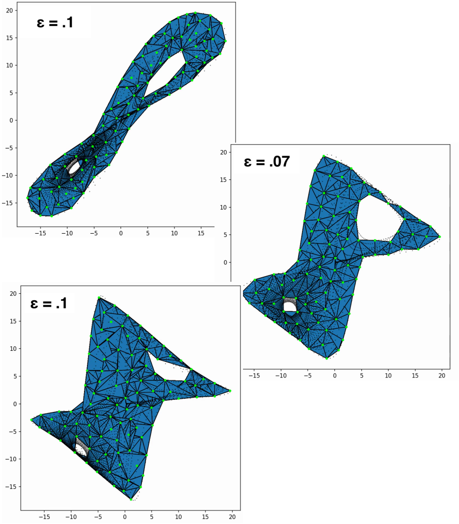
- $k_A = 5$
- penalty for opposite direction of travel!
- "outside-in, star-shaped" holes!
- maintains holes!



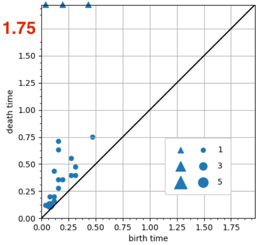
Novel Witness Complexes: Multiplicative Distortion

$$\boxed{d_M(w, l) = \frac{d_E}{1+k_M\cdot(1+\langle \hat{w}, \hat{l}\rangle)}}$$

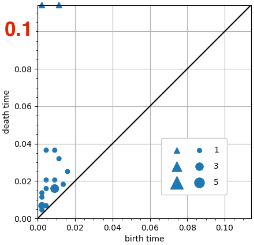
- $k_M = 10$
- bonus for parallel travel!
- "circularizes" ellipses!
- keeps holes open!



Persistence Diagrams: Additive / Multiplicative



$k_A = 5, \tau = 6$



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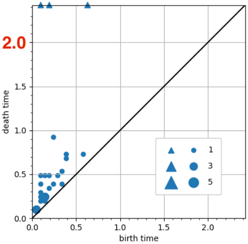
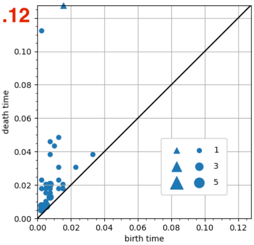


$k_A = 5, \tau = 24$



$k_M = 10, \tau = 24$

Final Remarks

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- Witness complexes are good; they reduce computation.
- Need to take care with time series reconstructions to get consistent topological signature.
- Important to have an automated method; requires metric: Wasserstein on PDs, weighted-L^2 on persistence rank functions.

V. Robins, K. Turner, Physica D Nonlinear Phenomena ('15)

Thanks for listening!

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