Burgers equation in the complex plane

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What this talk contains

Interesting instances of the appearance of Burgers equation in the complex plane in some stochastic models.

1. The semicircle law and Burgers equation.

2. (Real) Burgers turbulence and (complex) Burgers equation.

3. Harish Chandra's integral (HCIZ integral).

Another basic example: the Marchenko-Pastur law and method, not Burgers, closely related.

No new results -- mainly an advertisement of intriguing facts!

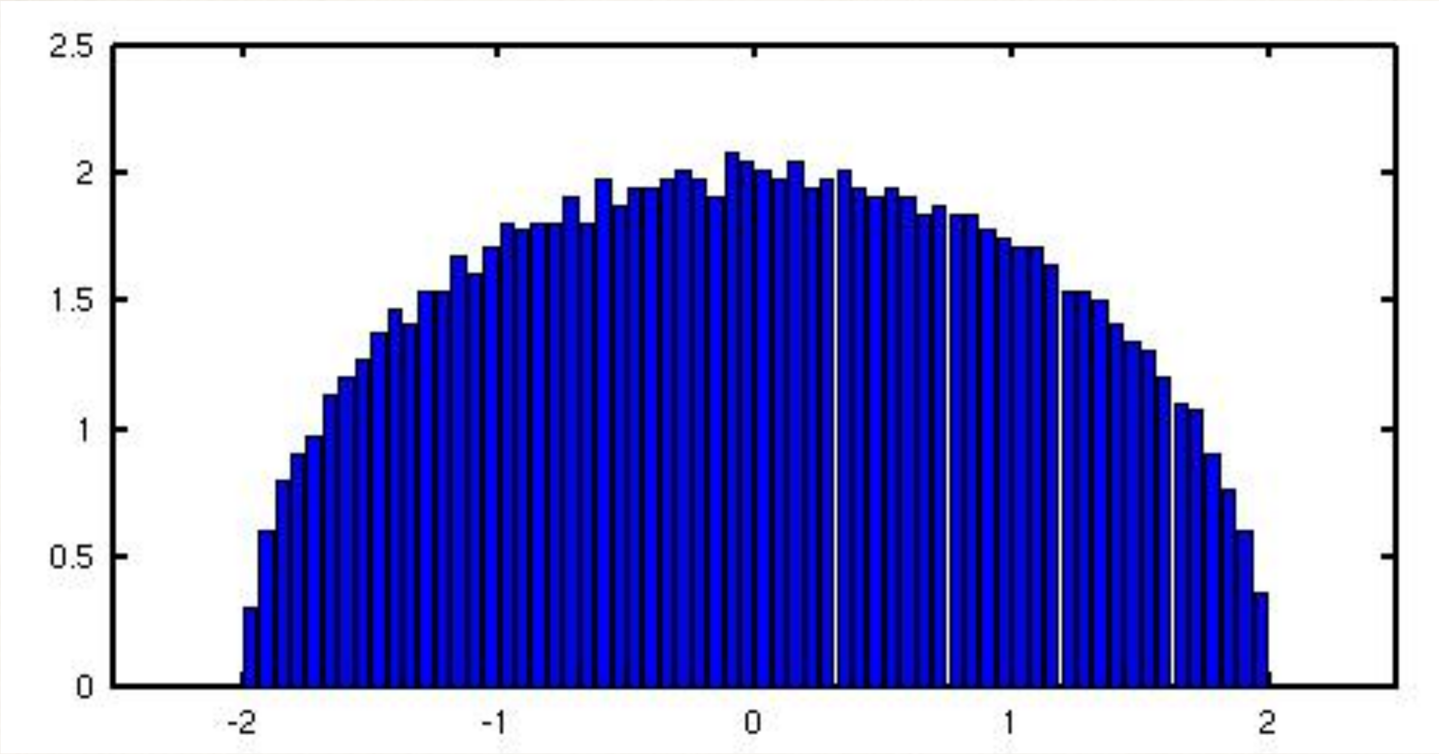
1. Burgers equation and the semicircle law

The semicircle law

A fundamental model in random matrix theory: GUE

$$p_{\mathrm{GUE}}(M) = \frac{1}{Z_N} e^{-\frac{N}{2}\mathrm{Tr}(M^2)}$$

Basic question: what is the law of the eigenvalues? What happens for large N?



