Large-scale sea ice modeling: Complexity and community

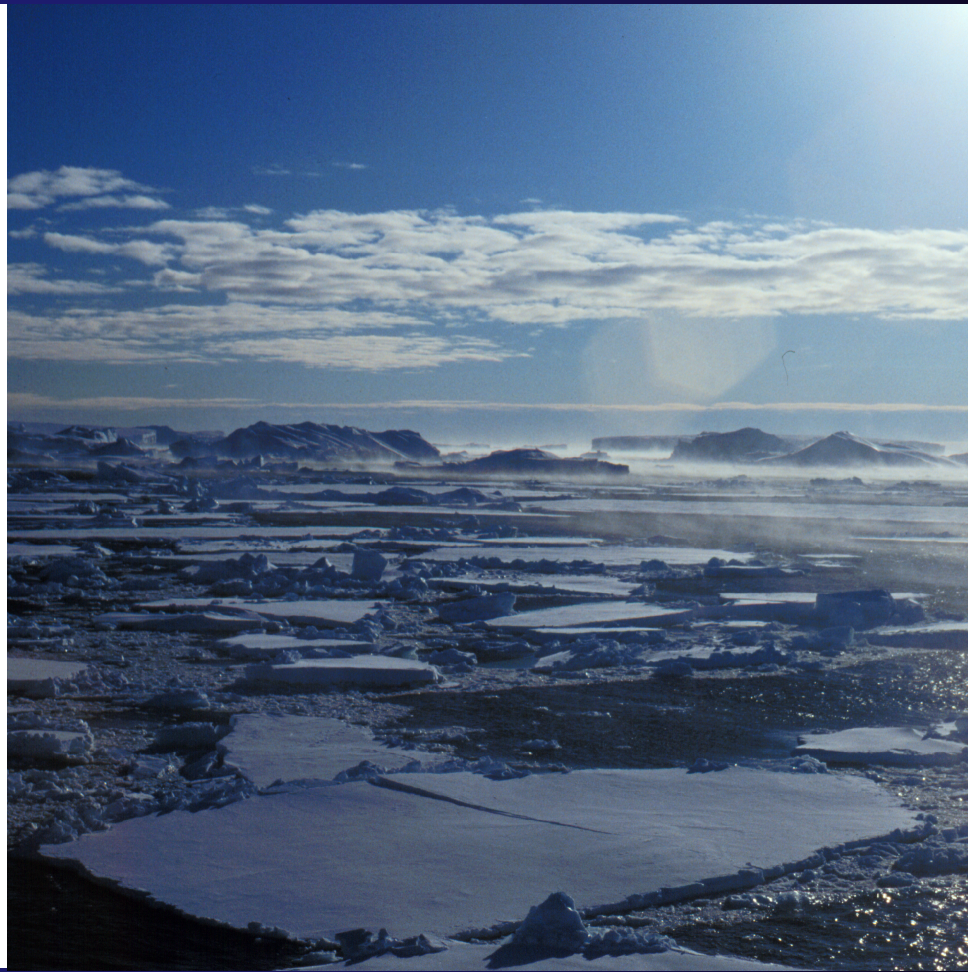
SIAM Conference on Mathematical and Computational Issues in the Geosciences, March 2019

Elizabeth Hunke

T-3 Fluid Dynamics and Solid Mechanics Deputy Group Leader

LANL Program Manager for DOE Climate and Environmental Sciences Division

Lead, CICE Consortium



Sea ice is frozen seawater



E. Hunke

Weddell Sea, Antarctica 1998

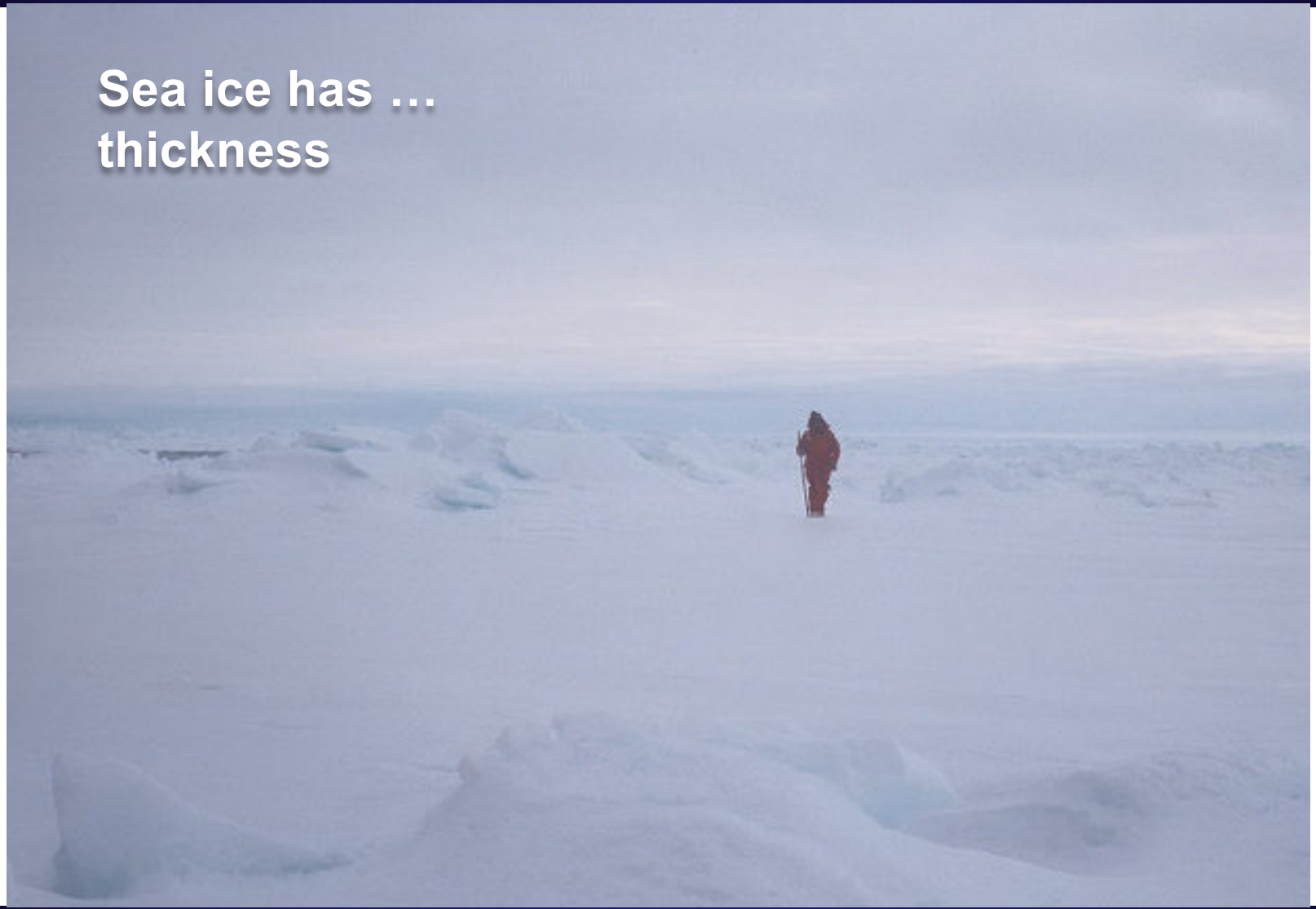
Sea ice has … horizontal area



E. Hunke

Weddell Sea, Antarctica 1998

Sea ice has …
thickness



Steve Ackley, Weddell Sea, 1998

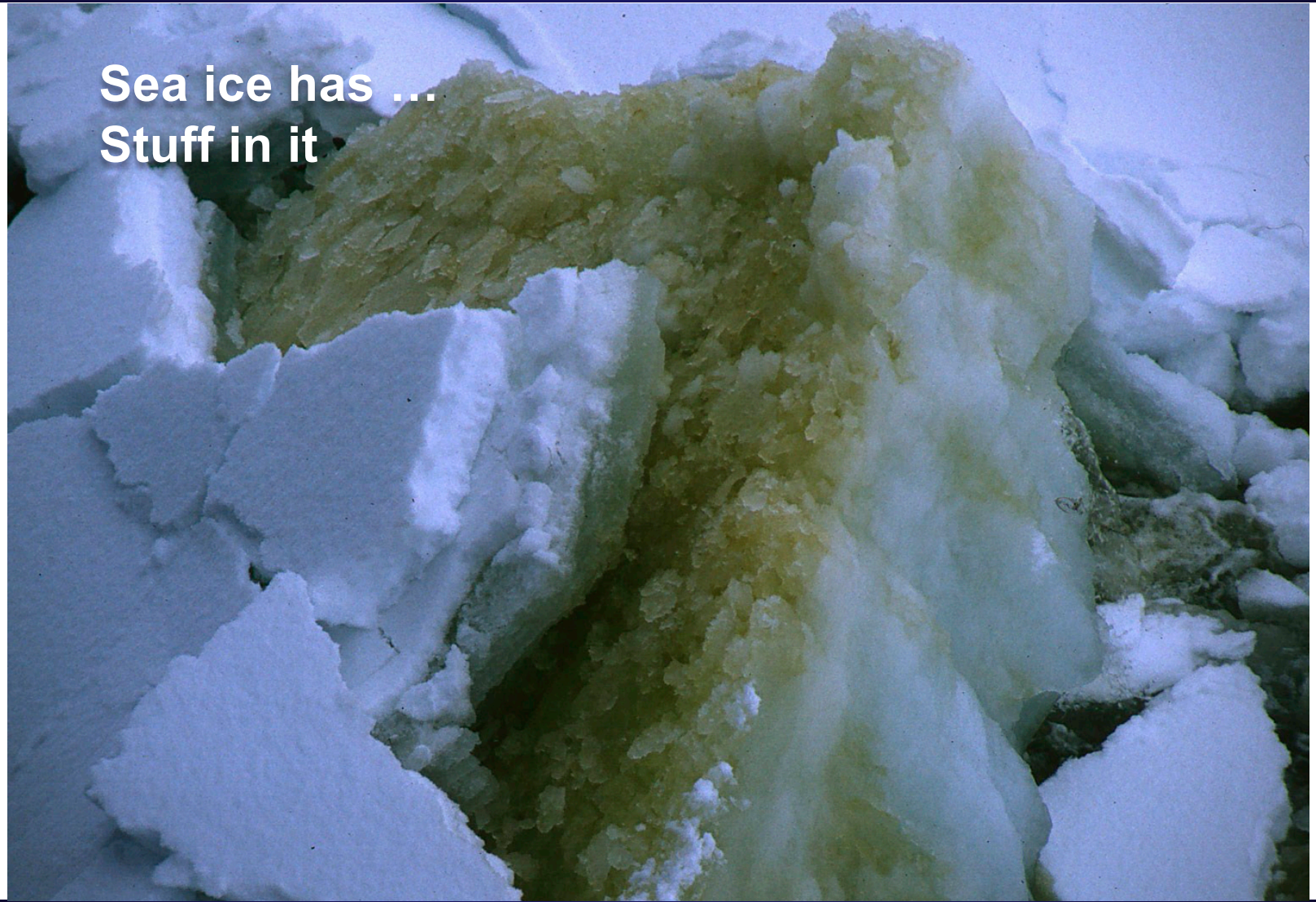


Sea ice moves

http://youtu.be/YBTVJIH6En8

``Ice in Motion'' by Kåre Holter Solhjell, near Svalbard


Sea ice has …
stuff on it



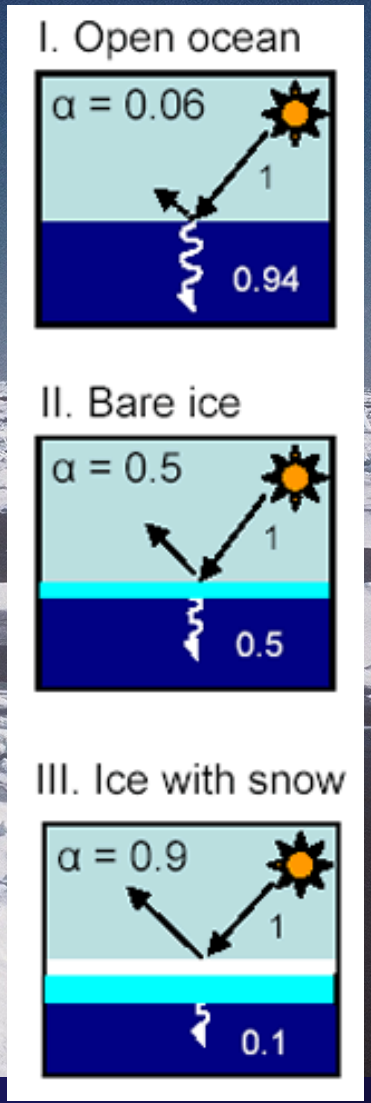
C. Haas

What's important?

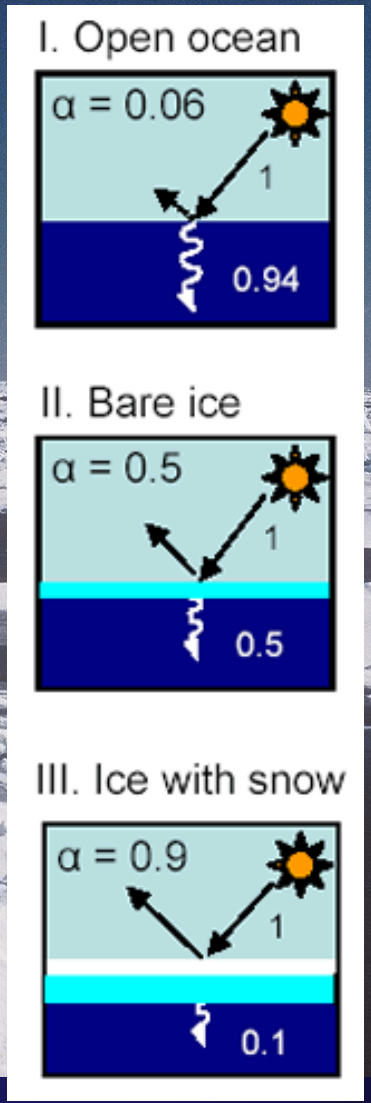


E. Hunke

What's important? Albedo

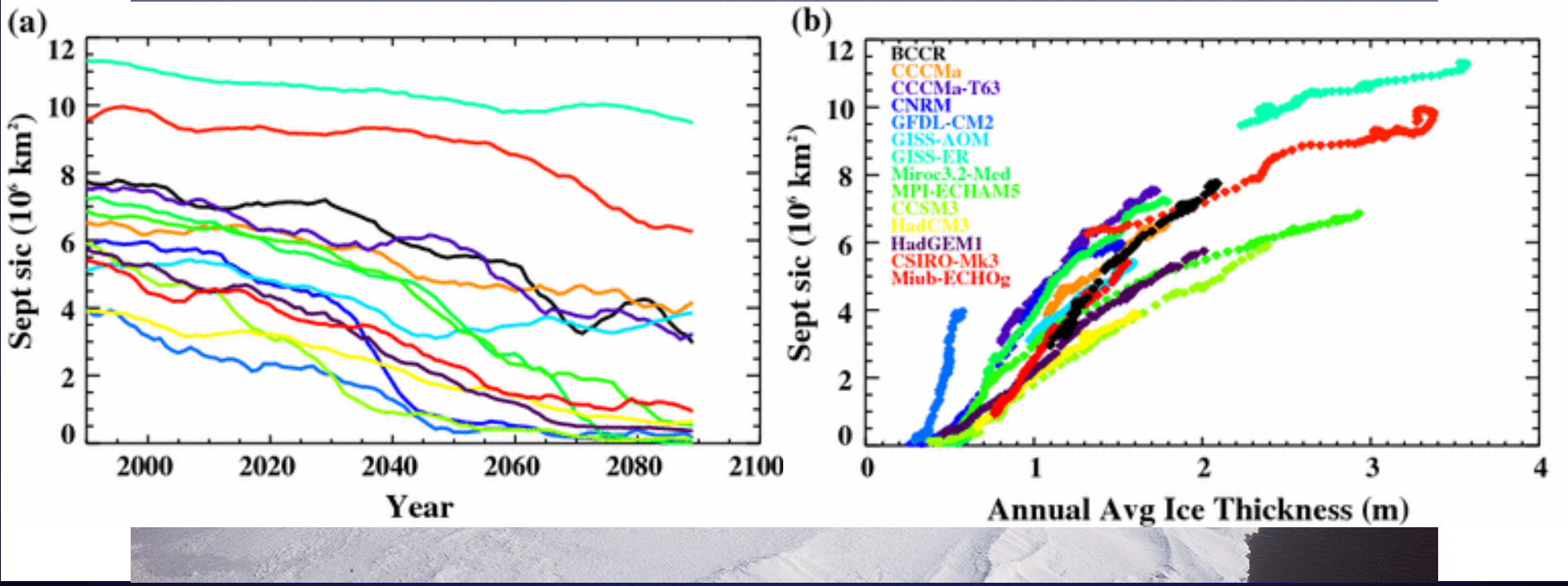


What's important? Albedo
What else?



