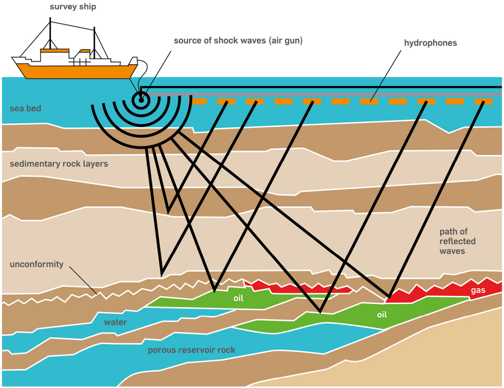
Nonlinear acoustic imaging via reduced order model backprojection

Alexander V. Mamonov[1],
Vladimir Druskin[2], Andrew Thaler[3] and Mikhail Zaslavsky[2]

[1]University of Houston,
[2]Schlumberger-Doll Research Center,
[3]The Mathworks, Inc.

Motivation: seismic oil and gas exploration



- **Seismic exploration**
- Seismic waves in the subsurface induced by sources (shots)
- Measurements of seismic signals on the surface or in a well bore
- Determine the acoustic or elastic parameters of the subsurface

Forward model: acoustic wave equation

- Consider an acoustic wave equation in the **time domain**

$$u_{tt} = \mathbf{A}u \quad \text{in } \Omega, \quad t \in [0, T]$$

with initial conditions

$$u\big|_{t=0} = u_0, \quad u_t\big|_{t=0} = 0$$

- The spatial operator $\mathbf{A} \in \mathbb{R}^{N \times N}$ is a fine grid discretization of

$$A(c) = c^2 \Delta$$

with the appropriate boundary conditions

- The solution is

$$u(t) = \cos(t\sqrt{-\mathbf{A}})u_0$$

Source model

- We stack all p sources in a single tall skinny matrix $\mathbf{S} \in \mathbb{R}^{N \times p}$ and introduce them in the initial condition

$$\mathbf{u}\big|_{t=0} = \mathbf{S}, \quad \mathbf{u}_t\big|_{t=0} = 0$$

- The solution matrix $\mathbf{u}(t) \in \mathbb{R}^{N \times p}$ is

$$\mathbf{u}(t) = \cos(t\sqrt{-\mathbf{A}})\mathbf{S}$$

- We assume the form of the source matrix

$$\mathbf{S} = q^2(\mathbf{A})\mathbf{C}\mathbf{E},$$

  where p columns of $\mathbf{E}$ are point sources on the surface, $q^2(\omega)$ is the Fourier transform of the source wavelet and $\mathbf{C} = \mathrm{diag}(c)$
- Here we take $q^2(\omega) = e^{\sigma\omega}$ with small σ so that $\mathbf{S}$ is localized near $\mathbf{E}$

Receiver and data model

- For simplicity assume that the sources and receivers are collocated
- Then the receiver matrix $\mathbf{R} \in \mathbb{R}^{N \times p}$ is

$$\mathbf{R} = \mathbf{C}^{-1}\mathbf{E}$$

- Combining the source and receiver we get the **data model**

$$\mathbf{F}(t; c) = \mathbf{R}^T \cos(t\sqrt{-\mathbf{A}(c)})\mathbf{S},$$

  a $p \times p$ matrix function of time
- The data model can be fully symmetrized

$$\mathbf{F}(t) = \mathbf{B}^T \cos\left(t\sqrt{-\widehat{\mathbf{A}}}\right)\mathbf{B},$$

  with $\widehat{\mathbf{A}} = \mathbf{C}\boldsymbol{\Delta}\mathbf{C}$ and $\mathbf{B} = q(\widehat{\mathbf{A}})\mathbf{E}$

