A Chromaticity-Brightness Model for Color Images Denoising

in collaboration with Luisa Mascarenhas and Rita Ferreira

Irene Fonseca

Department of Mathematical Sciences
Center for Nonlinear Analysis
Carnegie Mellon University
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Image restoration/ denoising . . . ROF Model (Rudin, Osher and Fatemi 1992)

$\Omega \subset \mathbb{R}^2$ open bounded domain, Liptschitz boundary . . . image domain

$u_0 : \Omega \to \mathbb{R}$ . . . (noisy) image

λ . . . tuning parameter

$$\min \left\{ |Du|(\Omega)| + \lambda \int_\Omega |u - u_0|^2 dx : u \in BV(\Omega), u - u_0 \in L^2(\Omega) \right\}$$

removes noise while preserving edges

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extended to higher order and/or vectorial setting (RGB color images)
- Gilles Aubert and Pierre Kornprobst 2006
- Tony Chan, Selim Esedoglu, Frederick Park and Andy Yip 2006

but . . .

blurring and stair-case effect

Fidelity term? Regularization term?

Here focus on the fidelity term

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• Yves Meyer 2001 ... images with oscillations often treated as texture or noise $\rightsquigarrow$ the G norm

$$\min\left\{|Du|(\Omega)| + \lambda||u-u_0||_G : u \in BV(\Omega), u-u_0 \in L^2(\Omega)\right\}$$

$$G(\Omega;\mathbb{R}^d) := \{v \in L^2(\Omega;\mathbb{R}^d) : v_i = \text{div}\xi_i, \xi \in L^\infty(\Omega;(\mathbb{R}^2)^d, \xi_i \cdot \nu = 0 \text{ on } \partial\Omega\}$$

$$||v||_G := \inf\{||\xi||_{L^\infty} : v_i = \text{div}\xi_i, \ldots\}$$

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$$||v||_G := \inf\{||\xi||_{L^\infty} : v_i = \text{div}\xi_i, \ldots\}$$

If $\Omega \subset \mathbb{R}^2$ is a domain with Lipschitz boundary

$$G(\Omega;\mathbb{R}^d) = \left\{v \in L^2(\Omega;\mathbb{R}^d) : \int_\Omega v(x)\,dx = 0\right\}$$

Chromaticity-Brightness, CB

$u_0 : \Omega \to [0, +\infty)^3 \setminus \{0\}$ ...color RGB image

$(u_0)_b := |u_0|$ ... intensity

$(u_0)_c := \frac{u_0}{|u_0|} \ldots \in S^2$ ... chromaticity

$$u_0 = (u_0)_b (u_0)_c$$

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$(u_0)_b := |u_0|$ ... intensity

$(u_0)_c := \frac{u_0}{|u_0|} \ldots \in S^2$ ... chromaticity

$$u_0 = (u_0)_b (u_0)_c$$

And in general

$$u = (u)_b (u)_c$$

$(u_0)_b \sim$ grey-scale image ... so use **Meyer's G-model**

$(u_0)_c \sim$ colored image ... so adopt a **Kang-March-type model** (Sung Ha Kang and Riccardo March 2007) ... weighted harmonic maps

$$\min \left\{ \int_\Omega g(|\nabla u_b^\sigma|) |\nabla u_c|^2 dx + \lambda \int_\Omega |u_c - (u_0)_c|^2 dx : u_c \in W^{1,2}(\Omega; S^2) \right\}$$

u_0 extended by zero outside Ω

$u_b{}^\sigma := G_\sigma \star (u_0)_b \ldots G_\sigma(x) := \frac{L}{\sigma} e^{-\frac{|x|^2}{4\sigma}}$, $A > 0, \sigma > 0$

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Kang-March

$$\min\left\{\int_\Omega g(|\nabla u_b^\sigma|)|\nabla u_c|^2 dx + \lambda\int_\Omega |u_c - (u_0)_c|^2 dx : u_c \in W^{1,2}(\Omega;S^2)\right\}$$

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Usually

$$g(t) \sim \frac{1}{1+\left(\frac{t}{a}\right)^2} \quad \text{or} \quad g(t) \sim e^{-\left(\frac{t}{a}\right)^2},\ a > 0$$

$g \sim 0$ where u_b^σ varies fast $\rightsquigarrow$ sharp transitions of u_c

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Kang-March

$$\min\left\{\int_\Omega g(|\nabla u_b^\sigma|)|\nabla u_c|^2dx+\lambda\int_\Omega|u_c-(u_0)_c|^2dx : u_c\in W^{1,2}(\Omega;S^2)\right\}$$