What's important? Albedo
What else? Volume

Holland, M.M., Serreze, M.C. and Stroeve, J. Climate Dynamics (2010)



Historical perspective on sea ice model evolution

- **Initial objective: effect on planetary albedo**
  - Sea ice (and ocean) were boundary conditions for the atmosphere
  - “modeled” only ice extent
  - no sub-grid-scale area, no thickness, no insulating effects between atmosphere and ocean, no motion

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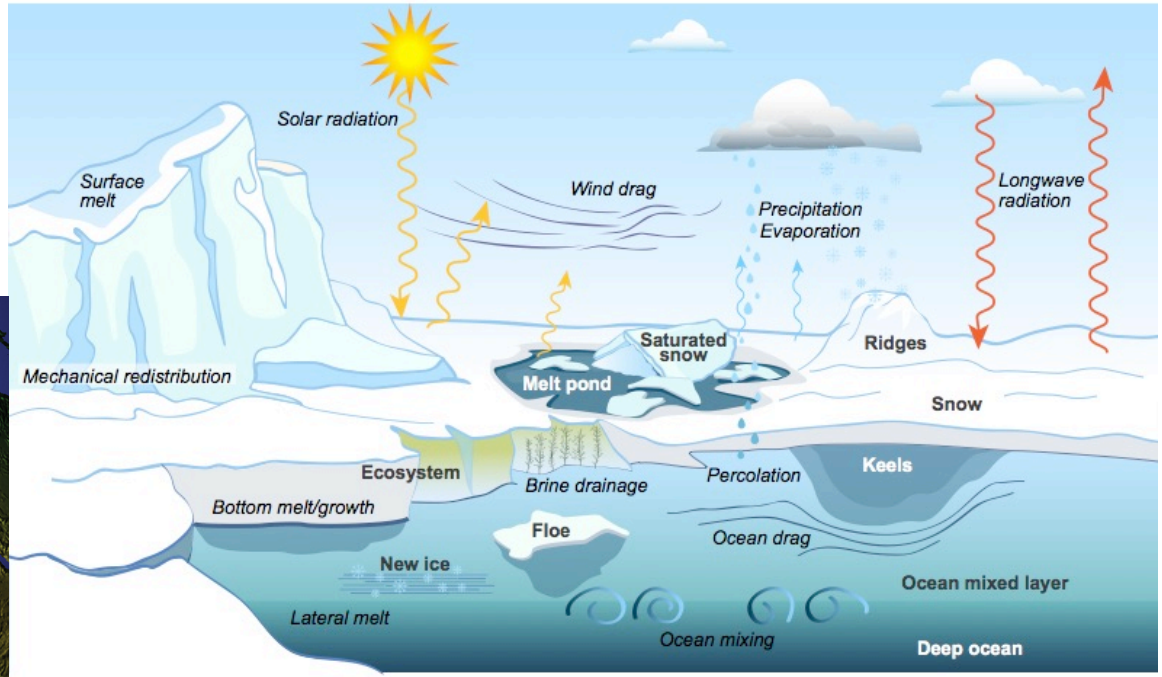
- **Need: albedo and thickness feedbacks → ice thickness distribution**

Physical processes that affect the area and thickness of sea ice



Ice motion





“Appropriate Complexity”

e.g., Larsen et al. 2016, French et al. 2016

evaluates and guides model sophistication based on

- time- and spatial-scales of emergent properties
- predictive needs of scientific or management problems
- validation potential
- interpretability
- tractability
- generality

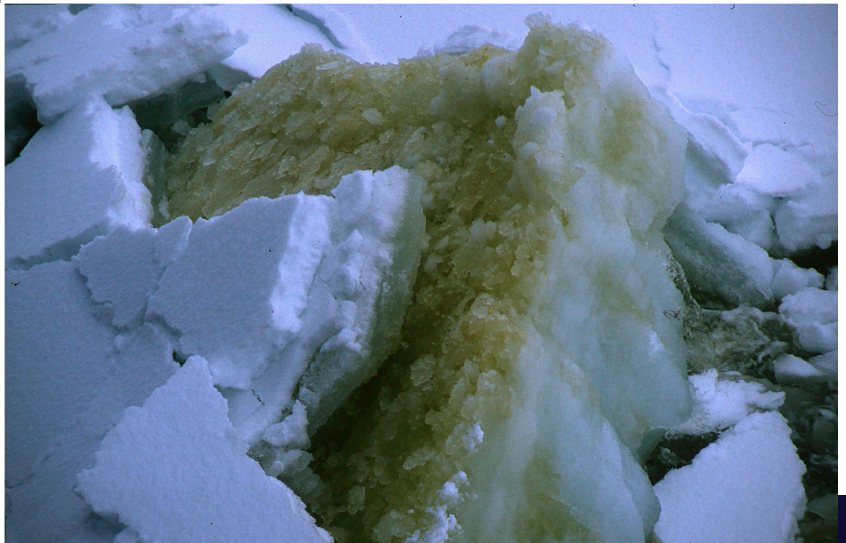
as functions of computational and representational detail



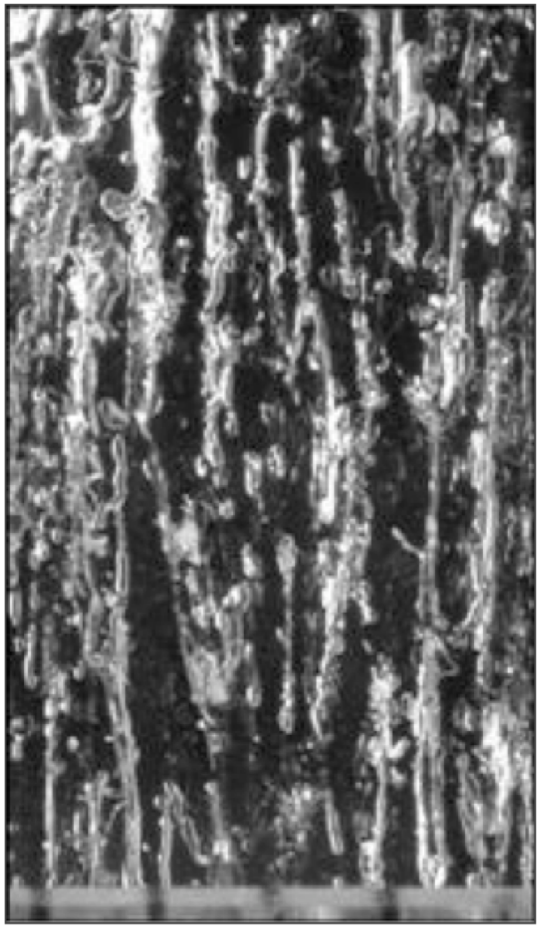
Sea ice ecosystem

Algae live in the liquid (brine) within sea ice and thrive where there are nutrients





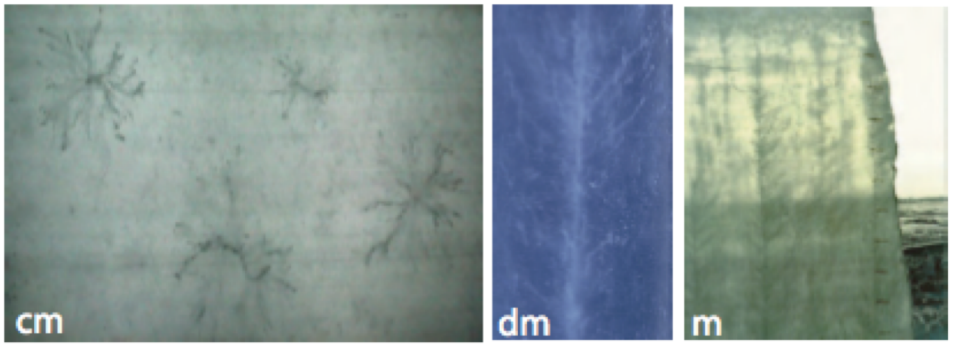
“Mushy” Thermodynamics



A photograph in natural light showing elongated tubes that form as brine pockets trapped between the ice crystals. The image is 5 millimeters in width. Photo courtesy of Ted Maksym, United States Naval Academy.

Equations	Variables
Conservation of Energy	Enthalpy
Conservation of Salt	Salinity
Ice-brine liquidus relation	Liquid fraction
Darcy flow through a porous medium	Vertical velocity

Brine channels

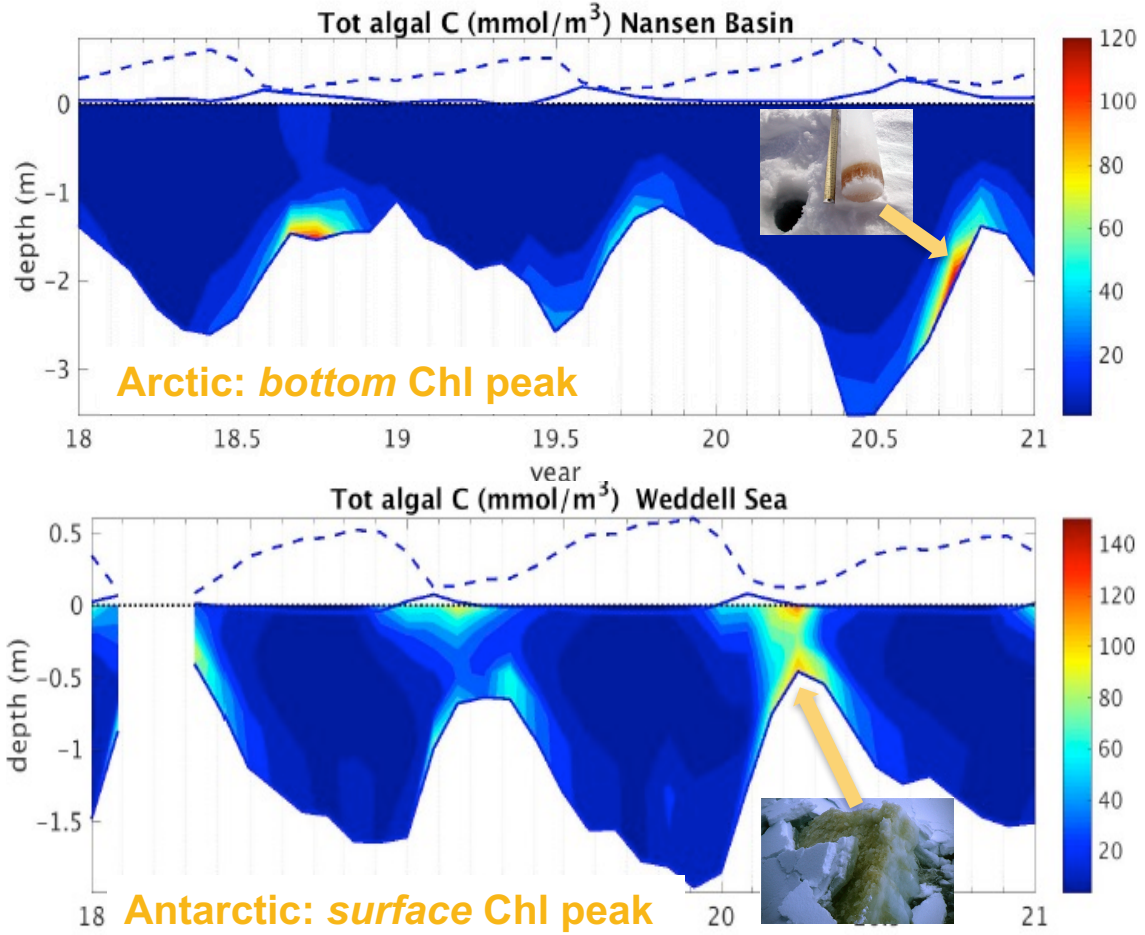


horizontal vertical

Courtesy K. Golden

Turner et al., JGR 2013

Coupled Ocean – Sea Ice Ecosystem



Algal types: diatoms, flagellates, *Phaeocystis*

DON/DOC: proteins, polysaccharides, lipids

Nutrients: nitrate/nitrite, silicate, ammonium, DMS, iron

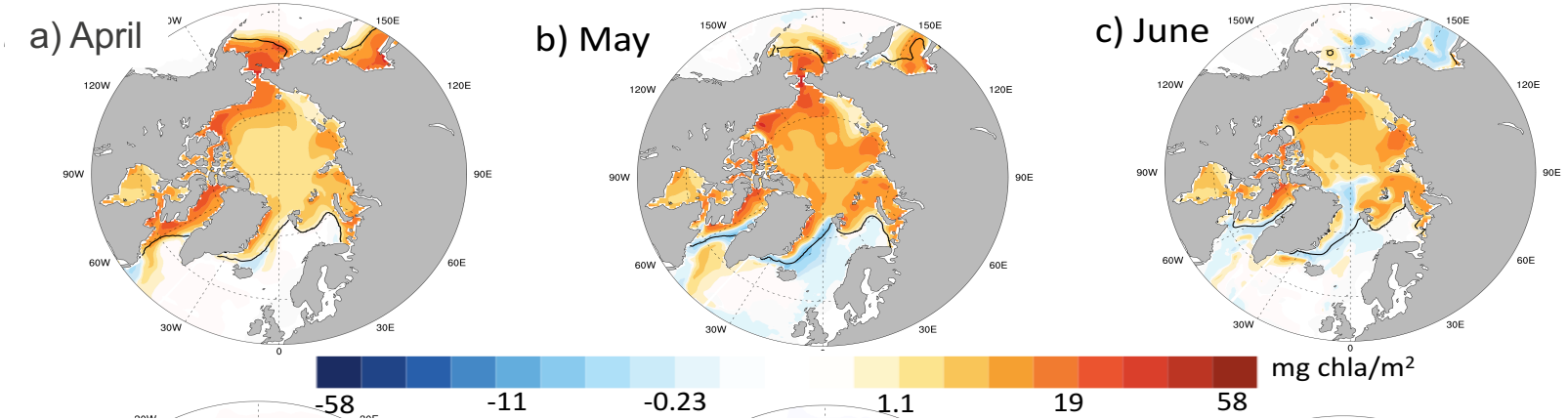
Aerosol & chlorophyll absorption alter ice growth, under-ice PAR via Delta-Eddington

POP + CICE

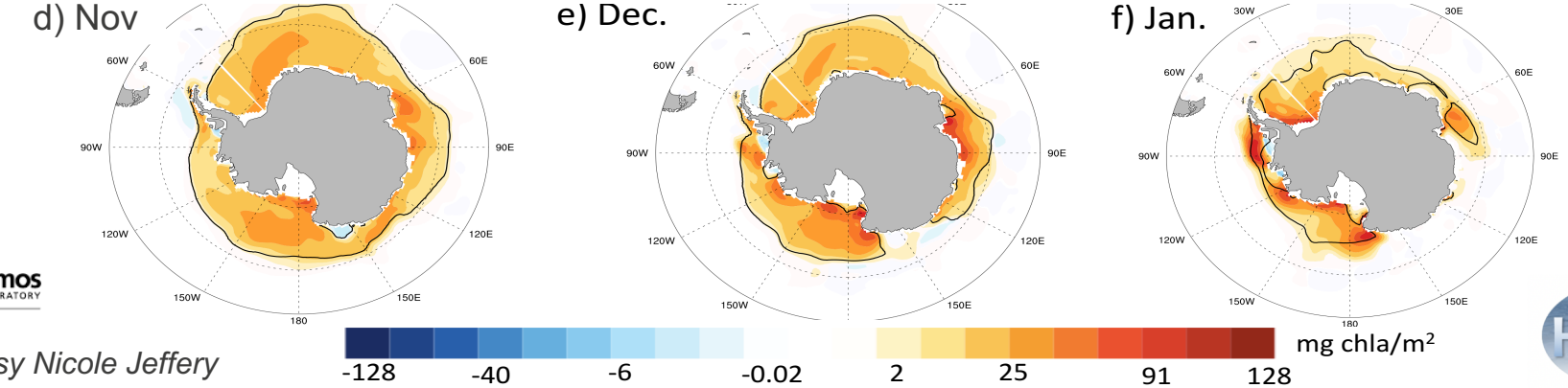
Courtesy Nicole Jeffery

Simulated difference in surface ocean + sea ice chlorophyll when algal biochemistry is included in sea ice

Arctic: net depletion of chl-a in some regions as sea ice BGC competes for nutrients.