$$f_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbf{1}_{|x|\le 2}.$$

Burgers equation with singular data

What is "the" solution to Burgers equation

$$g_t + g g_x = 0, \quad -\infty < x < \infty, t > 0,$$

with the following singular initial condition

$$g(x,0) = \frac{1}{x} \quad ?$$

Burgers equation with singular data

Answer:

$$g(x,t) = \frac{1}{\sqrt{t}}\, g_*\left(\frac{x}{\sqrt{t}}\right),$$

$$g_*(z) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\lambda - z}\, f_{\mathrm{sc}}(\lambda)\, d\lambda, \qquad z \in \mathbb{C}_+.$$

The value on the real axis is obtained by taking the principal value.

The basic idea

Consider Brownian motion in the space of Hermitian matrices, $M_N(t)$ and

$$g_N(z,t) = \frac{1}{N}\mathrm{tr}\left(M_N(t) - z\right)^{-1} = \frac{1}{N}\sum_{j=1}^{N} \frac{1}{\lambda_j(t) - z}.$$

Then the eigenvalues satisfy an SDE (Dyson's Brownian motion)

$$d\lambda_k = \sum_{j\neq k} \frac{dt}{\lambda_k - \lambda_j} + dB_k, \quad 1 \leq k \leq N.$$

Calculation: apply Ito's formula to $\log\det\left(M_N(t) - z\right), \quad z \in \mathbb{C}_+$

then let $z \longrightarrow \lambda_k$.

Herglotz (Pick) functions

Normalized analytic functions in the upper-half plane with positive imaginary part are Cauchy transforms of positive measures.

$$g(z,t) = \int_{\mathbb{R}} \frac{1}{x-z}\mu_t(dx)$$

Assume that the initial data for complex Burgers equation is of the above form, with g(x,0) given by the principal value. Then Burgers equation is equivalent to the kinetic equation (written in weak form):

$$\frac{d}{dt}\int_{\mathbb{R}} \varphi(s)\,\mu_t(ds) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\varphi'(y)-\varphi'(x)}{y-x}\,\mu_t(dx)\mu_t(dy).$$

Ref: S.V. Kerov, Asymptotic representation theory of the symmetric group, 2003.

2. (Real) Burgers turbulence and (Complex) Burgers equation.

"Closure in Burgers turbulence": Bertoin's theorem

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0, \quad x \in \mathbb{R}, t > 0.$$

Theorem 1. (Bertoin, CMP, 1996). Assume the initial data is a Levy process (in x) that may include a drift and Brownian motion, but only downward jumps.

Then for each t>0, the entropy solution remains a Levy process with downward jumps.

Remark 1. This theorem should be viewed as an invariant manifold theorem in the space of probability measures on the line.