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- u_b^σ ... a very smooth version of the brightness component ... should let $\sigma\to 0$
- $\inf_\Omega g(|\nabla u_b^\sigma|)>0$ for $\sigma>0$ ... hence compactness of minimizing sequences in $W^{1,2}(\Omega;\mathbb{R}^3)$

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Consider

$$\inf_{\substack{u_b \in W^{1,1}(\Omega), u_c \in W^{1,2}(\Omega;S^2), \\ u_b - (u_0)_b \in G(\Omega), u_0 - u_c u_b \in G(\Omega;\mathbb{R}^3)}} \Big\{ F_0(u_b u_c) + F_1(u_b) + F_2(u_c) \Big\}$$

where

$$F_0(u) := |Du|(\Omega) + \lambda_0 \|u - u_0\|_{G(\Omega;\mathbb{R}^3)}$$
$$u \in BV(\Omega;\mathbb{R}^3), u - u_0 \in G(\Omega;\mathbb{R}^3), \lambda_0 \in \mathbb{R}^+$$

$$F_1(u_b) := |Du_b|(\Omega) + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)}$$
$$u_b \in BV(\Omega), u_b - (u_0)_b \in G(\Omega), \lambda_b \in \mathbb{R}^+$$

$$F_2(u_c) := \int_\Omega g(|\nabla u_b|)|\nabla u_c|^2 \, dx + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \, dx$$
$$u_c \in W^{1,2}(\Omega; S^2), \lambda_c \in \mathbb{R}^+$$

That is . . .

$$\inf \left\{ \int_\Omega |\nabla(u_c u_b)|\,dx + \int_\Omega |\nabla u_b|\,dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c|^2\,dx \right.$$
$$\left. + \lambda_v \|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)} + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c - (u_0)_c|^2\,dx \right\}$$

where

- $u_b \in W^{1,1}(\Omega)$
- $u_c \in W^{1,2}(\Omega; S^2)$
- $u_b - (u_0)_b \in G(\Omega)$
- $u_0 - u_c u_b \in G(\Omega; \mathbb{R}^3)$

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where

- $u_b \in W^{1,1}(\Omega)$
- $u_c \in W^{1,2}(\Omega; S^2)$
- $u_b - (u_0)_b \in G(\Omega)$
- $u_0 - u_c u_b \in G(\Omega; \mathbb{R}^3)$

And will assume for some $0 < \alpha \leq \beta$

$$(u_0)_b,\ u_b \in [\alpha, \beta] \quad \text{a.e. in}\Omega$$

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Then

$$\alpha \int_\Omega |\nabla u_c|\, dx \leq \int_\Omega |\nabla (u_c u_b)|\, dx + \int_\Omega |\nabla u_b|\, dx$$

and if

$$\{(u_b^n, u_c^n)\}_{n\in\mathbb{N}} \subset \quad \Big\{(u_b, u_c) \in W^{1,1}(\Omega;[\alpha,\beta]) \times W^{1,2}(\Omega; S^2) : u_b - (u_0)_b \in G(\Omega), \\ u_b u_c - u_0 \in G(\Omega;\mathbb{R}^3)\Big\}$$

is a infimizing sequence then (up to a subsequence) there exist

- $\bar{u}_b \in BV(\Omega;[\alpha,\beta])$
- $\bar{u}_c \in BV(\Omega; S^2)$

such that

$$u_b{}^n \overset{\star}{\rightharpoonup} \bar{u}_b \text{ in } BV(\Omega)\,, \quad u_c{}^n \overset{\star}{\rightharpoonup} \bar{u}_c \text{ in } BV(\Omega;\mathbb{R}^3)$$

$$\bar{u}_b - (u_0)_b \in G(\Omega)\,, \bar{u}_b \bar{u}_c - u_0 \in G(\Omega;\mathbb{R}^3)$$

$$\lim_{n\to+\infty} F^{fid}(u_b^n, u_c^n) = F^{fid}(\bar{u}_b, \bar{u}_c)$$

where the Fidelity Term (sum of the three fidelity terms) is

$$F^{fid}(u_b, u_c) := \lambda_v \|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)} + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \, dx$$

So existence of minimizers . . . swlsc of the energy $\rightsquigarrow$ swlsc of the regularizing terms

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So existence of minimizers . . . swlsc of the energy $\rightsquigarrow$ swlsc of the regularizing terms