Antarctic: surface chl-a doubles when sea ice BGC is included in coupled ocean-ice simulations. Sea ice biochemistry supplies iron to the surface ocean.

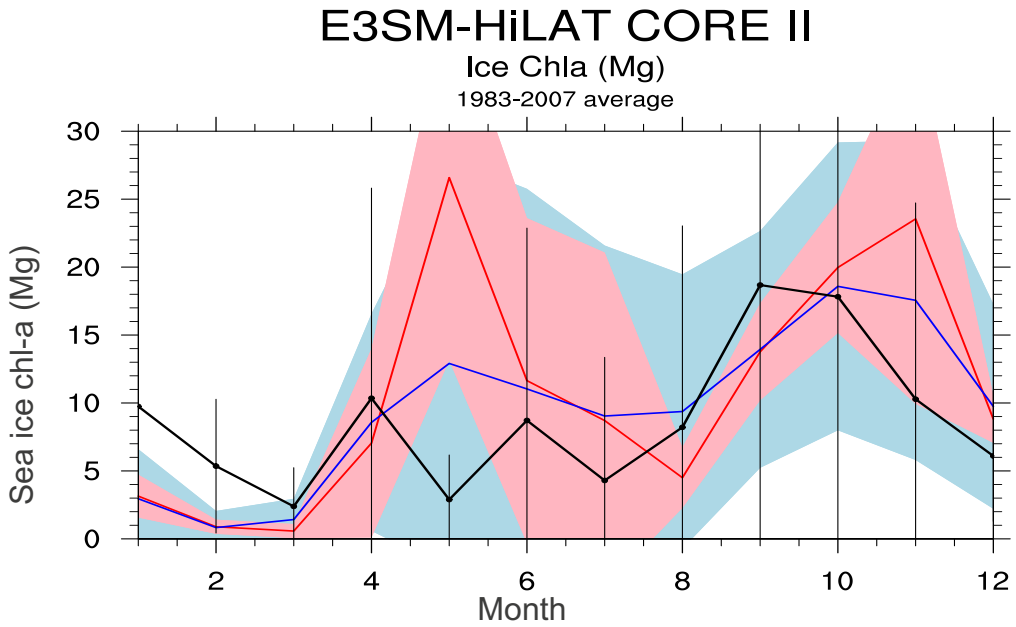


Los Alamos NATIONAL LABORATORY EST. 1943

Courtesy Nicole Jeffery

HiLAT RASM

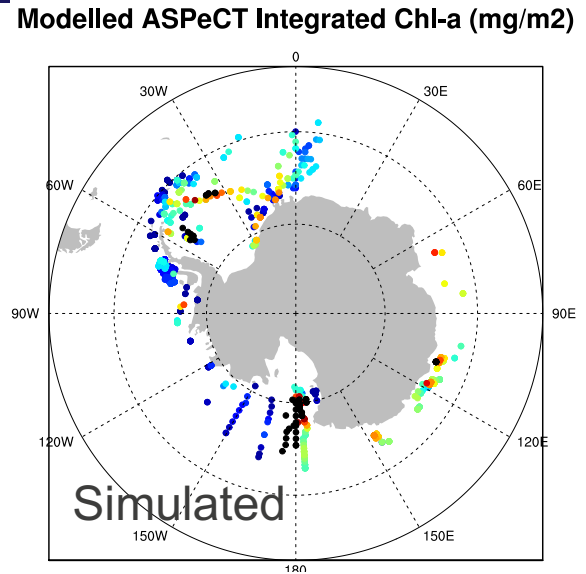
Historical simulations of total Southern Ocean sea ice chlorophyll are in good agreement with observations

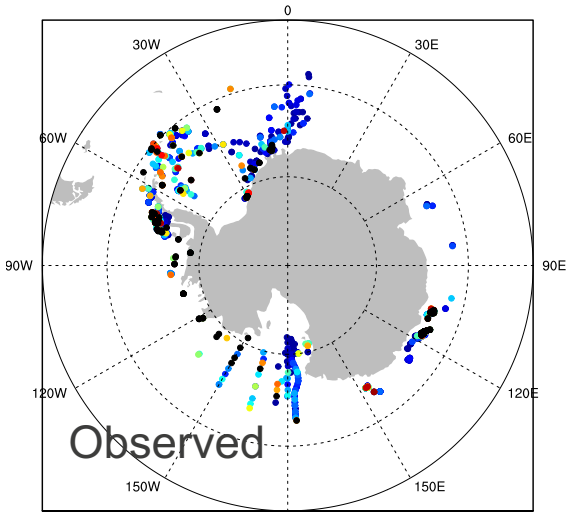


Obs. cycle from ASPeCT database (Worby et. al 2008)

Simulated cycle from all southern ocean sea ice grid cells

Simulated cycle from sea ice grid cells with ASPeCT Obs.





Courtesy N. Jeffery

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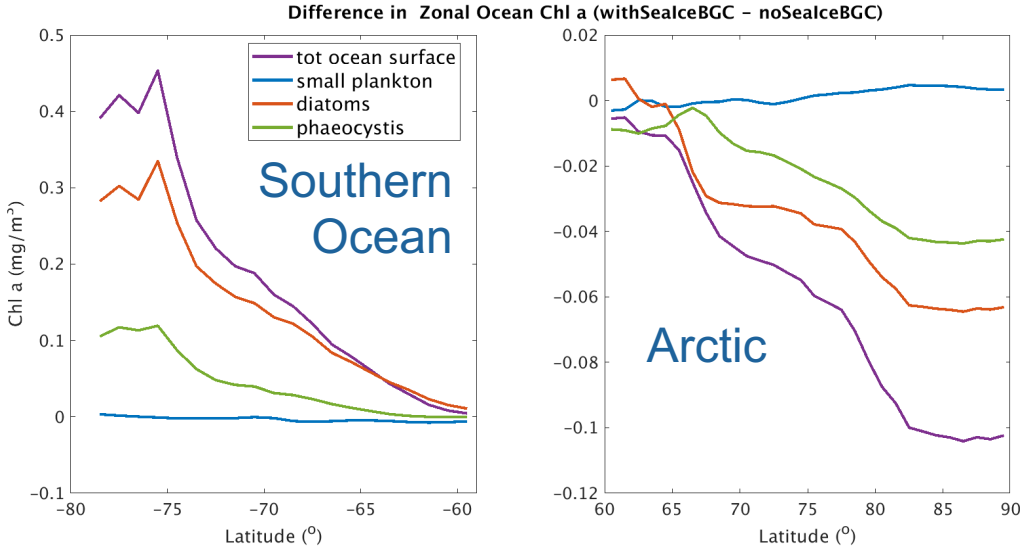
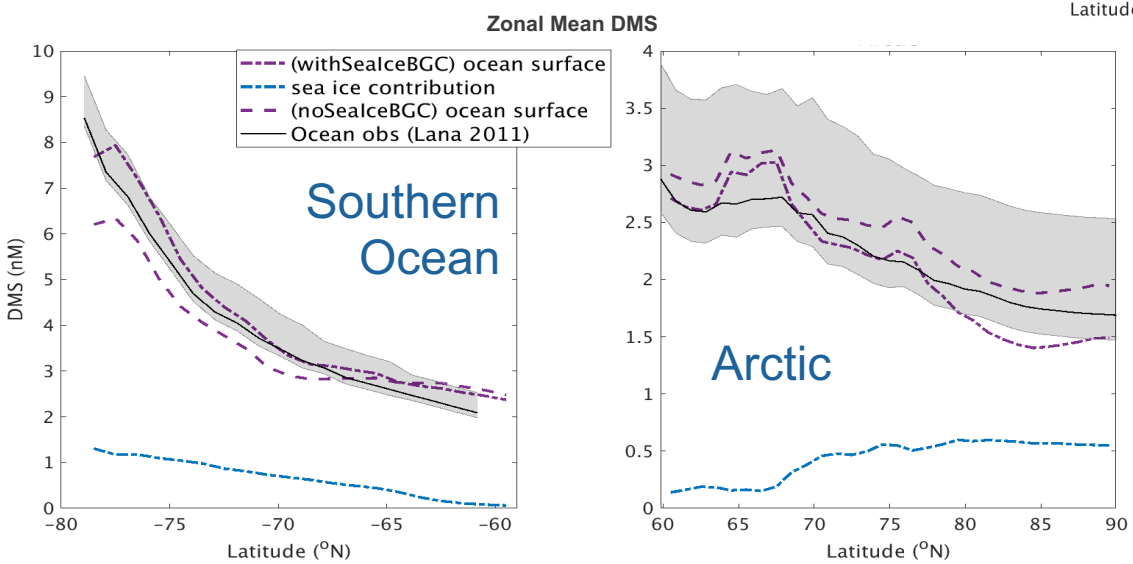
Southern Ocean vs Arctic: Contrasting Behavior

Sea ice BGC *enhances* surface chlorophyll and DMS production in the Southern ocean but *decreases* production in the Arctic





Avg surface chl-a is ~1 mg/m^3 in these regions

Enabling technologies:

- Delta-Eddington radiation
- Mushy thermodynamics
- Incremental remap advection
- Ecosystem

Los Alamos NATIONAL LABORATORY EST. 1943

Courtesy N. Jeffery



CICE Consortium

to enhance sea ice model development for and by the community

- Acceleration of scientific development
- Acceleration of R&D transfer to operational use
- Vehicle for collaboration and sharing

DEPARTMENT OF ENERGY · UNITED STATES OF AMERICA

NPS PRAESTANTIA PER SCIENTIAM NAVAL POSTGRADUATE SCHOOL

NSF

U.S. NAVAL RESEARCH LABORATORY

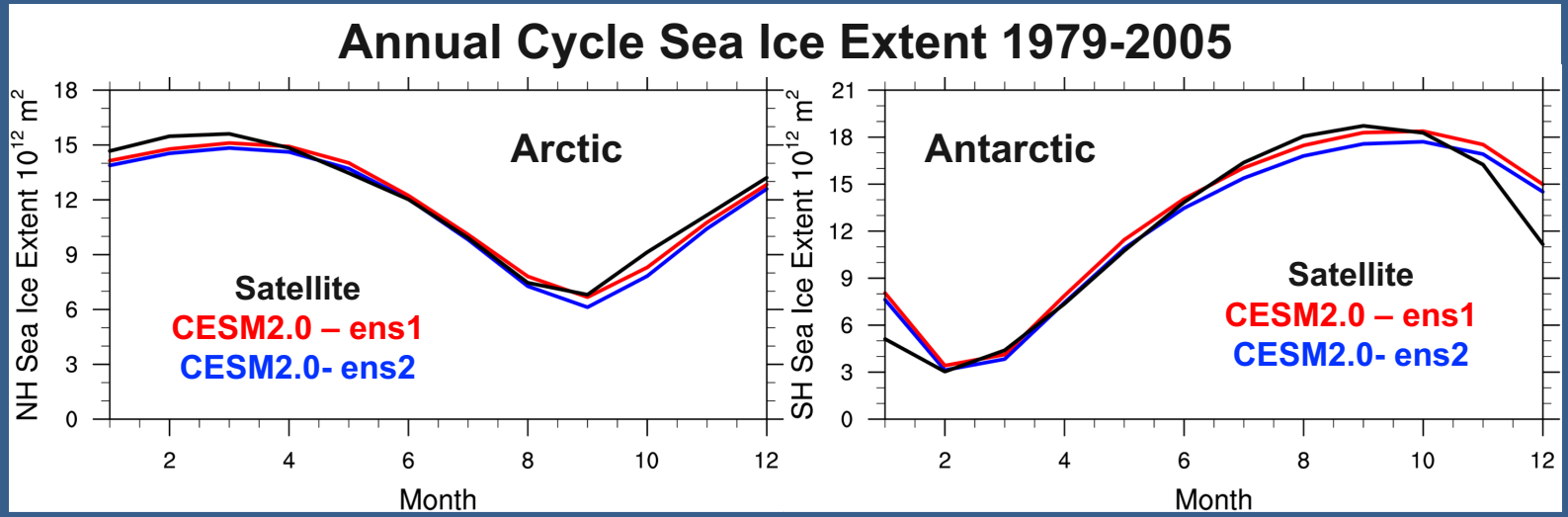
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION NOAA U.S. DEPARTMENT OF COMMERCE

DMI Danish Meteorological Institute

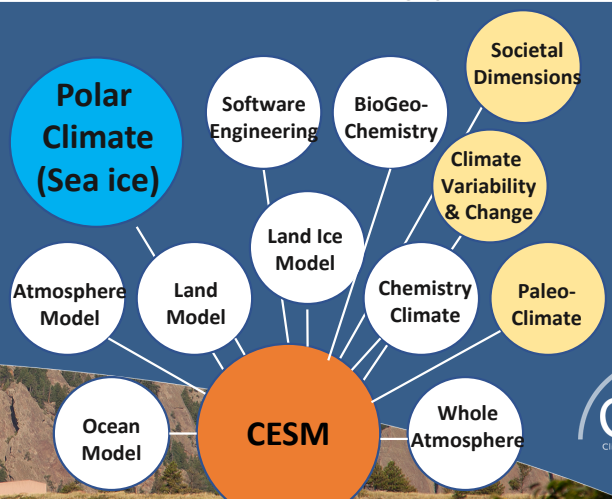
Environment Canada

NCAR UCAR

Community Earth System Model (CESM)



- >400 registered developers; >5000 registered users
- Current community developments (e.g. FSD) being incorporated.
- CICEv6 and Icepack will be used in CESM3 → in development!
- Opportunities for development (e.g. interaction with ice shelves) with MOM6 in CESM3