Simplest case: Burgers with monotone, compound Poisson initial data

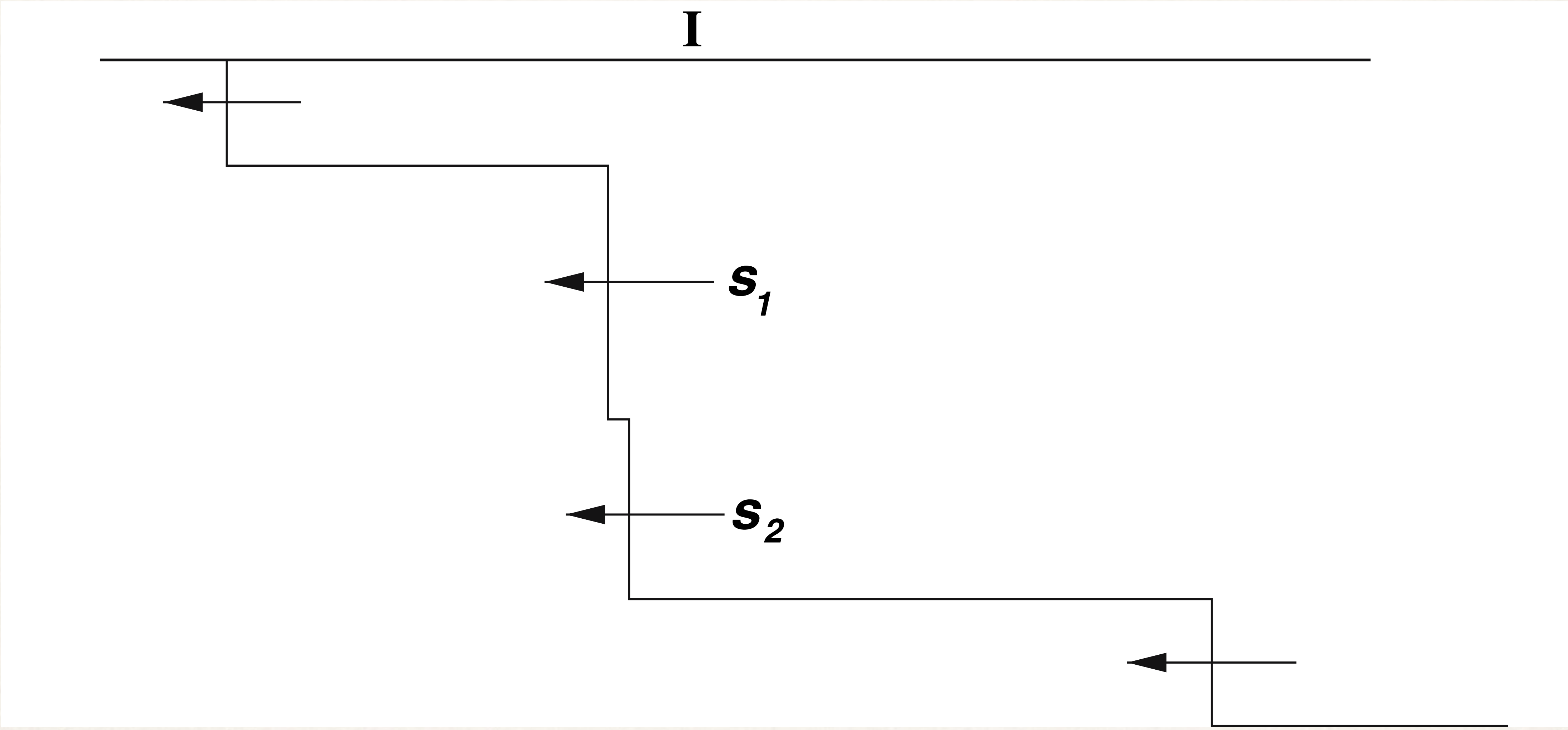