Seismic inversion and imaging

1. **Seismic inversion**: determine c from the knowledge of measured data $\mathbf{F}^\star(t)$ (full waveform inversion, FWI); highly nonlinear since $\mathbf{F}(\,\cdot\,; c)$ is nonlinear in c
   - Conventional approach: non-linear least squares (output least squares, OLS)
   $$\underset{c}{\text{minimize}}\, \|\mathbf{F}^\star - \mathbf{F}(\,\cdot\,; c)\|_2^2$$
   - Abundant local minima
   - Slow convergence
   - Low frequency data needed
2. **Seismic imaging**: estimate c or its discontinuities given $\mathbf{F}(t)$ and also a smooth kinematic model c_0
   - Conventional approach: linear migration (Kirchhoff, reverse time migration - RTM)
   - Major difficulty: multiple reflections

Reduced order models

- The data is always discretely sampled, say uniformly at $t_k = k\tau$
- The choice of τ is very important, optimally we want τ around Nyquist rate
- The discrete data samples are

$$\mathbf{F}_k = \mathbf{F}(k\tau) = \mathbf{B}^T \cos\left(k\tau\sqrt{-\widehat{\mathbf{A}}}\right)\mathbf{B} =$$

$$= \mathbf{B}^T \cos\left(k \arccos\left(\cos\tau\sqrt{-\widehat{\mathbf{A}}}\right)\right)\mathbf{B} = \mathbf{B}^T T_k(\mathbf{P})\mathbf{B},$$

where T_k is Chebyshev polynomial and the **propagator** is

$$\mathbf{P} = \cos\left(\tau\sqrt{-\widehat{\mathbf{A}}}\right)$$

- We want a **reduced order model** (ROM) $\widetilde{\mathbf{P}}$, $\widetilde{\mathbf{B}}$ that fits the measured data

$$\mathbf{F}_k = \mathbf{B}^T T_k(\mathbf{P})\mathbf{B} = \widetilde{\mathbf{B}}^T T_k(\widetilde{\mathbf{P}})\widetilde{\mathbf{B}}, \quad k = 0, 1, \ldots, 2n-1$$

Projection ROMs

- Projection ROMs are obtained from

$$\widetilde{\mathbf{P}} = \mathbf{V}^T \mathbf{P} \mathbf{V}, \quad \widetilde{\mathbf{B}} = \mathbf{V}^T \mathbf{B},$$

  where **V** is an orthonormal basis for some subspace
- How do we get a ROM that fits the data?
- Consider a matrix of **solution snapshots**

$$\mathbf{U} = [\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}] \in \mathbb{R}^{N \times np}, \quad \mathbf{u}_k = T_k(\mathbf{P})\mathbf{B}$$

Theorem (ROM data interpolation)

If span$(\mathbf{V}) =$ *span*$(\mathbf{U})$ *and* $\mathbf{V}^T\mathbf{V} = \mathbf{I}$ *then*

$$\mathbf{F}_k = \mathbf{B}^T T_k(\mathbf{P})\mathbf{B} = \widetilde{\mathbf{B}}^T T_k(\widetilde{\mathbf{P}})\widetilde{\mathbf{B}}, \quad k = 0, 1, \ldots, 2n-1,$$

where $\widetilde{\mathbf{P}} = \mathbf{V}^T\mathbf{P}\mathbf{V} \in \mathbb{R}^{np \times np}$ *and* $\widetilde{\mathbf{B}} = \mathbf{V}^T\mathbf{B} \in \mathbb{R}^{np \times p}$.

ROM from measured data

- We do not know the solutions in the whole domain **U** and thus **V** is unknown
- How do we obtain the ROM from just the data $\mathbf{F}_k$?
- The data does not give us **U**, but it gives us the **inner products**!
- A basic property of Chebyshev polynomials is

$$T_i(x)T_j(x) = \frac{1}{2}(T_{i+j}(x) + T_{|i-j|}(x))$$

- Then we can obtain

$$(\mathbf{U}^T\mathbf{U})_{i,j} = \mathbf{u}_i^T\mathbf{u}_j = \frac{1}{2}(\mathbf{F}_{i+j} + \mathbf{F}_{i-j}),$$

$$(\mathbf{U}^T\mathbf{P}\mathbf{U})_{i,j} = \mathbf{u}_i^T\mathbf{P}\mathbf{u}_j = \frac{1}{4}(\mathbf{F}_{j+i+1} + \mathbf{F}_{j-i+1} + \mathbf{F}_{j+i-1} + \mathbf{F}_{j-i-1})$$