NCAR is sponsored by National Science Foundation

CGD Climate & Global Dynamics a Laboratory of NCAR

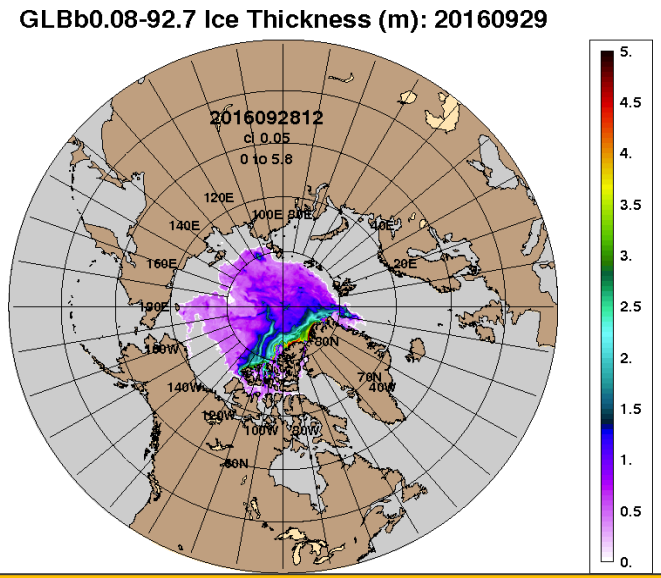
12/9/18

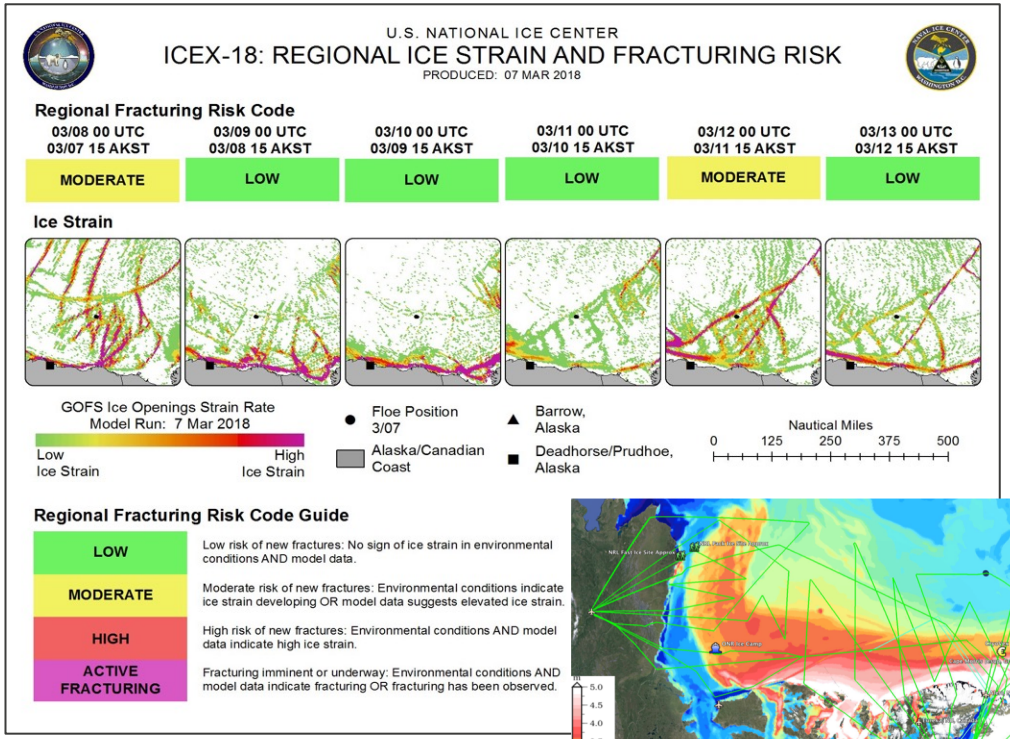
CICE: A foundational model for U.S. military and civilian forecasts

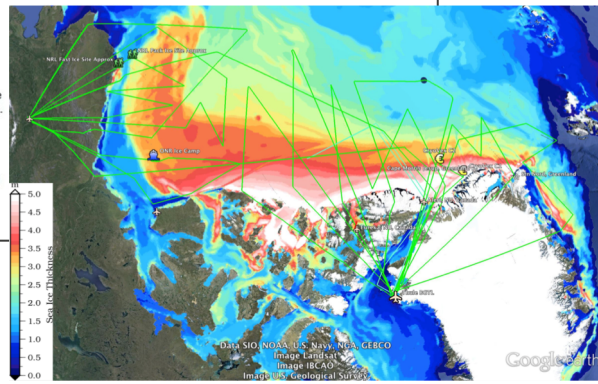
U.S. NAVAL RESEARCH LABORATORY

CICE-based, real-time sea ice strain and fracture risk product developed by the U.S. National Ice Center to support the 2018 ICEX experiment, using the U.S. Navy's Global Ocean Forecast System (GOFS 3.1). Supports US Navy's submarine readiness.

Global Ocean Forecast System
GOFS 3.1 Ice Thickness







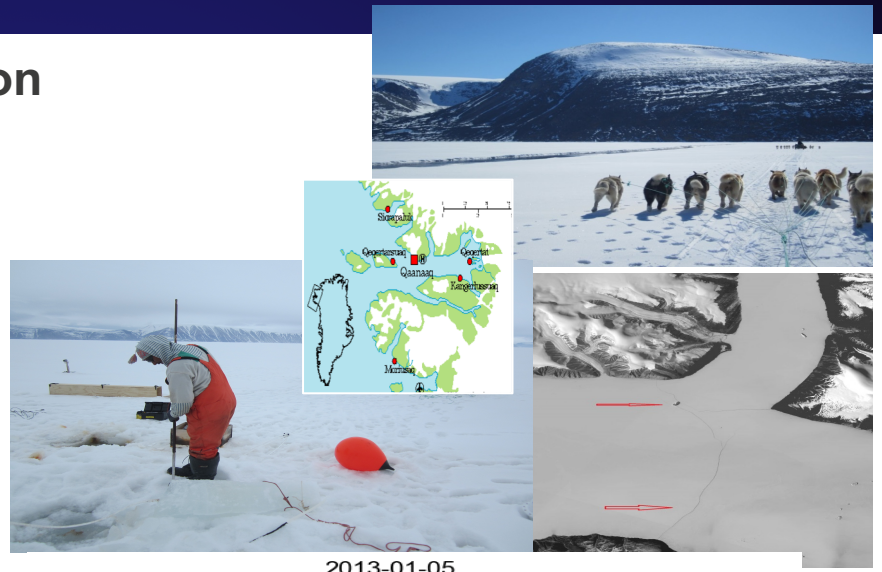
Preflight sea ice analysis for planning NASA IceBridge flights

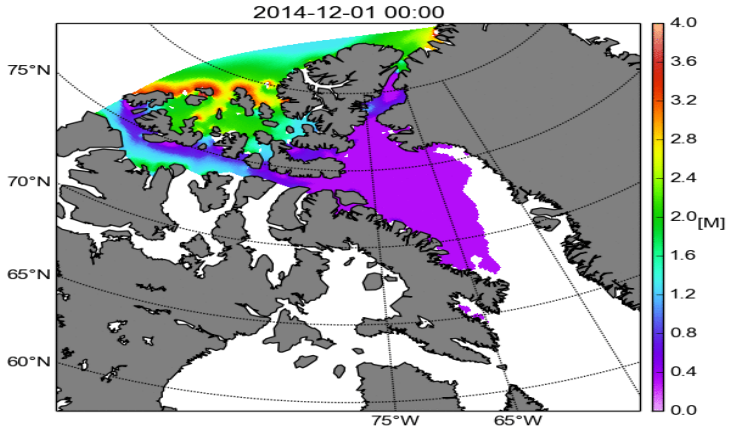
- NRL Nowcast/7-day forecasts of ice concentration, ice thickness, ice drift, SST, SSS and ocean currents in both hemispheres
- Products pushed daily to the U.S. National Ice Center (NIC) and NOAA
- Navy GOFS3.1 and Navy ESPC output data provided for YOPP: https://doi.pangaea.de/10.1594/PANGAEA.890343

Danish Operational Sea Ice Information

for Greenlandic waters

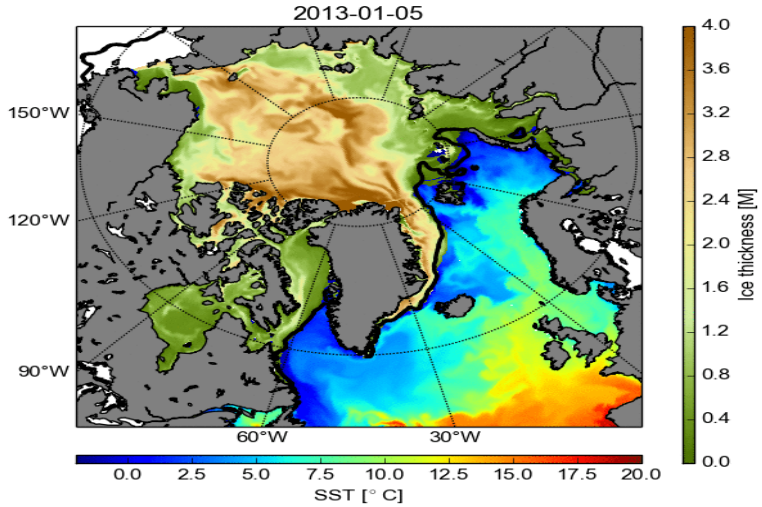
- **Users range from local communities to shipping**
- **Provide in-situ observations, ice charts and remotely sensed data to users and for model validation**
- **CICE model products include**
  - Ice thickness
  - Ice concentration
  - Ice convergence
  - Ice drift
  - Ice temperature
- **Scales range from local to pan-Arctic**





Local, high-resolution domain

1 km resolution



Regional Arctic hindcasts

10 m resolution

Courtesy T. Rasmussen

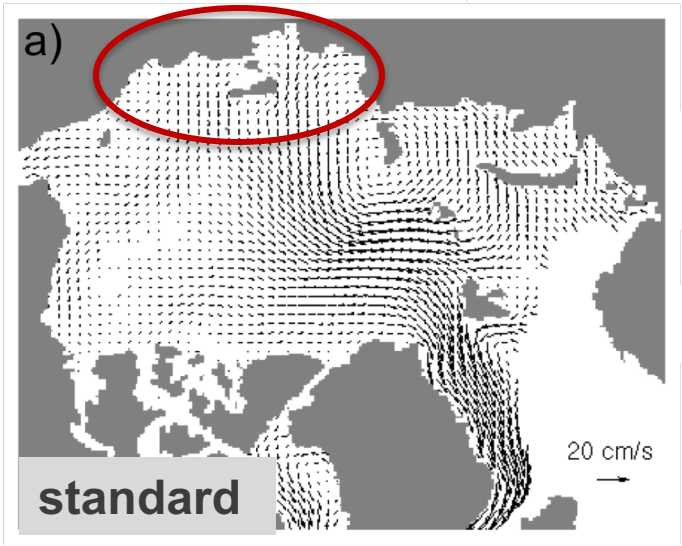
DMI Danish Meteorological Institute

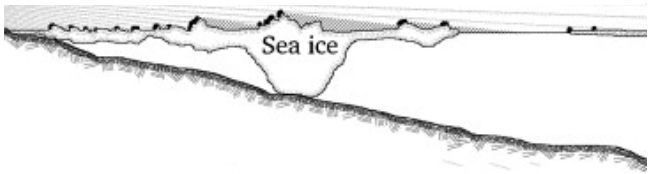
CICE version 6

Landfast sea ice

NIC data
15 April 2002

Simulation by a
standard VP model



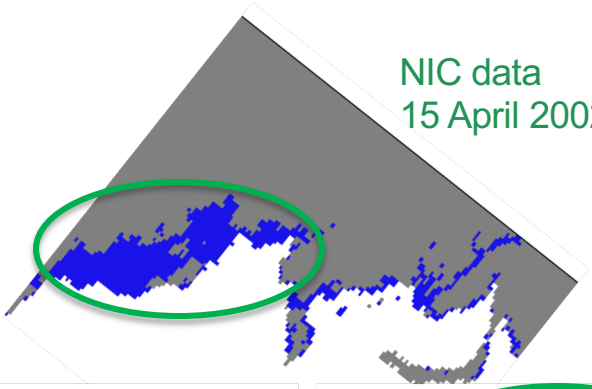


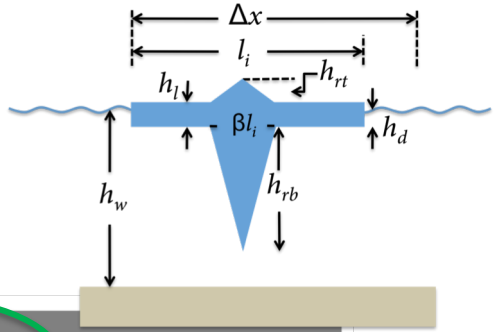
Mahoney et al., *Cold Reg. Sci. Technol.*, 2017