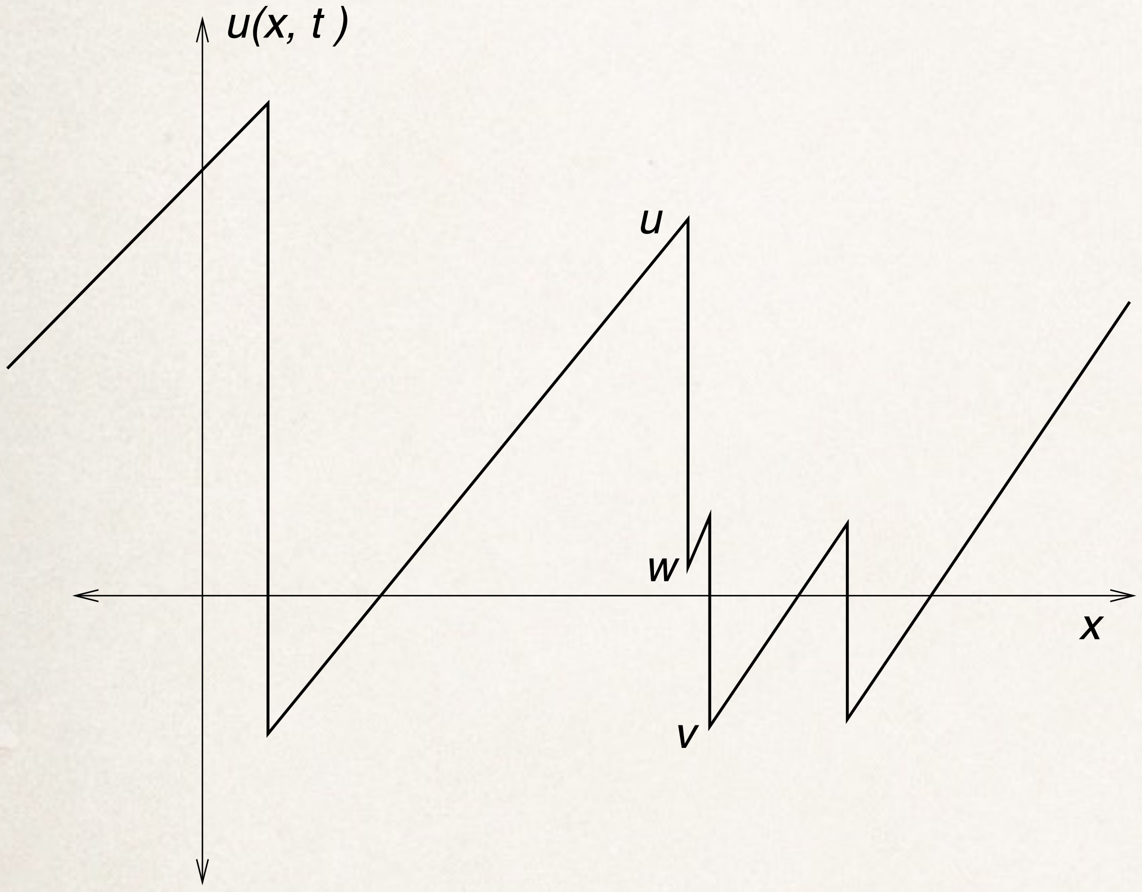
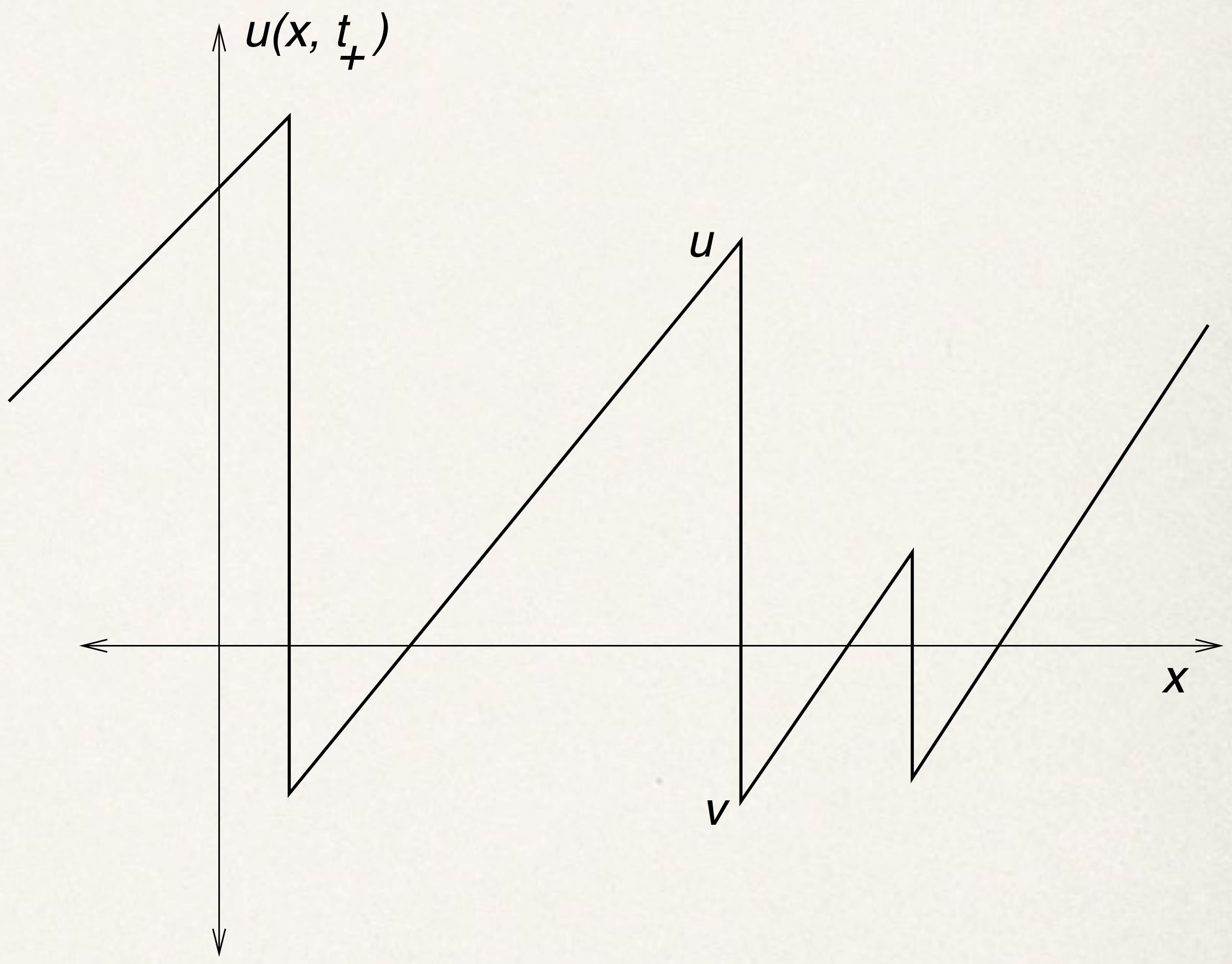