GOAL: Find an integral representation for

$$\inf \left\{ \liminf_{n\to\infty} \int_\Omega h(u_b^n, u_c^n, \nabla u_b^n, \nabla u_c^n) \, dx : \quad \begin{array}{l} u_b^n \in W^{1,1}(\Omega; [\alpha, \beta]), \\ u_b^n \rightharpoonup u_b \text{ in } W^{1,1}(\Omega), \\ u_c^n \in W^{1,2}(\Omega; S^2), \\ u_c^n \rightharpoonup u_c \text{in } W^{1,1}(\Omega; \mathbb{R}^3) \end{array} \right\}$$

$$h(r, s, \xi, \eta) := |\xi| + g(|\xi|)|\eta|^2 + |s \otimes \xi + r\eta|$$

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In general,

$$(\xi, \eta) \mapsto h(r, s, \xi, \eta) = |\xi| + g(|\xi|)|\eta|^2 + |s \otimes \xi + r\eta|$$

is not quasiconvex

Moreover, for $(r, s) \in [\alpha, \beta] \times S^2$, h satisfies the
non-standard growth conditions

$$\frac{1}{C}(|\xi| + |\eta|) \leq h(r, s, \xi, \eta) \leq C(1 + |\xi| + |\eta|^2),$$

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$$\frac{1}{C}(|\xi| + |\eta|) \leq h(r, s, \xi, \eta) \leq C(1 + |\xi| + |\eta|^2),$$

which leads us to ... the gap problem !
concerning the unconstrained setting

- I. F., Jan Malý 1997
- I. F., Giovanni Leoni and Stefan Müller 2004
- Giuseppe Mingione and Domenico Mucci 2005

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And more!

Admissible sequences must satisfy

$$u_b^n - (u_0)_b \in G(\Omega), \quad u_b^n u_c^n - u_0 \in G(\Omega; \mathbb{R}^3)$$

or, equivalently,

$$\int_\Omega (u_b^n - (u_0)_b)\, dx = 0, \quad \int_\Omega (u_b^n u_c^n - u_0)\, dx = 0$$

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Challenge: To construct a recovery sequence that simultaneously satisfies the manifold constraint and the average restrictions

So . . . singularly perturb the average constraints
Study the asymptotic behavior as $\varepsilon \to 0^+$ of

$$\inf_{(u_b,u_c)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2)} \left\{F^{reg}(u_b,u_c) + F_\varepsilon^{fid}(u_b,u_c)\right\}$$

where

$$F^{reg}(u_b,u_c) := \int_\Omega |\nabla u_b|\, dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c|\, dx + \int_\Omega |\nabla(u_c u_b)|\, dx$$

standard growth conditions: $\int_\Omega g(|\nabla u_b|)|\nabla u_c|^2\, dx \rightsquigarrow \int_\Omega g(|\nabla u_b|)|\nabla u_c| dx$

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The original fidelity term

$$F^{fid}(u_b,u_c) := \lambda_v \|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)} + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c - (u_0)_c|^2\,dx$$

$$\begin{aligned}
F^{fid}_\varepsilon(u_b,u_c) := \quad & \lambda_v \left\| u_b u_c - u_0 - \fint_\Omega (u_b u_c - u_0)\,dx \right\|_{G(\Omega;\mathbb{R}^3)} \\
& + \frac{1}{\varepsilon}\left| \int_\Omega (u_b u_c - u_0)\,dx \right| \\
& + \lambda_b \left\| u_b - (u_0)_b - \fint_\Omega (u_b - (u_0)_b)\,dx \right\|_{G(\Omega)} \\
& + \frac{1}{\varepsilon}\left| \int_\Omega (u_b - (u_0)_b)\,dx \right| \\
& + \lambda_c \int_\Omega |u_c - (u_0)_c|^2\,dx
\end{aligned}$$

Good news:

1. in the limit as $\varepsilon \to 0^+$ we will recover the functional F^{fid}
2. pairs (u_b,u_c) satisfying $u_b - (u_0)_b \in G(\Omega)$ and $u_b u_c - u_0 \in G(\Omega;\mathbb{R}^3)$.

Notation

Recall

$$F^{reg}(u_b, u_c) := \int_\Omega |\nabla u_b|\, dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c|\, dx + \int_\Omega |\nabla (u_c u_b)|\, dx$$
$$= \int_\Omega f(u_b(x), u_c(x), \nabla u_b(x), \nabla u_c(x))\, dx$$

where $f : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2} \to [0, +\infty)$

$$f(r, s, \xi, \eta) := |\xi| + g(|\xi|)|\eta| + |s \otimes \xi + r\eta|$$

$g : [0, +\infty) \to (0, 1]$ ... non-increasing, Lipschitz

$g(0) = 1$ and $\lim_{t\to+\infty} g(t) = 0$

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$g : [0, +\infty) \to (0, 1]$ . . . non-increasing, Lipschitz