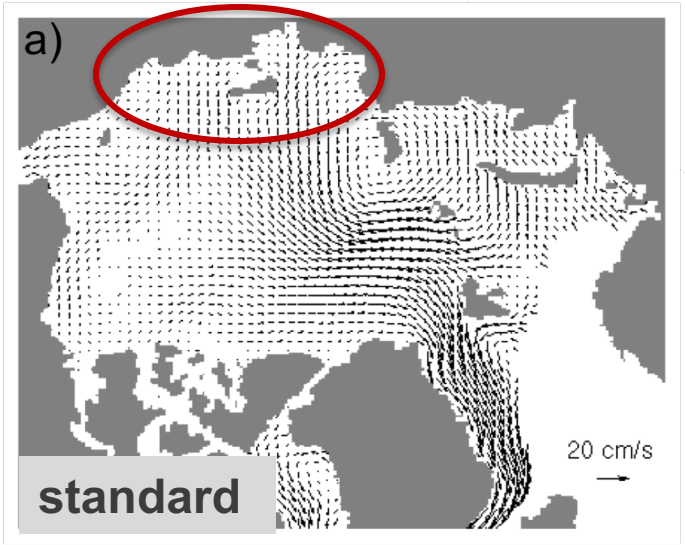
Environment Canada

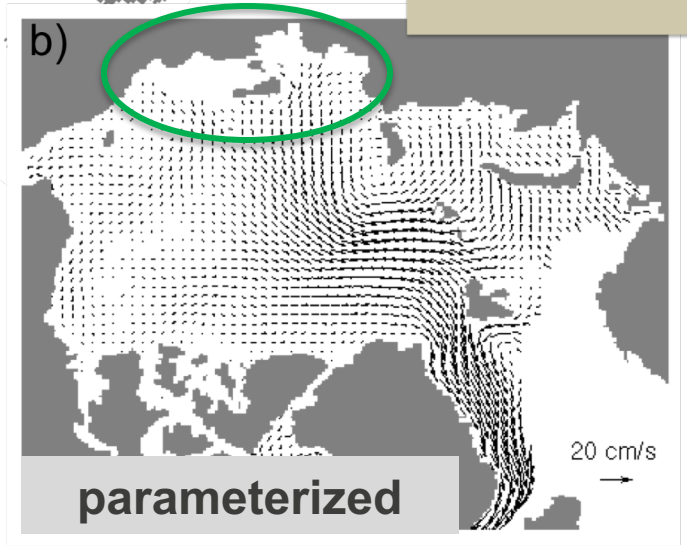
CICE version 6

Standard model vs. model with grounding parameterization









Environment Canada

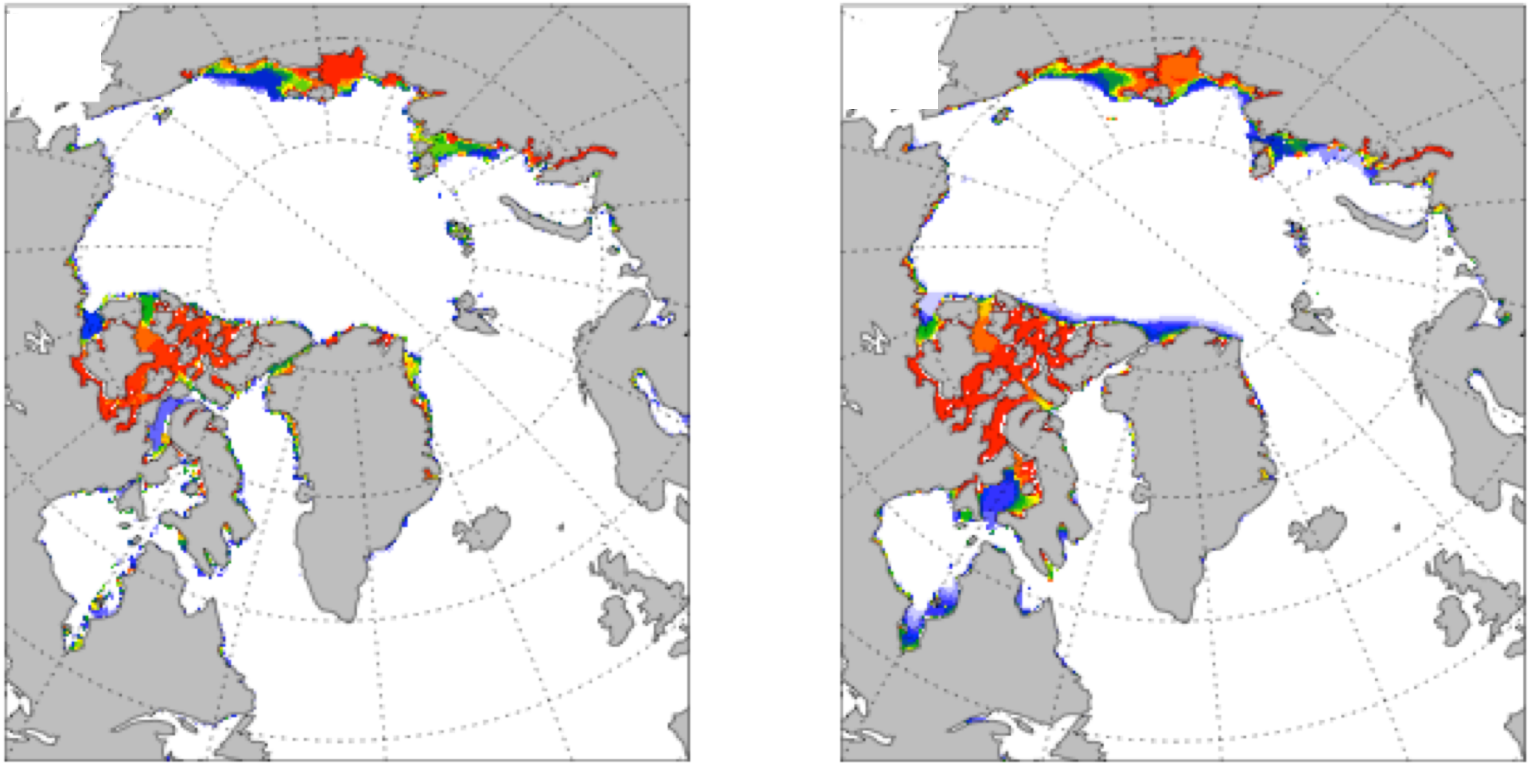
Lemieux et al. *J. Geophys. Res. Oceans*, 2016

CICE version 6

CICE with grounding and tensile strength

Observation

Model



0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Frequency (0 = Never, 1 = Always)

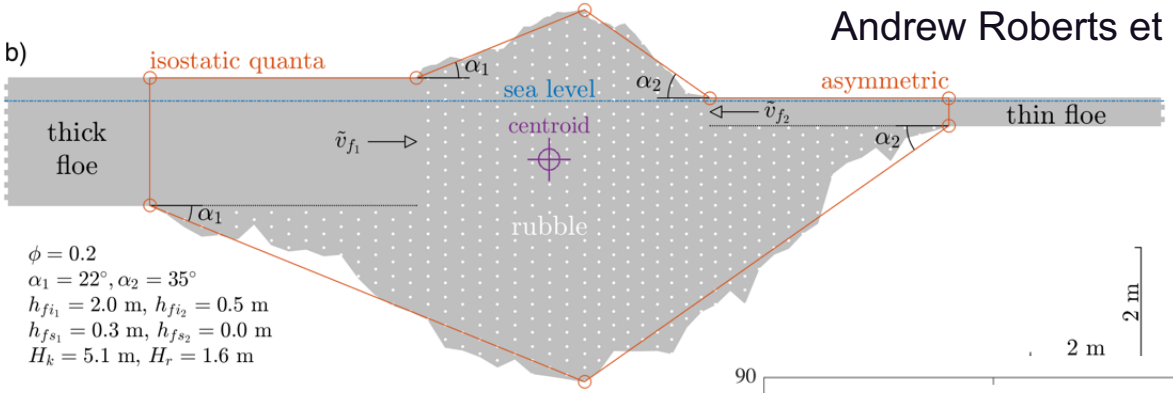
2005-2007
(Jan-May)

Environment Canada

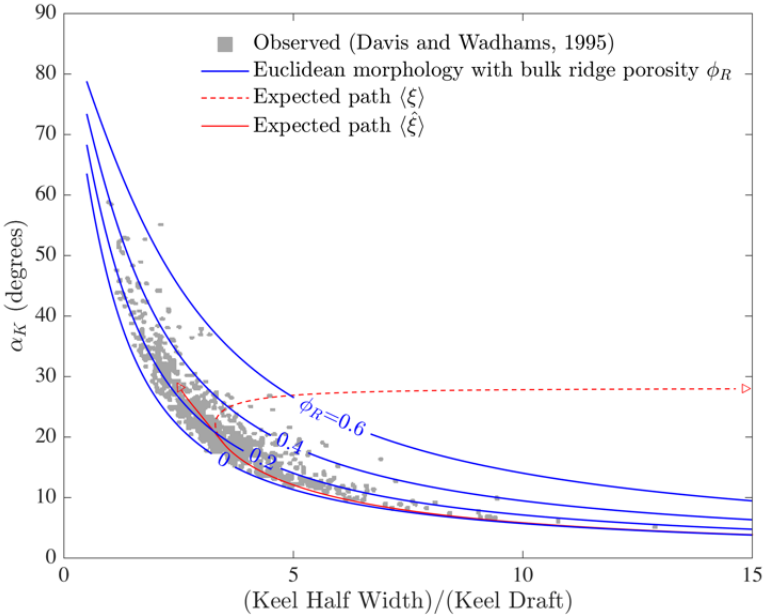
Lemieux et al. *J. Geophys. Res. Oceans*, 2016

A variational formulation for sea ice ridging

Andrew Roberts et al., JAMES 2019



Using fundamental Lagrangian mechanics and concepts from soil mechanics



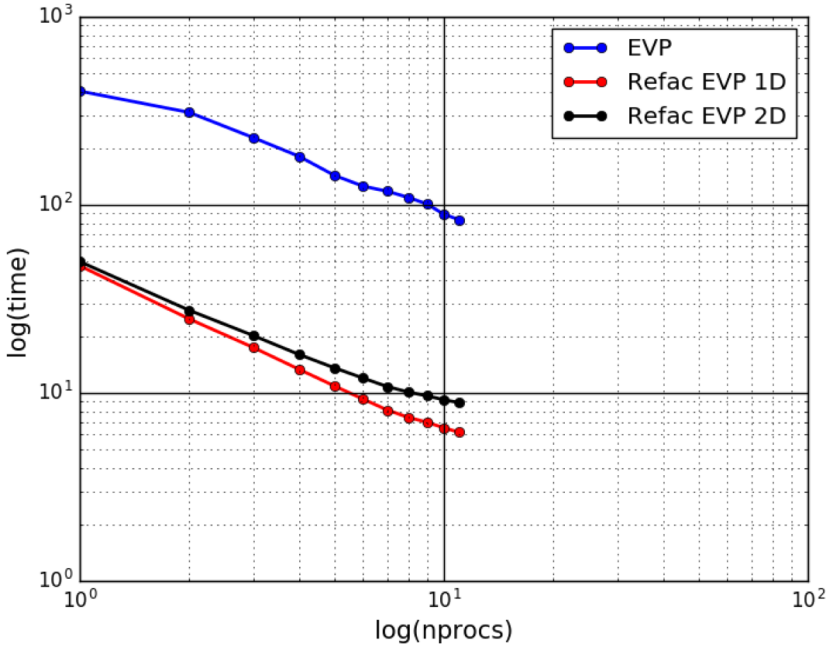
coming soon to

CICE Consortium

Refactored elastic-viscous-plastic dynamics

DMI
Danish Meteorological Institute

- Goal: **Improve efficiency of dynamics model code**
- Method
  - Reduce memory use
  - Improve parallelization
  - Skip MPI



Future plans:

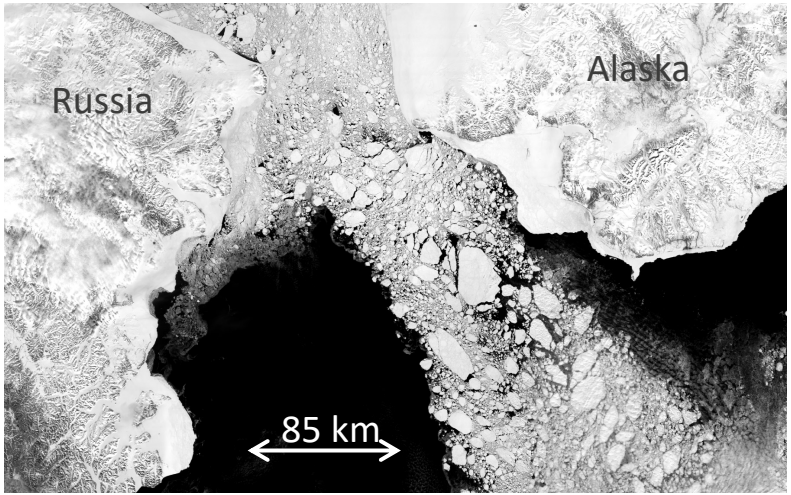
Integrate icebergs as anchor points for landfast ice

Floe size distribution

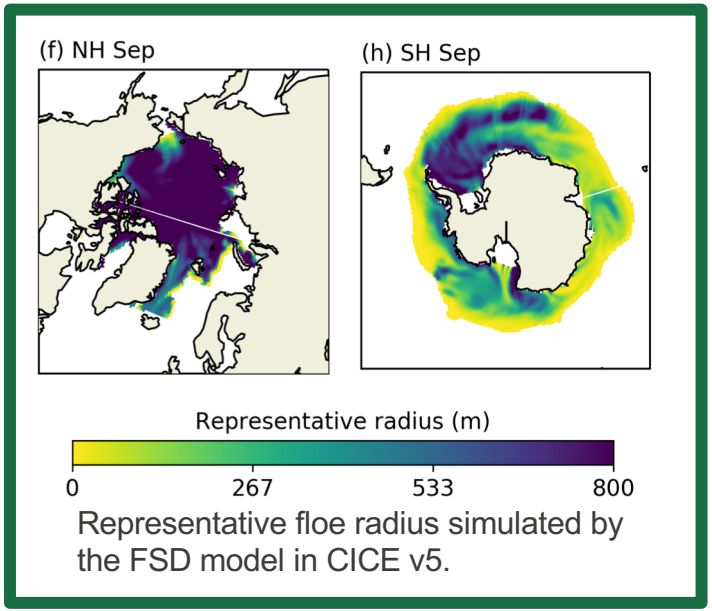
Sea ice is composed of distinct pieces, known as floes.

Floe sizes span a wide range of scales, described statistically by their floe size distribution (FSD).





Floes in the Bering Sea (MODIS)



Representative floe radius simulated by the FSD model in CICE v5.

coming soon to

CICE Consortium

Roach, L. A., Horvat, C., Dean, S. M., & Bitz, C. M. (2018). An emergent sea ice floe size distribution in a global coupled ocean--sea ice model. *J. Geophys. Res. Oceans.*

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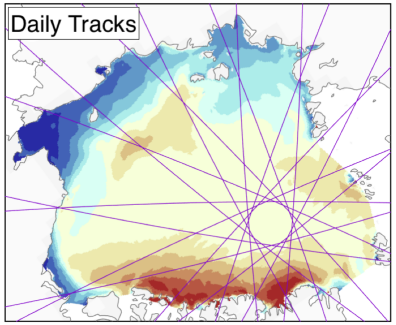
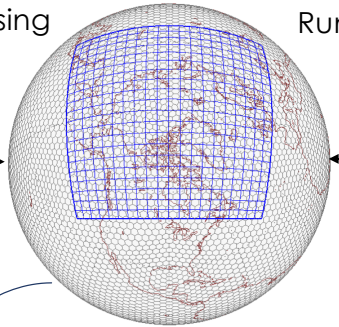
Satellite Emulator

'Flying' an altimetric satellite through the model



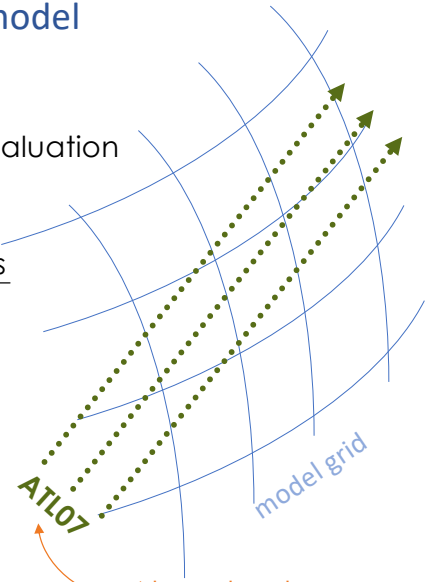
Preprocessing

grid mapping



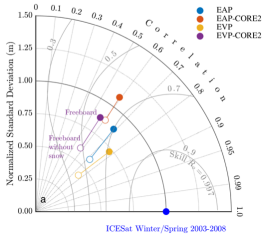
Runtime evaluation

statistics $g(h,\phi_R)$



Along-track freeboard from ICESat-2

End Result is a set of skill scores for different simulations obtained in run time.