Fig. 1. Binary clustering of shocks

More generally, can include a deterministic drift in the Levy process





The Laplace exponent: definition.

In general, in order to understand the statistics of a process, we must understand the joint distribution at n-points. For Levy processes, all of this information collapses into one function -- the Laplace exponent.

This is simplest to explain under the assumption that all jumps are downward.

$$\mathbb{E}\left(e^{qu(x)}\right) = e^{-x\psi(q)}, \quad q \in \mathbb{C}_+, \quad x \geq 0.$$

$$\psi(q) = \int_0^\infty \left(e^{-qs} - 1 + qs\right) \Lambda(ds).$$

The Laplace exponent: examples.

(a) Poisson process of rate λ in space, and jumps of unit size downwards.

$$\mathbb{E}\left(e^{qu(x)}\right) = e^{\lambda x(1-e^{-q})}.$$

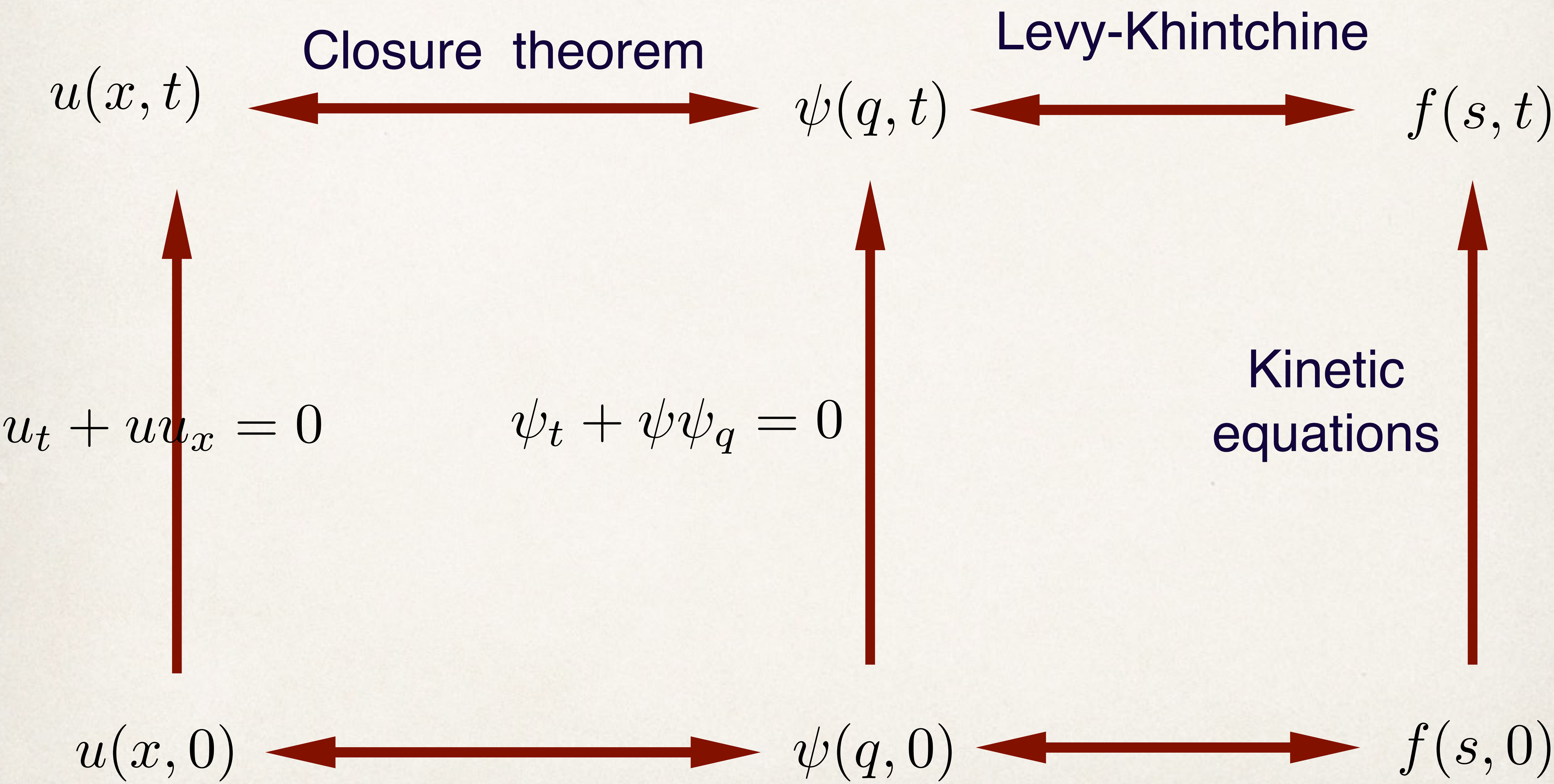
(b) Compound Poisson process with independent downward jumps with pdf f.