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Recession function

$$f^\infty(r, s, \xi, \eta) := \limsup_{t\to+\infty} \frac{f(r, s, t\xi, t\eta)}{t}$$
$$= \limsup_{t\to+\infty} \big(|\xi| + g(t|\xi|)|\eta| + |r\eta + s \otimes \xi|\big)$$
$$= |\xi| + \chi_{\{0\}}(|\xi|)|\eta| + |r\eta + s \otimes \xi|$$

Tangential Quasiconvex Envelope of f $T_s(S^2)$... tangential space to S^2 at s

$$\mathcal{Q}_T f(r,s,\xi,\eta) := \inf\left\{ \int_Q f(r,s,\xi+\nabla\varphi(y),\eta+\nabla\psi(y))\,dy : \varphi\in W_0^{1,\infty}(Q),\ \psi\in W_0^{1,\infty}(Q;T_s(S^2)) \right\}$$

Recession Function of $\mathcal{Q}_T f$

$$(\mathcal{Q}_T f)^\infty(r,s,\xi,\eta) := \limsup_{t\to+\infty} \frac{Q_T f(r,s,t\xi,t\eta)}{t}$$

Jump Energy Density

$a,b\in[\alpha,\beta]\times S^2$, $\nu\in S^1$, Q_ν ... unit cube in $\mathbb{R}^2$ centered at the origin and with two faces orthogonal to ν

$$\begin{aligned} K(a,b,\nu) &:= \inf\left\{ \int_{Q_\nu} f^\infty(\varphi(y),\psi(y),\nabla\varphi(y),\nabla\psi(y))\,dy : (\varphi,\psi)\in\mathcal{P}(a,b,\nu) \right\} \\ &= \inf\left\{ \int_{Q_\nu} \left(|\nabla\varphi(y)| + |\nabla(\varphi\psi)(y)| + \chi_{\{0\}}(|\nabla\varphi|)|\nabla\psi|\right)dy : (\varphi,\psi)\in\mathcal{P}(a,b,\nu) \right\} \end{aligned}$$

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Relaxation of $F^{reg}(u_b, u_c)$

Recall

$$F^{reg}(u_b, u_c) := \int_\Omega |\nabla u_b|\, dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c|\, dx + \int_\Omega |\nabla(u_c u_b)|\, dx$$

extend it to $F : L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3) \to [0, +\infty]$

$$F(u_b, u_c) := \begin{cases} F^{reg}(u_b, u_c) & \text{if } (u_b, u_c) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2), \\ +\infty & \text{otherwise,} \end{cases}$$

for $(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$

Looking for the lower semicontinuous envelope of F

$\mathcal{F} : L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3) \to [0, +\infty]$

$$\mathcal{F}(u_b, u_c) := \inf \Big\{ \liminf_{n \to +\infty} F(u_b^n, u_c^n) : n \in \mathbb{N},\ (u_b^n, u_c^n) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3),$$
$$u_b^n \to u_b \text{ in } L^1(\Omega),\ u_c^n \to u_c \text{ in } L^1(\Omega; \mathbb{R}^3) \Big\}$$

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Integral Representation of $F^{reg}(u_b, u_c)$

Theorem

$$\mathcal{F}(u_b, u_c) = \begin{cases} F^{reg,sc^-}(u_b, u_c) & \textit{if } (u_b, u_c) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2), \\ +\infty & \textit{otherwise} \end{cases}$$

for $(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$, *where* $F^{reg,sc^-} : BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2) \to \mathbb{R}$

$$F^{reg,sc^-}(u_b, u_c) := \int_\Omega \mathcal{Q}_T f(u_b(x), u_c(x), \nabla u_b(x), \nabla u_c(x))\, dx$$
$$+ \int_{S_{(u_b,u_c)}} K\big((u_b, u_c)^+(x), (u_b, u_c)^-(x), \nu_{(u_b,u_c)}(x)\big)\, d\mathcal{H}^1(x)$$
$$+ \int_\Omega (\mathcal{Q}_T f)^\infty(u_b(x), u_c(x), C_1(x), C_{2,3}(x))\, |d|D^c(u_b, u_c)|(x)$$

- C_1 ... *first row of* $C := \frac{dD^c(u_b,u_c)}{d|D^c(u_b,u_c)|}$
- $C_{2,3}$ ... 3×2 *matrix, last two rows of* C

Main Theorem with $\{\varepsilon_n\}_{n\in\mathbb{N}} \to 0^+$, $\{\delta_n\}_{n\in\mathbb{N}} \to 0^+$

$X := \{(u_b, u_c) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2) : u_b - (u_0)_b \in G(\Omega), u_b u_c - u_0 \in G(\Omega; \mathbb{R}^3)\}$

btw ... it is nonempty ...