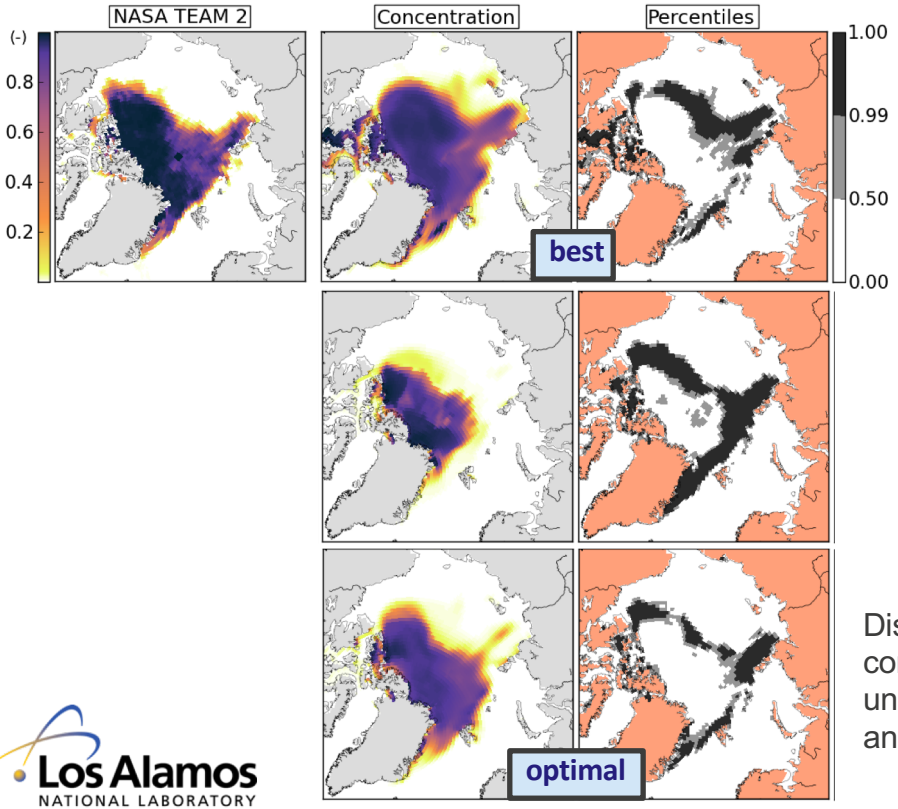
coming soon to CICE Consortium

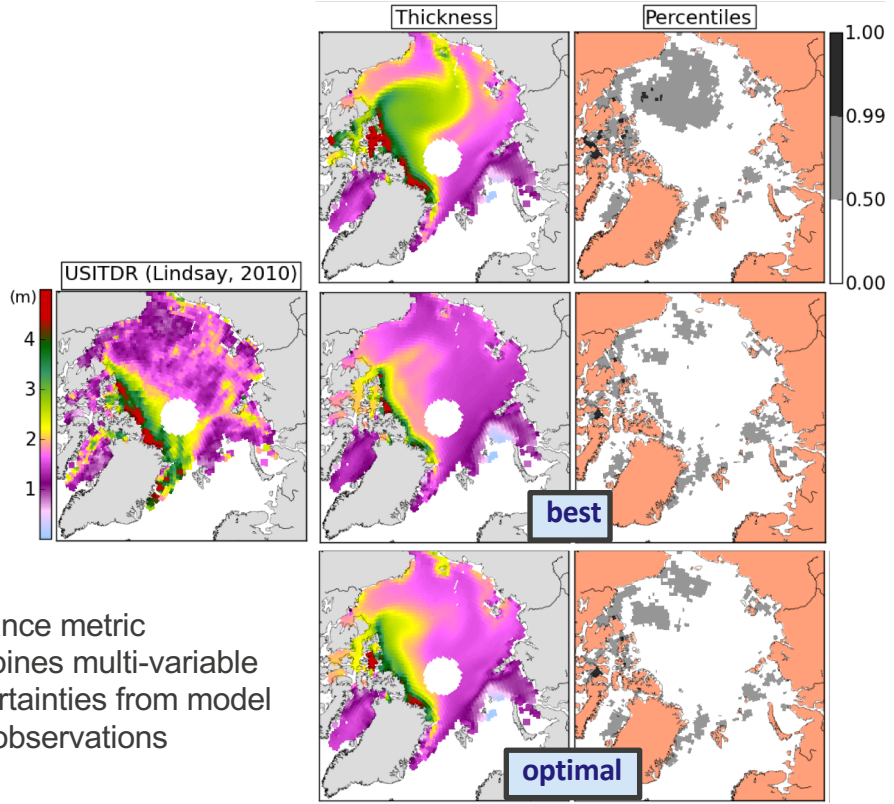
Courtesy A. Roberts

Probabilistic Validation of Sea Ice Models

September 2007 sea ice concentration

March 2007 sea ice thickness





Distance metric combines multi-variable uncertainties from model and observations

Los Alamos NATIONAL LABORATORY EST.1943

*Urrego-Blanco, Hunke, Urban, Jeffery, Turner, Langenbrunner, and Booker, JGR-Oceans, **122**, 2017.*

CICE Quality Control

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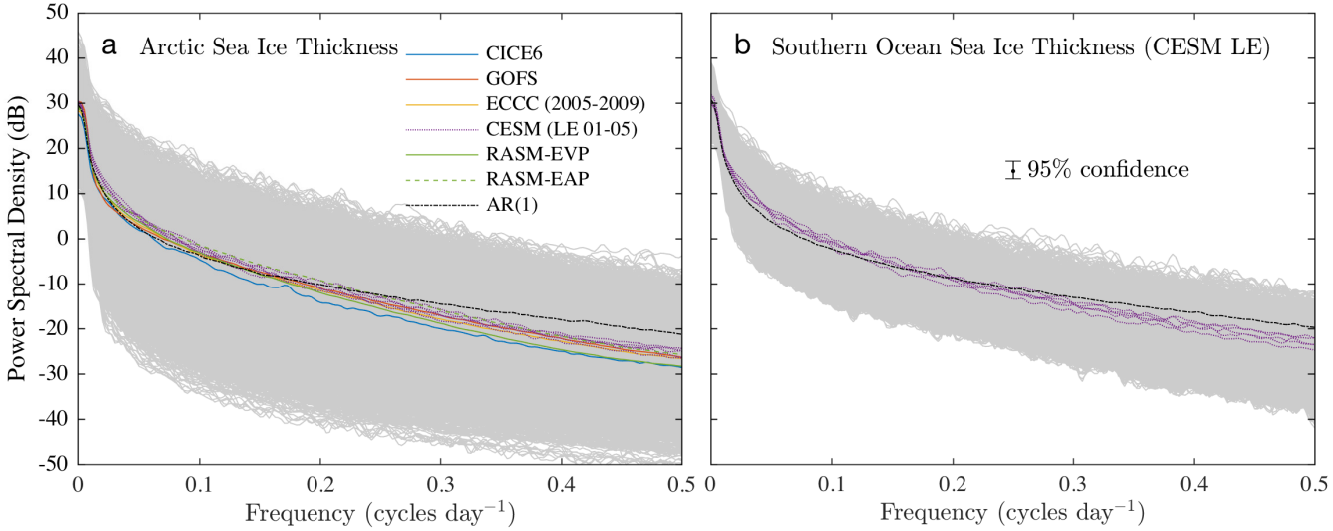
Categorize code modifications based on test simulations

1. Bit-For-Bit (BFB)
2. Non-BFB, but not climate changing
3. Non-BFB, climate changing, requires review and checking
4. A new model improvement or addition, requiring review and checking



Objective, statistical tests allow code modifications to be quickly evaluated for potential bugs etc. Automated, standardized testing enables community participation in sea ice model development. The metrics provide objective guidance for assessing new physics and code functionality.

Roberts, A. F., Hunke, E. C., Allard, R., Bailey, D. A., Craig, A. P., Lemieux, J., & Turner, M. D. (2018). Quality control for community-based sea-ice model development. *Philosophical Transactions of the Royal Society A*, *376*, 17. doi:10.1098/rsta.2017.0344

CICE version 6 code structure



github.com/CICE-Consortium/CICE

CICEcore (dynamical core)

- Momentum
- Stress
- Transport
- Grid/parallelization
- I/O
- Etc

github.com/CICE-Consortium/Icepack

Icepack (column physics)

- Ice thickness distribution
- Thermodynamics
- Ridging
- Biogeochemistry







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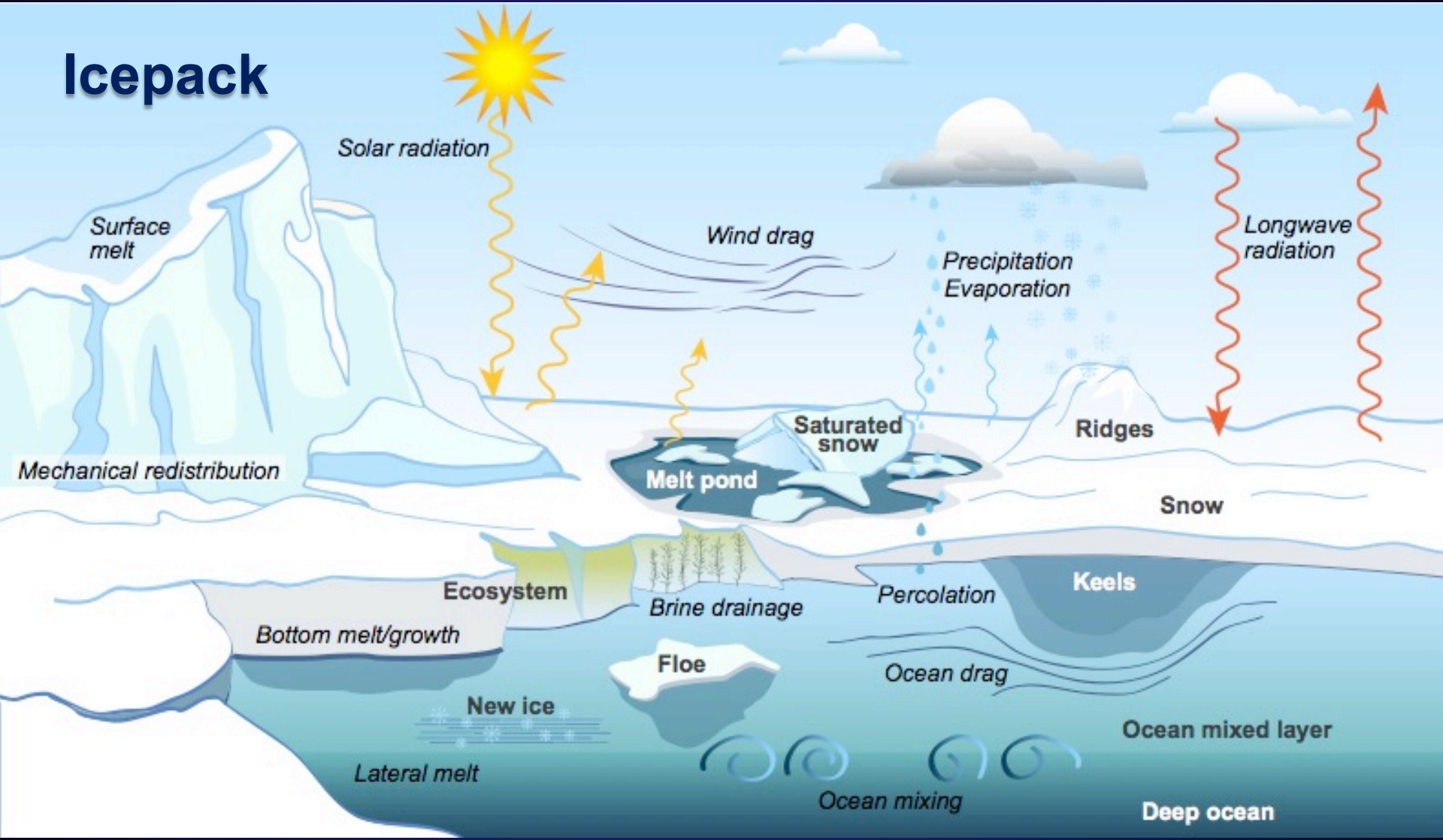
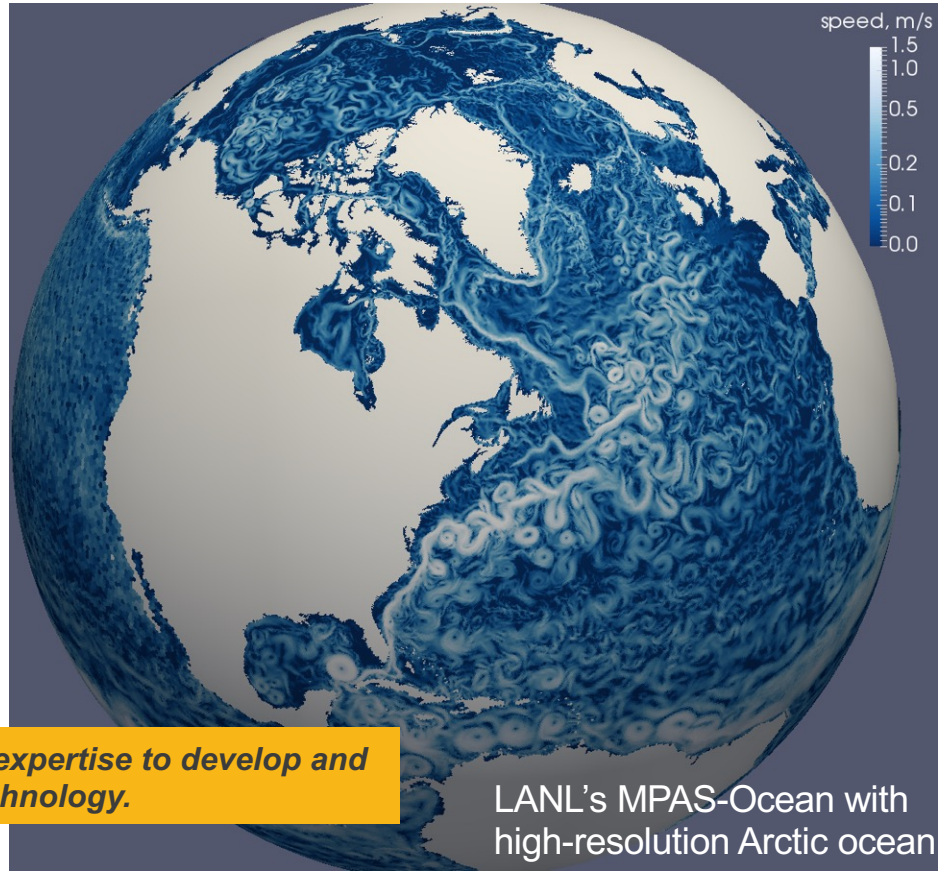
DMI
Danish Meteorological Institute

Icepack


Solar radiation
Surface melt
Wind drag
Precipitation
Evaporation
Longwave radiation
Saturated snow
Ridges
Mechanical redistribution
Melt pond
Snow
Ecosystem
Brine drainage
Percolation
Keels
Bottom melt/growth
Floe
Ocean drag
New ice
Ocean mixed layer
Lateral melt
Ocean mixing
Deep ocean

US Dept. of Energy invests in science and technology to understand Arctic system dynamics and change

- **E3SM: Energy Exascale Earth System Model**
  - Computationally advanced coupled model, that bridges weather to climate temporal projections, at spatial scales relevant to national security needs.
- **HiLAT-RASM:** High Latitude Application and Testing of coupled models
  - High resolution coupled (land, atmosphere, ocean, sea ice, ice shelf, ice sheet) Arctic/Antarctic system modeling
- **NGEE-Arctic:** Next Generation Ecosystem Experiment-Arctic
  - Improved representation of Arctic land processes and their interactions with the coupled Arctic system
- **ARM deployments:** Atmosphere Radiation Measurements
  - Arctic and Antarctic deployments
- **CICE Consortium**
  - Sea ice modeling for research and operational forecasting