$$\mathbb{E}\left(e^{qu(x)}\right) = e^{\lambda x \int_0^\infty (1-e^{-qu}) f(u) du}.$$

(c) Brownian motion.

$$\mathbb{E}\left(e^{qu(x)}\right) = e^{xq^2}.$$

Evolution of Levy processes with downward jumps



Example: solutions to Burgers equation with Brownian motion initial data

If u(x,0) is Brownian motion, then $\psi(q,0) = q^2.$

$$\psi(q,t) = \frac{1}{t^2}\psi_*(qt), \quad \psi_*(q) = q + \frac{1}{2} - \sqrt{q + \frac{1}{4}}.$$

The Laplace exponent can be inverted to obtain the jump (shock) statistics

$$f_*(s) = \frac{1}{\sqrt{2\pi s^3}}\, e^{-s}.$$

Convergence to this self-similar solution requires study of characteristics in right-half-plane, not just real half line.

(M,Pego, SIMA (2005), Srinivasan, SIMA (2011).

Mapping Burgers turbulence to the semicircle law

$$g_*(z) = \frac{\psi_*(q)}{q}, \quad z = 2 + \frac{1}{q}.$$

The Laplace exponent can be inverted to obtain the jump (shock) statistics

$$\frac{g(z/\sqrt{t}, t)}{\sqrt{t}} = \frac{\psi(q/t, t)}{q/t}.$$

I have no "stochastic process" explanation for this transformation !

3. Harish-Chandra's integral

Harish-Chandra's integral (Itzykson-Zuber integral)

$$A = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_N)\,, \quad B = \mathrm{diag}\,(\beta_1, \ldots, \beta_N)\,.$$

$$\int_{U(N)} e^{t\,\mathrm{tr}(AUBU^*)}\, dU = c_N \frac{\det\left(e^{t\alpha_i\beta_j}\right)_{1\le i,j\le N}}{t^{N(N-1)/2}\, \triangle(A)\triangle(B)}.$$

$$\triangle(A) = \prod_{j<k}(\alpha_k - \alpha_j), \quad \triangle(B) = \prod_{j<k}(\beta_k - \beta_j).$$

HCIZ integral and the heat equation

The fundamental solution to the heat equation in the space of Hermitian matrices is

$$K(t; M_A, M_B) = \frac{1}{(2\pi t)^{N^2/2}} e^{-\frac{1}{2t}\operatorname{tr}(M_A - M_B)^2}.$$

$$M_A = UAU^*, \quad M_B = VBV^*$$

$$\operatorname{tr}(M_A M_B) = \operatorname{tr} UAU^*VBV^* = \operatorname{tr} AU^*VB(U^*V)^*$$

Cannot simultaneously diagonalize both matrices.