Theorem

- $$\min_{(u_b,u_c)\in X}\Big(F^{reg,sc^-}(u_b,u_c) + F^{fid}(u_b,u_c)\Big) = \lim_{n\to\infty}\inf_{(u_b,u_c)}\Big(F^{reg}(u_b,u_c) + F^{fid}_{\varepsilon_n}(u_b,u_c)\Big)$$

- *If* $(\bar{u}_b^n, \bar{u}_c^n) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2)$ *is a δ_n-minimizer of* $F^{reg} + F^{fid}_{\varepsilon_n}$*, i.e.,*

$$F^{reg}(\bar{u}_b^n, \bar{u}_c^n) + F^{fid}_{\varepsilon_n}(\bar{u}_b^n, \bar{u}_c^n) \leq \inf_{(u_b,u_c)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2)}\Big(F^{reg}(u_b,u_c) + F^{fid}_{\varepsilon_n}(u_b,u_c)\Big) + \delta_n,$$

then $\{(\bar{u}_b^n, \bar{u}_c^n)\}_{n\in\mathbb{N}}$ *is sequentially, relatively compact with respect to the weak-$\star$ convergence in* $BV(\Omega) \times BV(\Omega; \mathbb{R}^3)$

Main Theorem cont.

Theorem (Cont.)

• *If* $(\bar{u}_b, \bar{u}_c)$ *is a cluster point of* $\{(\bar{u}_b^n, \bar{u}_c^n)\}_{n\in\mathbb{N}}$, *then* $(\bar{u}_b, \bar{u}_c) \in X$ *is a minimizer of* $(F^{reg,sc^-} + F^{fid})$ *in* X *and*

$$F^{reg,sc^-}(\bar{u}_b, \bar{u}_c) + F^{fid}(\bar{u}_b, \bar{u}_c) = \limsup_{n\to\infty} \left(F^{reg}(\bar{u}_b^n, \bar{u}_c^n) + F^{fid}_{\varepsilon_n}(\bar{u}_b^n, \bar{u}_c^n) \right)$$

• *If the whole sequence* $\{(\bar{u}_b^n, \bar{u}_c^n)\}_{n\in\mathbb{N}}$ *weakly-*$\star$ *converges to* $(\bar{u}_b, \bar{u}_c)$ *in* $BV(\Omega) \times BV(\Omega; \mathbb{R}^3)$, *then the limit superior above is actually a limit.*

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What is new . . .

The relaxation result falls within . . . lower semicontinuity and/or integral representations of lower semicontinuous envelopes for

$$u \mapsto \int_\Omega f(x, u(x), \nabla u(x))\, dx$$

$u \in W^{1,p}(\Omega; \mathcal{M})$, $\mathcal{M} \subset \mathbb{R}^d$ is a (sufficiently) smooth, m-dimensional manifold

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E.g., liquid crystals, micromagnetic, magnetostrictive materials,

- Bernard Dacorogna, IF, Jan Malý, Konstantina Trivisa 1999
- Roberto Alicandro, Antonio Esposito and Chiara Leone 2007
- Jean-François Babadjian and Vincent Millot 2010
- Jerry Ericksen 1990
- Domenico Mucci 2009
- Haïm Brézis, Jean-Michel Coron and Elliot Lieb 1986
- and others

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What is new . . .

Key ingredients are
• density of smooth functions in $W^{1,1}(\Omega; \mathcal{M})$
• projection lemma (as in Alicandro, Esposito and Leone 2007, and also Virga 1994)

BUT
as opposed to Alicandro, Esposito and Leone 2007, Babadjian and Millot 2010, Mucci 2009, etc.

• given $(r, s) \in [\alpha, \beta] \times S^2$, $(\xi, \eta) \in \mathbb{R}^2 \times [T_s(S^2)]^2 \mapsto f(r, s, \xi, \eta) \in \mathbb{R}^+$
is NEVER tangential quasiconvex
• our manifold $\mathcal{M} = [\alpha, \beta] \times S^2$ has boundary
• the recession function f^∞ does not satisfy a hypothesis of the type

$$|f(r, s, \xi, \eta) - f^\infty(r, s, \xi, \eta)| \le C(1 + |(\xi, \eta)|^{1-m})$$

for some $C > 0$ and $m \in (0, 1)$, for a.e. (r, s) and for all (ξ, η)

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The Tangential Quasiconvex Envelope