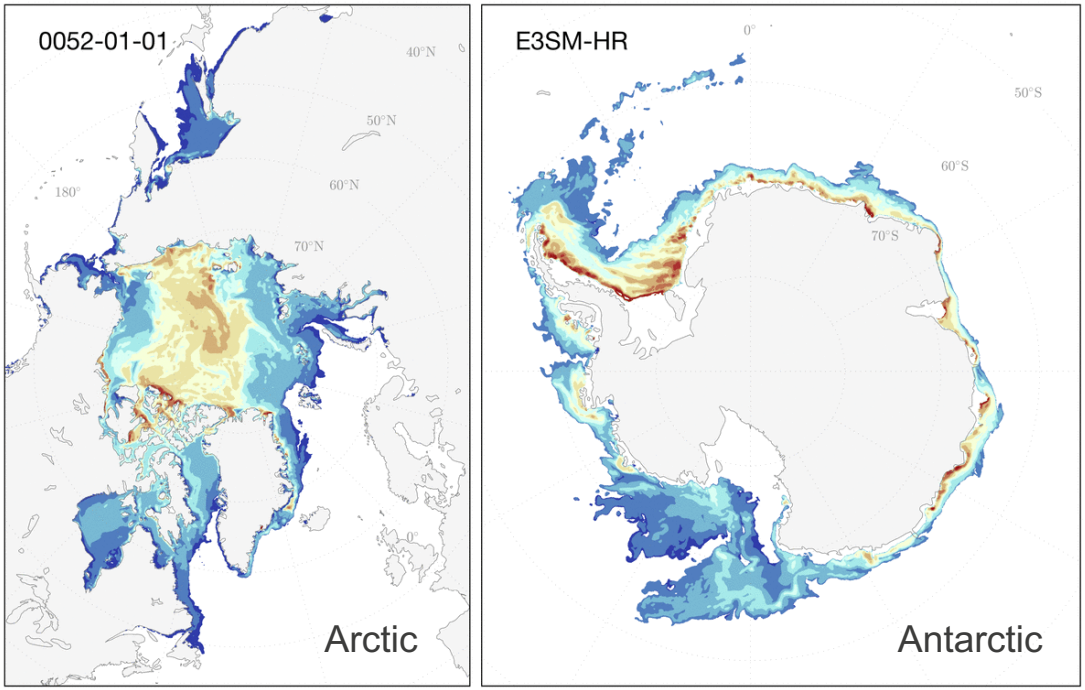


LANL's MPAS-Ocean with high-resolution Arctic ocean

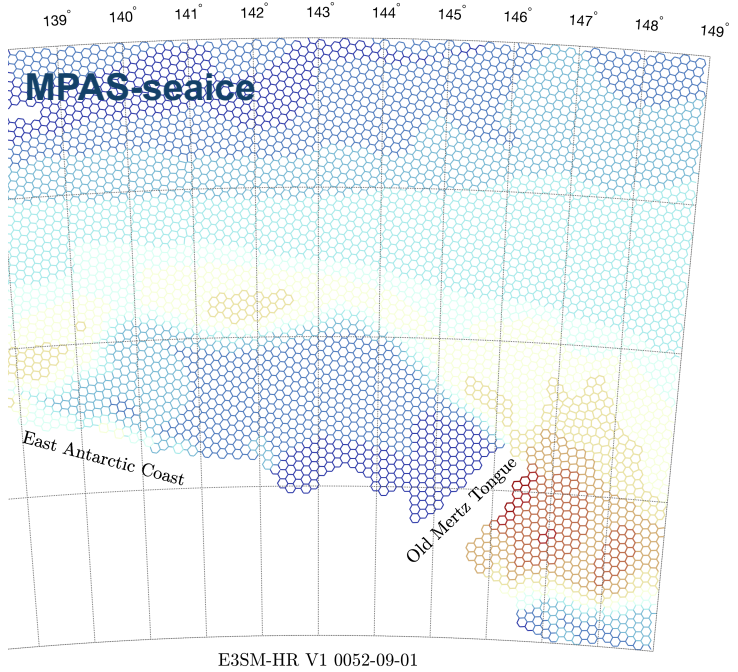
The DOE National Labs provide the facilities, workforce and expertise to develop and apply next generation climate observation and prediction technology.

Los Alamos Leads Cryosphere Modeling for DOE

Fully Coupled High-Resolution E3SM V1



Single annual cycle of sea ice thickness from a 50+ year fully coupled simulation using the HighResMip 1950 repeated year protocol



E3SM unstructured grid surrounding the Mertz Glacier Polynya in the East Antarctic, colored according to ice thickness.

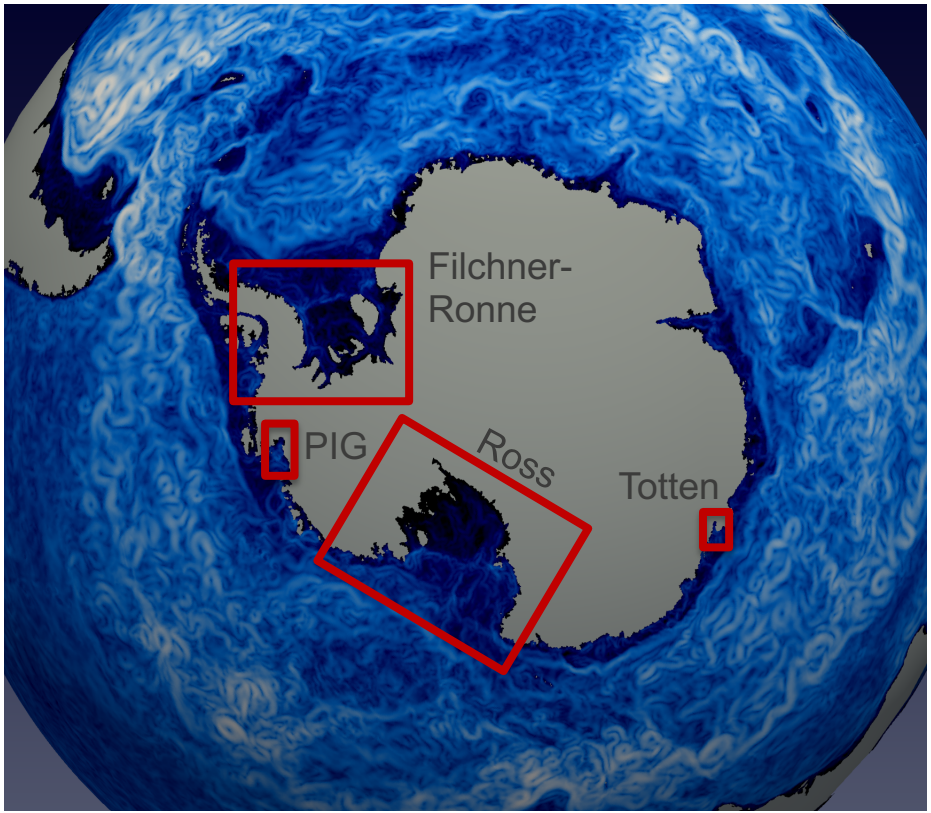
Resolution:

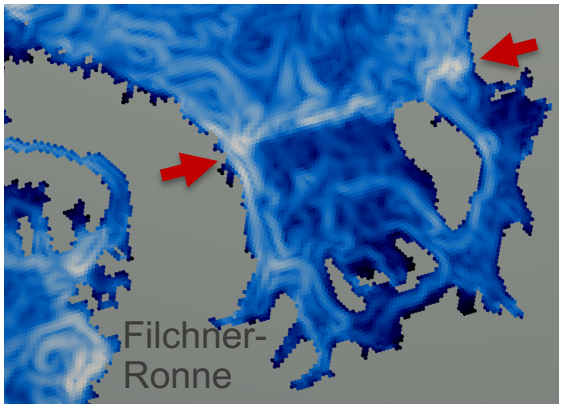
- atmosphere, land 25 km
- ocean, sea ice: 6 km near the poles expanding to 18 km near the equator

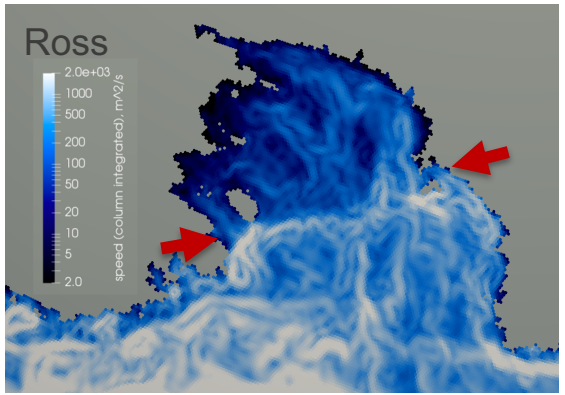
E3SM Energy Exascale Earth System Model

Key Contributors: Peter Caldwell, Luke VanRoekel, Azamat Mametjanov, Adrian Turner, Chris Golaz, Milena Veneziani, Wuyin Lin, Mat Maltrud, Jon Wolfe, Andrew Roberts

Ice shelf cavities in the fully coupled model







Column integrated ocean velocity

Red arrows indicate the edge of ice shelves

Key Contributors: *Mark Petersen, Xylar Asay-Davis, Darin Comeau, Matthew Hoffman, Jon Wolfe, Steve Price*

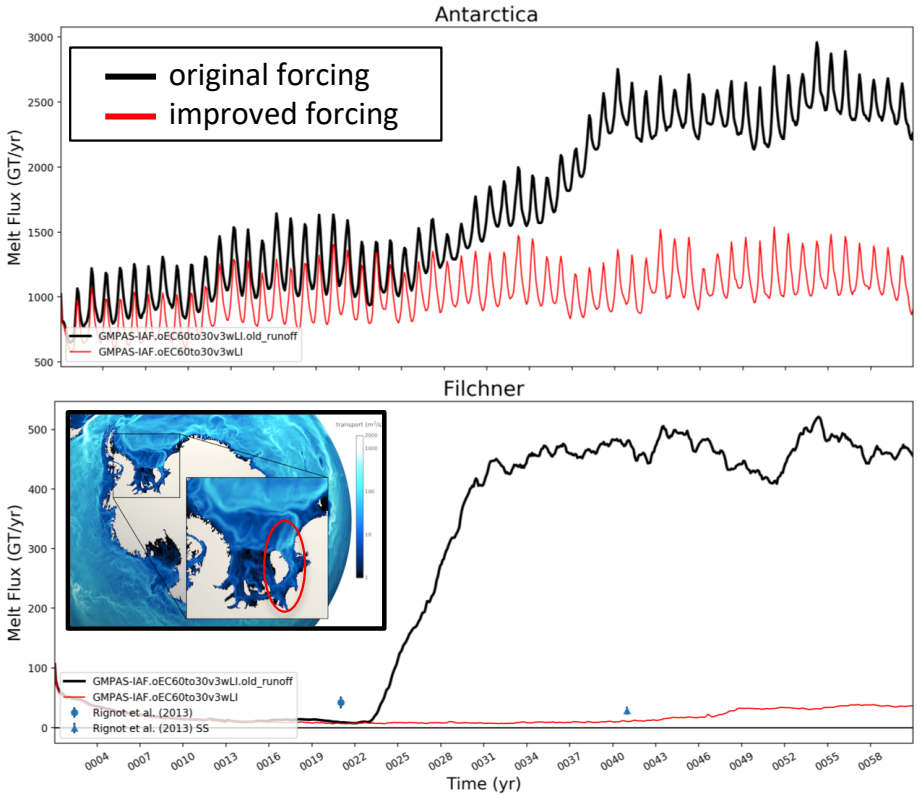
Improving Antarctic Freshwater Forcing with Data Icebergs

Problem in coupled simulations:
A "snow-capping" scheme defines solid ice runoff from land

- routes excess precip to nearest coastal grid cell
- causes biases in timing, magnitude, and location of fluxes
- causes double counting due to melting in active ice shelf cavities
- feedbacks lead to excessive ice-shelf melt rates

Solution:
model icebergs and/or their effects

Impacts on ice shelf melt rates



Key Contributors: D. Comeau, M. Hoffman, J. Wolfe



“Appropriate Complexity”

In practice:

- **the physical system and the model are both complex**
- **sea ice model uses are highly diverse**
- **sea ice and climate models balance complexity with computational expense to address certain questions**
- **new developments enable solutions to new questions**

We progress more quickly through collaboration

Thank You

Elizabeth Hunke
eclare@lanl.gov

An incomplete set of sea ice equations

$$\frac{\partial g}{\partial t} = -\nabla \cdot (g\mathbf{u}) - \frac{\partial}{\partial h}(fg) + \psi \tag{1}$$

$$\int_0^\infty g\, dh = 1 \tag{2}$$

$$\frac{\partial}{\partial t}(a_{in}) + \nabla \cdot (a_{in}\mathbf{u}) = 0 \tag{3}$$

$$\frac{\partial v_{in}}{\partial t} + \nabla \cdot (v_{in}\mathbf{u}) = 0 \tag{4}$$

$$\frac{\partial v_{sn}}{\partial t} + \nabla \cdot (v_{sn}\mathbf{u}) = 0 \tag{5}$$

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An incomplete set of sea ice equations

$$\frac{\partial e_{ink}}{\partial t} + \nabla \cdot (e_{ink}\mathbf{u}) = 0 \quad (6)$$

$$\frac{\partial e_{snk}}{\partial t} + \nabla \cdot (e_{snk}\mathbf{u}) = 0 \quad (7)$$

$$\frac{\partial (a_{in} T_n)}{\partial t} + \nabla \cdot (a_{in} T_n \mathbf{u}) = 0$$

$$\frac{\partial (v_{in} T_n)}{\partial t} + \nabla \cdot (v_{in} T_n \mathbf{u}) = 0$$

$$\frac{\partial (v_{sn} T_n)}{\partial t} + \nabla \cdot (v_{sn} T_n \mathbf{u}) = 0$$

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An incomplete set of sea ice equations

$$b(h) = \frac{\exp[-G(h)/a^*]}{a^*[1 - \exp(-1/a^*)]}$$

$$G(h) = \int_0^h g(h')dh'$$

$$a_{Pn} = \frac{\exp(-G_{n-1}/a^*) - \exp(-G_n/a^*)}{1 - \exp(-1/a^*)}$$

$$f_m^{\text{area}} = \exp[-(H_{m-1} - H_{\min})/\lambda] - \exp[-(H_m - H_{\min})/\lambda]$$

$$\begin{aligned} f_m^{\text{vol}} &= \{(H_{m-1} + \lambda)\exp[-(H_{m-1} - H_{\min})/\lambda] \\ &- (H_m + \lambda)\exp[-(H_m - H_{\min})/\lambda]\}/[H_{min} + \lambda] \end{aligned}$$

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$$R_{\text{net}} = \frac{C_s}{2}(\Delta - |D_D|) - \min(D_D, 0)$$

$$\begin{aligned} a_{lvl} + a_{rdg} &= a_i \\ v_{lvl} + v_{rdg} &= v_i \end{aligned}$$

$$P = C_f\, C_p\, \beta \sum_{n=1}^{N_C} \left[-a_{Pn}\, h_n^2 + \frac{a_{Pn}}{k_n} \left(H_{\text{min}}^2 + 2H_{\text{min}}\lambda + 2\lambda^2 \right) \right]$$

$$C_P = (g/2)(\rho_i/\rho_w)(\rho_w - \rho_i)$$

$$\beta = R_{\text{tot}}/R_{\text{net}} > 1$$

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$$m\frac{\partial \mathbf{u}}{\partial t} = \nabla \cdot \sigma + \vec{\tau}_a + \vec{\tau}_w - \hat{k} \times mf\mathbf{u} - mg\nabla H_\circ$$

$$\begin{aligned}
\frac{1}{E}\frac{\partial \sigma_1}{\partial t} + \frac{\sigma_1}{2\zeta} + \frac{P}{2\zeta} &= D_D = \dot{\epsilon}_{11} + \dot{\epsilon}_{22} \\
\frac{1}{E}\frac{\partial \sigma_2}{\partial t} + \frac{\sigma_2}{2\eta} &= D_T = \dot{\epsilon}_{11} - \dot{\epsilon}_{22}, \\
\frac{1}{E}\frac{\partial \sigma_{12}}{\partial t} + \frac{\sigma_{12}}{2\eta} &= \frac{1}{2}D_S = \dot{\epsilon}_{12},
\end{aligned}$$

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$$
\begin{aligned}
E &= \frac{\zeta}{T} \\
T &= E_\circ \Delta t \\
\dot{\epsilon}_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\
\zeta &= \frac{P}{2\Delta} \\
\eta &= \frac{P}{2\Delta e^2} \\
\Delta &= \left[D_D^2 + \frac{1}{e^2}\left(D_T^2 + D_S^2\right)\right]^{1/2}
\end{aligned}
$$