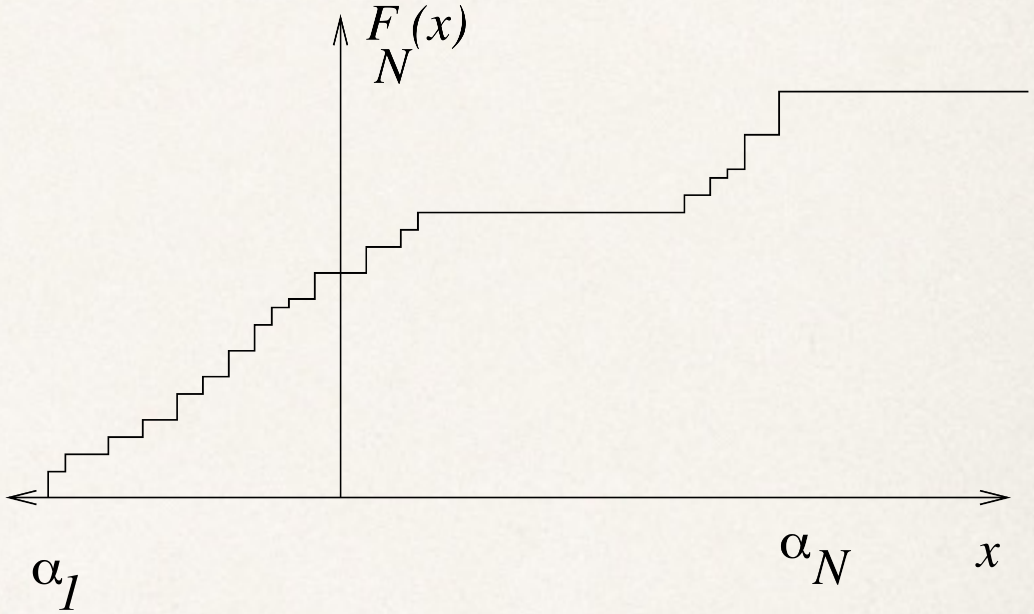
Tao's blog: search "HCIZ integral".

The large N limit.

Assume the spectral measures converge to limiting probability measures:

$$\lim_{N\to\infty} \frac{\{\#\alpha_j \in (a,b)\}}{N} = \int_a^b \alpha(dx),$$

$$\lim_{N\to\infty} \frac{\{\#\beta_j \in (a,b)\}}{N} = \int_a^b \beta(dx),$$



Main question: what are the asymptotics of the HCIZ integral for large N?

First-order asymptotics (Matytsin, 1997; Guionnet, 2006).

$$\lim_{N\to\infty} \frac{1}{N^2} \log Z_N(A_N, B_N) = S(\alpha, \beta) + \ldots,$$

... terms are omitted for clarity. The main term S is the action integral for

$$f_t + f f_x = 0, \quad x \in \mathbb{R}, \quad t \in (0,1),$$

$$f(x,t) = v(x,t) + i\rho(x,t), \quad v = \mathcal{H}\rho$$

with boundary conditions:

$$\rho\big|_{t=0} = \alpha, \quad \rho\big|_{t=1} = \beta.$$

Elliptic or hyperbolic ?

The linear equation $$u_t + c u_x = 0$$

is hyperbolic (i.e. a transport equation), when c is real.

However, if $c = i = \sqrt{-1}$ this is the Cauchy-Riemann equation.

In general, if c has non-zero imaginary part, the equation is elliptic, not hyperbolic.

Thus, complex Burgers changes type when the density vanishes.

Equations in real variables

$$f(x,t) = v(x,t) + i\rho(x,t), \quad v = \mathcal{H}\rho$$

is equivalent to:

$$\partial_t \rho + \partial_x (\rho v) = 0,$$

$$\partial_t (\rho v) + \partial_x \left(\rho v^2\right) = -\partial_x p,$$

$$p = -\frac{\pi^2}{3}\rho^3.$$

Equation switches type when density vanishes.

Mass transport scenarios