Inspired by Dacorogna, F., Malý and Trivisa 1999

Lemma

$r \in [\alpha, \beta]$, $s \in S^2$, $\xi \in \mathbb{R}^2$, $\eta \in [T_s(S^2)]^2$

$$\mathcal{Q}_T f(r, s, \xi, \eta) = \mathcal{Q}\tilde{f}(r, s, \xi, \eta)$$

where

$$\mathcal{Q}\tilde{f}(r, s, \xi, \eta) := \inf \Big\{ \int_Q \tilde{f}(r, s, \xi + \nabla\varphi(y), \eta + \nabla\psi(y))\, dy : \varphi \in W_0^{1,\infty}(Q),\ \psi \in W_0^{1,\infty}(Q; \mathbb{R}^3) \Big\}$$

$$\tilde{f}(r, s, \xi, \eta) := \begin{cases} f(\tilde{r}, \tilde{s}, \xi, P_{\tilde{s}}\eta)\, \phi(|s|) & \text{if } s \in \mathbb{R}^3 \setminus \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

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More About $\tilde{f}(r, s, \xi, \eta)$

$$P_s\eta := (\mathbb{I}_{3\times 3} - s \otimes s)\eta$$

projection of $\mathbb{R}^{3\times 2}$ onto $[T_s(S^2)]^2$ (resp., of $\mathbb{R}^3$ onto $T_s(S^2)$)

$$\tilde{r} := \begin{cases} \alpha & \text{if } r \le \alpha, \\ r & \text{if } \alpha \le r \le \beta, \\ \beta & \text{if } r \ge \beta, \end{cases} \qquad \tilde{s} := \frac{s}{|s|},$$

$\phi \in C^\infty(\mathbb{R}; [0,1])$ . . . cut-off function s. t.

$$\phi(t) = \begin{cases} 1 & \text{if} \quad t \ge 1 \\ 0 & \text{if} \quad t \le \frac{3}{4} \end{cases}$$

For all $r \in [\alpha, \beta]$, $s \in S^2$, $\xi \in \mathbb{R}^2$, and $\eta \in [T_s(S^2)]^2$

$$\tilde{f}(r, s, \xi, \eta) = f(r, s, \xi, \eta).$$

Remark. There does **NOT** exist $(r, s) \in [\alpha, \beta] \times S^2$ for which

$$(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^{3\times 2} \mapsto \tilde{f}(r, s, \xi, \eta)$$

is quasiconvex.

Proof: Road Map

Blow-up method ... but with several road blocks ...

1. Localization of the Energy: $(u, v) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2)$

$$A \in \mathcal{A}(\Omega) \mapsto \mathcal{F}(u, v; A) := \inf \Big\{ \liminf_{n \to +\infty} \int_A f(u_n(x), v_n(x), \nabla u_n(x), \nabla v_n(x))\, dx : n \in \mathbb{N}, (u_n, v_n) \in W^{1,1}(A; [\alpha, \beta]) \times W^{1,1}(A; S^2), u_n \to u \text{ in } L^1(A), v_n \to v \text{ in } L^1(A; \mathbb{R}^3) \Big\}$$

2. Prove that $\mathcal{F}(u, v; \cdot)$ is the restriction of a Radon measure on Ω to $\mathcal{A}(\Omega)$ a. c. wrt $|D(u, v)|$
3. Look at the Radon-Nikodym derivatives, e.g., $(u, v) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2)$. For $\mathcal{L}^2$ a.e. $x_0 \in \Omega$

$$\frac{d\mathcal{F}(u, v; \cdot)}{d\mathcal{L}^2}(x_0) = \mathcal{Q}_T f(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0))$$

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Projection function $\pi_y : \overline{B(0,1)} \setminus \{y\} \to S^2$ (Alicandro, Esposito and Leone 2007)

$$\pi_y(s) := y + \frac{-y\cdot(s-y) + \sqrt{(y\cdot(s-y))^2 + |s-y|^2(1-|y|^2)}}{|s-y|^2}(s-y)$$

projects $s \in \overline{B(0,1)} \setminus \{y\}$ onto S^2 along the direction $s-y$

$$\pi_{y|S^2} = \mathbb{I}_{S^2}, \quad \nabla\pi_y(s)w = w \quad \text{for } s \in S^2,\ w \in T_s(S^2)$$

Lemma

$A \in \mathcal{A}(\Omega)$, $v \in W^{1,1}(A; \overline{B(0,1)}) \cap C^\infty(A;\mathbb{R}^3)$. *There exists* $y \in B\left(0,\frac{1}{2}\right)$ *s. t.* $\pi_y \circ v \in W^{1,1}(A;S^2) \cap C^\infty(A;S^2)$

$$\int_A |\nabla(\pi_y \circ v)|\,dx \le C\int_A |\nabla v|\,dx.$$

and then approximate with same trace on the boundary:

Lemma

$A \in \mathcal{A}_\infty(\Omega)$, $w = (u,v) \in BV(A;[\alpha,\beta]\times S^2)$.
There exists a sequence $\{\bar{w}_n\}_{n\in\mathbb{N}} \subset W^{1,1}(A;[\alpha,\beta]\times S^2) \cap C^\infty(A;\mathbb{R}\times\mathbb{R}^3)$ *s. t.*

1. $\bar{w}_n = w$ *on* ∂A *for all* $n \in \mathbb{N}$
2. $\lim_{n\to\infty} \|\bar{w}_n - w\|_{L^1(A;\mathbb{R}\times\mathbb{R}^3)} = 0, \quad \limsup_{n\to\infty} \int_A |\nabla\bar{w}_n(x)|\,dx \le \widetilde{C}|Dw|(A)$

Upper Bound for $\mathcal{F}$

$(u, v) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2)$. Then for $\mathcal{L}^2$ a.e. $x_0 \in \Omega$

$$\frac{d\mathcal{F}(u, v; \cdot)}{d\mathcal{L}^2}(x_0) \leq \mathcal{Q}_T f(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0))$$

Fix $\varepsilon > 0$. Let $\varphi_\epsilon \in W_0^{1,\infty}(Q)$, $\psi_\varepsilon \in W_0^{1,\infty}(Q; T_{v(x_0)}(S^2))$, extended by periodicity to the whole $\mathbb{R}^2$, be such that

$$\mathcal{Q}_T f(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0)) + \varepsilon \geq \int_Q f(u(x_0), v(x_0), \nabla u(x_0) + \nabla \varphi_\varepsilon(y), \nabla v(x_0) + \nabla \psi_\varepsilon(y))\, dy$$

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$\{\varsigma_k\}_{k\in\mathbb{N}}$ ... decreasing sequence of positive real numbers s. t.

$$B(x_0, 2\varsigma_k) \subset \Omega, \quad |Du|(\partial B(x_0,\varsigma_k)) = |Dv|(\partial B(x_0,\varsigma_k)) = 0$$

$\{\rho_n\}_{n\in\mathbb{N}}$ ... standard mollifiers for $\delta = 1/n$

$$u_n(x) := u * \rho_n, \quad v_n := v * \rho_n$$

Use Lemma:
$v_{n,k} := \pi_{y_{n,k}} \circ v_n \in W^{1,1}(B(x_0,\varsigma_k); S^2) \cap C^\infty(\overline{B(x_0,\varsigma_k)}; \mathbb{R}^3)$,
$y_{n,k} \in B(0, 1/2)$ s. t.

$$\int_{A^\varepsilon_{n,k}} |\nabla v_{n,k}(x)|\, dx \le C_\star \int_{A^\varepsilon_{n,k}} |\nabla v_n(x), dx.$$

where

$$A^\varepsilon_{n,k} := \{x \in B(x_0,\varsigma_k) \colon \text{dist}(v_n(x), S^2) > \delta_\varepsilon/2\}$$

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$$A^\varepsilon_{n,k} := \{x \in B(x_0, \varsigma_k)\colon \text{dist}(v_n(x), S^2) > \delta_\varepsilon/2\}$$

with $\delta_\varepsilon > 0$ s. t.

$$s_1, s_2 \in B(v(x_0), \delta_\varepsilon) \Rightarrow |\nabla\Pi(s_1) - \nabla\Pi(s_2)| \leq \frac{\rho_\varepsilon}{2b_\varepsilon}.$$

$$b_\varepsilon := 1 + |\nabla v(x_0)| + \|\nabla\psi_\varepsilon\|_\infty$$

$$|\xi_1|, |\xi_2|, |\eta_1|, |\eta_2| \leq a_\varepsilon,\ |\xi_1 - \xi_2|, |\eta_1 - \eta_2| \leq \rho_\varepsilon \ \Rightarrow$$
$$|f(u(x_0), v(x_0), \xi_1, \eta_1) - f(u(x_0), v(x_0), \xi_2, \eta_2)| \leq \varepsilon$$

$$a_\varepsilon := \max\{2 + 2|\nabla u(x_0)| + \|\nabla\varphi_\varepsilon\|_\infty, (\|\nabla\Pi\|_\infty + 1)(2 + 2|\nabla v(x_0)| + \|\nabla\psi_\varepsilon\|_\infty)\}$$