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$$\mathbf{A} = \int_{\mathbb{S}} \vartheta(\mathbf{r})\mathbf{r}\mathbf{r}d\mathbf{r}.$$

$$\vartheta(z) = \omega_1 \exp(-\omega_2 z^2)$$

$$\frac{D\mathbf{A}}{Dt} = \mathbf{F}_{iso}(\mathbf{A}) + \mathbf{F}_{frac}(\mathbf{A}, \sigma)$$

$$\begin{aligned} \frac{\partial A_{11}}{\partial t} &= -k_t \left(A_{11} - \frac{1}{2} \right) + M_{11} \\ \frac{\partial A_{12}}{\partial t} &= -k_t A_{12} + M_{12} \end{aligned}$$

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$$\boldsymbol{\sigma}^b(\mathbf{r}, h) = P_r(h)\boldsymbol{\sigma}_r^b(\mathbf{r}) + P_s(h)\boldsymbol{\sigma}_s^b(\mathbf{r})$$

$$\boldsymbol{\sigma}^{EAP}(h) = P_r(h) \int_{\mathbb{S}} \vartheta(\mathbf{r}) \left[\boldsymbol{\sigma}_r^b(\mathbf{r}) + k\boldsymbol{\sigma}_s^b(\mathbf{r})\right] d\mathbf{r}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} + \frac{\boldsymbol{\sigma}}{2T} = \frac{\boldsymbol{\sigma}^{EAP}}{2T}$$

$$\alpha_r(\theta) = \frac{\sigma_{ij}^r \dot{\epsilon}_{ij}}{P_r|\dot{\boldsymbol{\epsilon}}|} \qquad \alpha_s(\theta) = \frac{\sigma_{ij}^s \dot{\epsilon}_{ij}}{P_s|\dot{\boldsymbol{\epsilon}}|}$$

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$$C_d = \frac{NcS_c^2 \gamma L_y H}{2S_T} \left[\frac{\ln(H/z_0)}{\ln(z_{ref}/z_0)} \right]^2$$

$$S_c = (1 - \exp(-s_l D/H))^{1/2}$$

$$\texttt{Cdn\_skin} = a_i \left(1 - m_{(s/k)} \frac{H_{(s/k)}}{D_{(s/k)}} \right) c_{s(s/k)}, \text{ if } \frac{H_{(s/k)}}{D_{(s/k)}} \geq \frac{1}{m_{(s/k)}}$$

$$H_s = 2 \frac{v_{rdg}}{a_{rdg}} \left(\frac{\alpha \tan \alpha_k R_d + \beta \tan \alpha_s R_h}{\phi_r \tan \alpha_k R_d + \phi_k \tan \alpha_s R_h^2} \right)$$

$$D_s = 2H_s \frac{a_i}{a_{rdg}} \left(\frac{\alpha}{\tan \alpha_s} + \frac{\beta}{\tan \alpha_k} \frac{R_h}{R_d} \right)$$

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$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right) + w\frac{\partial q_{br}}{\partial z} + F$$

$$K = \phi K_{br} + (1-\phi)K_i$$

$$\phi = \frac{S}{S_{br}}$$

$$\begin{aligned} q = &\quad \phi q_{br} &&+ (1-\phi) q_i \\ = &\quad \phi \rho_w c_w T &&+ (1-\phi)(\rho_i c_i T - \rho_i L_0) \end{aligned}$$

$$\rho(S_{br}) = 1000.3 + 0.78237 S_{br} + 2.8008 \times 10^{-4} S_{br}^2$$

$$\Pi(\phi) = 3 \times 10^{-8}(\phi - \phi_\Pi)$$

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$$\frac{1}{S} = \frac{1}{1000} + \frac{1}{Z}$$

$$S_{br} = \frac{(T + J_1)}{(T/1000 + L_1)} l_0 + \frac{(T + J_2)}{(T/1000 + L_2)} (1 - l_0)$$

$$l_0 = \begin{cases} 1 & \text{if} \quad T \geq T_0 \\ 0 & \text{if} \quad T < T_0 \end{cases}$$

$$J_{1,2} = \frac{b_{1,2}}{a_{1,2}}$$

$$L_{1,2} = \frac{(1 + b_{1,2}/1000)}{a_{1,2}}$$

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$$\frac{\partial S}{\partial t} = w\frac{\partial S_{br}}{\partial z} + G$$

$$\left.\frac{\partial S}{\partial t}\right|_{rapid} = w(z)\frac{\partial S_{br}}{\partial z}$$

$$\left.\frac{\partial q}{\partial t}\right|_{rapid} = w(z)\frac{\partial q_{br}}{\partial z}$$

$$w_k = \max_{j=k,n}\left(\tilde{w}_j\right)$$

$$\tilde{w}(z) = w\left(\frac{Ra(z) - Ra_c}{Ra(z)}\right)$$

$$Ra(z) = \frac{g\Delta\rho\Pi(h-z)}{\kappa\eta}$$

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$$w(z)\Delta x^2 = A_m\left(-\frac{\Delta P}{l} + B_m\right)$$

$$\begin{aligned}
\frac{\Delta P}{l} &= \frac{A_p B_p + A_m B_m}{A_m + A_p} \\
A_m &= \frac{\Delta x^2}{\eta}\frac{n}{\sum_{k=1}^{n}\frac{1}{\Pi(k)}} \\
B_m &= -\frac{g}{n}\sum_{k=1}^{n}\rho(k) \\
A_p &= \frac{\pi a^4}{8\eta} \\
B_p &= -\rho_p g
\end{aligned}$$

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$$\left.\frac{\partial S(z)}{\partial t}\right|_{slow} = -\lambda(S(z) - S_c)$$

$$\lambda = S^* \max\left(\frac{T_{bot} - T_{sf}}{h}, 0\right)$$

$$S_c(z) = \phi_c S_{br}(z)$$

$$\left.\frac{\partial q}{\partial t}\right|_{flush} = w_f \frac{\partial q_{br}}{\partial z}$$

$$\left.\frac{\partial S}{\partial t}\right|_{flush} = w_f \frac{\partial S_{br}}{\partial z}$$

$$w_f = \frac{\overline{\Pi}\rho_w g \Delta h}{h\eta},$$

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$$(1-\phi_i)L_0\rho_i\frac{\partial h}{\partial t} = K\left.\frac{\partial T}{\partial z}\right|_i - K_w\left.\frac{\partial T}{\partial z}\right|_w$$

$$q\,\delta h = \begin{cases} (F_0 - F_{ct})\,\Delta t & \text{if } F_0 > F_{ct} \\ 0 & \text{otherwise} \end{cases}$$

$$q\,\delta h = (F_{cb} - F_{bot})\,\Delta t,$$

$$F_{cb} = \frac{K_{i,N+1}}{\Delta h_i}(T_{iN} - T_f).$$

$$(\rho L_v - q)\delta h = F_l \Delta t,$$

$$\rho_i h_i + \rho_s h_s = \rho_w h_i.$$

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$$F_0 = F_s + F_l + F_{L\downarrow} + F_{L\uparrow} + (1-\alpha)(1-i_0)F_{sw}$$

$$\begin{aligned}
F_s &= C_s\left(\Theta_a - T_{sf}^K\right) \\
F_l &= C_l\left(Q_a - Q_{sf}\right) \\
Q_{sf} &= \left(q_1/\rho_a\right)\exp\left(-q_2/T_{sf}^K\right) \\
F_{bot} &= -\rho_w c_w c_h u_* (T_w - T_f) \\
u_* &= \sqrt{\left|\vec{\tau}_w\right|/\rho_w} \\
f_{snow} &= \frac{h_s}{h_s + h_{snowpatch}}
\end{aligned}$$

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$$\Upsilon = \frac{\kappa g z_\circ}{u^{*2}} \left(\frac{\Theta^*}{\Theta_a \left(1 + 0.606 Q_a\right)} + \frac{Q^*}{1/0.606 + Q_a} \right)$$

$$\begin{aligned} u^* &= c_u \left| \vec{U}_a \right| \\ \Theta^* &= c_\theta \left(\Theta_a - T_{sfc} \right) \\ Q^* &= c_q \left(Q_a - Q_{sfc} \right) \end{aligned}$$

$$\chi = \left(1 - 16\Upsilon\right)^{0.25}$$

$$\psi_m = 2 \ln \left[0.5(1 + \chi)\right] + \ln \left[0.5(1 + \chi^2)\right] - 2 \tan^{-1} \chi + \frac{\pi}{2}$$

$$\psi_s = 2 \ln \left[0.5(1 + \chi^2)\right] \text{ or}$$

$$\psi_m = \psi_s = -\left[0.7\Upsilon + 0.75 \left(\Upsilon - 14.3\right) \exp\left(-0.35\Upsilon\right) + 10.7\right]$$

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$$
\begin{aligned}
c'_u &= \frac{c_u}{1 + c_u \left(\lambda - \psi_m\right) / \kappa} \\
c'_\theta &= \frac{c_\theta}{1 + c_\theta \left(\lambda - \psi_s\right) / \kappa} \\
c'_q &= c'_\theta \\
\lambda &= \ln\left(z_\circ / z_{ref}\right)
\end{aligned}
$$

$$
\begin{aligned}
C_l &= \rho_a \left(L_{vap} + L_{ice}\right) u^* c_q \\
C_s &= \rho_a c_p u^* c^*_\theta + 1 \\
\vec{\tau}_a &= \frac{\rho_a u^{*2} \vec{U}_a}{|\vec{U}_a|}
\end{aligned}
$$

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$$F_{lw\downarrow} = \epsilon\sigma T_a^4 (1 - 0.261 \exp\left(-7.77 \times 10^{-4} T_a^2\right)(1 + 0.275 f_{cld})$$

$$F_{sw\downarrow} = \frac{1353 \cos^2 Z}{10^{-3}(\cos Z + 2.7)e_a + 1.085 \cos Z + 0.1}\left(1 - 0.6 f_{cld}^3\right)$$

$$\vec{\tau}_w = c_w \rho_w \left|\vec{U}_w - \vec{u}\right| \left[\left(\vec{U}_w - \vec{u}\right)\cos\theta + \hat{k} \times \left(\vec{U}_w - \vec{u}\right)\sin\theta\right]$$

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$$\frac{\partial}{\partial t}(a_{lvl}a_i) + \nabla \cdot (a_{lvl}a_i\mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}(a_{pnd}a_{lvl}a_i) + \nabla \cdot (a_{pnd}a_{lvl}a_i\mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}(h_{pnd}a_{pnd}a_{lvl}a_i) + \nabla \cdot (h_{pnd}a_{pnd}a_{lvl}a_i\mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}(h_{ipnd}a_{pnd}a_{lvl}a_i) + \nabla \cdot (h_{ipnd}a_{pnd}a_{lvl}a_i\mathbf{u}) = 0$$

$$\rho_i L \frac{\partial h_i}{\partial t} = k_i \frac{\partial T_i}{\partial z} \sim k_i \frac{\Delta T}{h_i}$$

$$\Delta h_i = \begin{cases} \sqrt{\beta \Delta t}/2 & \text{if } h_i = 0 \\ \beta \Delta t / 2h_i & \text{if } h_i > 0 \end{cases}$$

$$\beta = \frac{2k_i \Delta T}{\rho_i L}$$

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$$\Delta h_i = -\min\left(\frac{\max(F_\circ, 0)\Delta t}{\rho_i L}, h_i\right)$$

$$\Delta V_{frz} = -\Delta h_i a_{pnd} a_{lvl} a_i \rho_i / \rho_0$$

$$a_{pnd}^{eff} = \left(1 - \min\left(h_{ps}/h_{s1}, 1\right)\right) a_{pnd} a_{lvl}$$

$$f_{frac} = \min\left(-\frac{\rho_i L \Delta h_i}{F_\circ \Delta t}, 1\right)$$

$$r_p = \frac{V_p}{V_p + V_s \rho_s / \rho_0} < 0.15$$

$$V_{mx} = h_{mx} a_p = \left(\frac{\rho_0 - \rho_s}{\rho_0}\right) h_s a_p$$

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$$h_p^{eff} = \left(\frac{\rho_0}{\rho_0 - \rho_s}\right) h_{pnd}$$

$$h_s^{eff} = h_s(1 - a_p) + (h_s - h_p^{eff})a_p$$

$$h_p^{eff} = \frac{\rho_0 h_{pnd} + \rho_s h_s}{\rho_0}$$

$$\rho_i h_i + \rho_s h_s + \rho_0 h_p = \rho_w d \leq \rho_w h_i$$

$$\Delta V_{perm} = -a_i d_p \min\left(h_{pnd}, \frac{pP\Delta t}{\mu h_i}\right)$$

$$a'_{pnd}(a_{lvl} + \Delta a_{lvl})(a_i + \Delta a_i) = a_{pnd} a_{lvl} a_i$$

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$$\frac{\partial}{\partial t}(h_{pnd}a_{pnd}a_i) + \nabla \cdot (h_{pnd}a_{pnd}a_i\mathbf{u}) = 0$$

$$\sum_n a_{pnd}^{t+\Delta t}(n)a_i^{t+\Delta t}(n) = \sum_n a_{pnd}^{t}(n)a_i^{t}(n)$$

$$a_{pnd}^{t+\Delta t}(m) = \frac{\sum_n a_{pnd}^{t}(n)a_i^{t}(n) - \sum_{n\neq m} a_{pnd}^{t+\Delta t}(n)a_i^{t+\Delta t}(n)}{a_i^{t+\Delta t}(m)}$$

$$= \frac{a_{pnd}^{t}(m)a_i^{t}(m) + \sum_{n\neq m} \Delta\left(a_{pnd}a_i\right)^{t+\Delta t}}{a_i^{t+\Delta t}(m)}$$

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$$\frac{dT_b}{dt} = w_b \frac{\Delta T_b}{\Delta z} + R_b(T_j : j = 1, \ldots, N_b)$$

$$w_b = -p_g \left| m_1 + m_2 \frac{dh}{dt} - m_3 \left(\frac{dh}{dt} \right)^2 \right|$$

$$w_b = p_m \left| m_2 \frac{dh}{dt} - m_3 \left(\frac{dh}{dt} \right)^2 \right|$$

$$\Delta T_b = T_b / h_{sk} - T_{io}$$

$$w_b \frac{\Delta N_a}{\Delta z} = \frac{N_o h_{sk} / \phi_{sk} - N_a}{\Delta t}$$

$$w_b \frac{\Delta T_b}{\Delta z} = \begin{cases} T_b |dh_i/dt| / h_{sk} & \text{if } |dh_i/dt| \Delta t / h_{sk} < 1 \\ T_b / \Delta t & \text{otherwise} \end{cases}$$

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$$h_b' = h_b(t) + \Delta h_b|_{h_i, h_s}$$

$$w = -\frac{\Pi^* \bar{\rho} g}{\mu} \frac{h_p}{h_i}$$

$$\Pi^* = \left(\frac{1}{h} \sum_{i=1}^{N} \frac{\Delta z_i}{\Pi_i} \right)^{-1}$$

$$h_{sl} = \frac{\bar{\rho}}{\rho_w} h_i + \frac{\rho_s}{\rho_w} h_s$$

$$\frac{\partial h_b}{\partial t} = -w h_p$$

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$$(\Delta h_b)_{top} = \frac{\rho_i}{\rho_o} \cdot \begin{cases} -(\Delta h_i)_{top} & \text{if } |(\Delta h_i)_{top}| < h_i - h_b \\ h_i - h_b & \text{otherwise} \end{cases}$$

$$\Delta h_b|_{h_i,h_s} = (\Delta h_b)_{top} - (\Delta h_i)_{bot} - \frac{\rho_s}{\rho_o}\Delta h_s$$

$$h_b(t + \Delta t) = h_b(t) + \Delta h_i$$

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