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cut-off functions

$\zeta_1 \in C_c^\infty(\mathbb{R};[0,1])$, $\|\zeta_1'\|_\infty \leq 2/\delta_\varepsilon$

$$\zeta_1(r) = \begin{cases} 1 & r \in \left(-\frac{\delta_\varepsilon}{4}, \frac{\delta_\varepsilon}{4}\right), \\ 0 & r \notin \left(-\frac{\delta_\varepsilon}{2}, \frac{\delta_\varepsilon}{2}\right) \end{cases}$$

$\zeta_2 \in C_c^\infty(\mathbb{R}^3;[0,1])$, $\|\nabla\zeta_2\|_\infty \leq 2/\delta_\varepsilon$

$$\zeta_2(s) = \begin{cases} 1 & s \in B\left(0, \frac{\delta_\varepsilon}{4}\right), \\ 0 & s \notin B\left(0, \frac{\delta_\varepsilon}{2}\right) \end{cases}$$

$$u_{n,k}^\varepsilon(x) := u_n(x) + \frac{1}{n}\zeta_1(u_n(x) - u(x_0))\,\varphi_\varepsilon(nx)$$

$$v_{n,k}^\varepsilon(x) := v_{n,k}(x) + \frac{1}{n}\zeta_2(v_{n,k}(x) - v(x_0))\,\psi_\varepsilon(nx)$$

$$\bar{u}_{n,k}^\varepsilon(x) := \Phi_n(u_{n,k}^\varepsilon(x)), \bar{v}_{n,k}^\varepsilon(x) := \begin{cases} v_{n,k}(x) & \text{if } |v_{n,k}(x) - v(x_0)| \geq \dfrac{\delta_\varepsilon}{2}, \\ \Pi(v_{n,k}^\varepsilon(x)) & \text{if } |v_{n,k}(x) - v(x_0)| < \dfrac{\delta_\varepsilon}{2}. \end{cases}$$

$\Phi_n : \mathbb{R} \to \mathbb{R} \ldots$ projection of $[\alpha - \|\varphi_\varepsilon\|_\infty/n, \beta + \|\varphi_\varepsilon\|_\infty/n]$ onto $[\alpha, \beta]$

$$\Phi_n(r) := \frac{n(\beta-\alpha)r + (\beta+\alpha)\|\varphi_\varepsilon\|_\infty}{n(\beta-\alpha) + 2\|\varphi_\varepsilon\|_\infty}.$$

$$|\nabla \bar{u}^\varepsilon_{n,k}(x)| \leq C_\varepsilon(1 + |\nabla u_n(x) - \nabla u(x_0)|)$$

$$|\nabla \bar{v}^\varepsilon_{n,k}(x)| \leq C_\varepsilon(1 + |\nabla v_{n,k}(x) - \nabla v(x_0)|)$$

$\{\bar{u}^\varepsilon_{n,k}\}_{n\in\mathbb{N}}$ and $\{\bar{v}^\varepsilon_{n,k}\}_{n\in\mathbb{N}}$ are admissible sequences for $\mathcal{F}(u, v; B(x_0; \varsigma_k))$

$$\frac{d\mathcal{F}(u,v;\cdot)}{d\mathcal{L}^2}(x_0) \leq \limsup_{k\to\infty} \limsup_{n\to\infty} \fint_{B(x_0,\varsigma_k)} f(u(x_0), v(x_0), \nabla \bar{u}^\varepsilon_{n,k}(x), \nabla \bar{v}^\varepsilon_{n,k}(x))\, dx$$

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…and after a few estimates conclude that

$$
\begin{aligned}
&\limsup_{k\to\infty}\limsup_{n\to\infty} ⨍_{B(x_0,\varsigma_k)} f\big(u(x_0), v(x_0), \nabla \bar{u}^{\varepsilon}_{n,k}(x), \nabla \bar{v}^{\varepsilon}_{n,k}(x)\big)\,dx \\
&\le \limsup_{k\to\infty}\limsup_{n\to\infty} ⨍_{B(x_0,\varsigma_k)} f\big(u(x_0), v(x_0), \nabla u(x_0) + \nabla\varphi_\varepsilon(nx), \\
&\qquad\qquad\qquad\qquad\qquad \nabla v(x_0) + \nabla\psi_\varepsilon(nx)\big)\,dx + \varepsilon \\
&\quad = \int_Q f\big(u(x_0), v(x_0), \nabla u(x_0) + \nabla\varphi_\varepsilon(y), \nabla v(x_0) + \nabla\psi_\varepsilon(y)\big)\,dy + \varepsilon \\
&\quad \le \mathcal{Q}_T f\big(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0)\big) + 2\varepsilon
\end{aligned}
$$

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