ROM from measured data

- Suppose $\mathbf{U}$ is orthogonalized by a **block QR** (Gram-Schmidt) procedure
$$\mathbf{U} = \mathbf{V}\mathbf{L}^T, \text{ equivalently } \mathbf{V} = \mathbf{U}\mathbf{L}^{-T},$$
where $\mathbf{L}$ is a **block Cholesky** factor of the Gramian $\mathbf{U}^T\mathbf{U}$ known from the data
$$\mathbf{U}^T\mathbf{U} = \mathbf{L}\mathbf{L}^T$$
- The projection is given by
$$\widetilde{\mathbf{P}} = \mathbf{V}^T\mathbf{P}\mathbf{V} = \mathbf{L}^{-1}\left(\mathbf{U}^T\mathbf{P}\mathbf{U}\right)\mathbf{L}^{-T},$$
where $\mathbf{U}^T\mathbf{P}\mathbf{U}$ is also known from the data
- The use of Cholesky for orthogonalization is essential, (block) lower triangular structure is the linear algebraic equivalent of **causality**

Image from the ROM

- How to extract an image form the ROM?
- ROM is a projection, we can use **backprojection**
- If *span*(**U**) is suffiently rich, then columns of $\mathbf{V}\mathbf{V}^T$ should be good approximations of δ**-functions**, hence

$$\mathbf{P} \approx \mathbf{V}\mathbf{V}^T \mathbf{P} \mathbf{V}\mathbf{V}^T = \mathbf{V}\widetilde{\mathbf{P}}\mathbf{V}^T$$

- Problem: snapshots **U** in the whole domain are unknown, so are orthogonalized snapshots **V**
- In **imaging** we have a rough idea of **kinematics**, i.e. we know approximately the **travel times**
- This is equivalent to knowing a **kinematic model**, a smooth non-reflective sound speed c_0
- Once c_0 is fixed, we know everything associated with it

$$\widehat{\mathbf{A}}_0, \quad \mathbf{P}_0, \quad \mathbf{U}_0, \quad \mathbf{V}_0, \quad \widetilde{\mathbf{P}}_0$$

Approximate backprojection

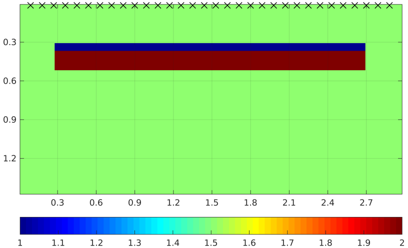
- We take the backprojection $\mathbf{P} \approx \mathbf{V}\widetilde{\mathbf{P}}\mathbf{V}^T$ and make another approximation replacing unknown $\mathbf{V}$ with the kinematic model basis $\mathbf{V}_0$:
$$\mathbf{P} \approx \mathbf{V}_0\widetilde{\mathbf{P}}\mathbf{V}_0^T$$
- For the kinematic model we know the basis exactly
$$\mathbf{P}_0 \approx \mathbf{V}_0\widetilde{\mathbf{P}}_0\mathbf{V}_0^T$$
- If δ_x is a δ-function centered at point x, then
$$\mathbf{P}\delta_x = \cos\left(\tau\sqrt{-\widehat{\mathbf{A}}}\right)\delta_x = w(\tau),$$
where $w(t)$ is a solution to
$$w_{tt} = \widehat{\mathbf{A}}w, \quad w(0) = \delta_x, \quad w_t(0) = 0,$$
i.e. it is a **Green's function** $G(x, \cdot, \tau)$

Green's function and imaging

- Diagonal entries of **P** are Green's function evaluated at the same point
$$G(x,x,\tau) = \delta_x^T \mathbf{P} \delta_x$$
- By taking the **diagonals** of backprojections we may extract the **approximate** Green's functions
$$G(\cdot,\cdot,\tau) - G_0(\cdot,\cdot,\tau) = \mathrm{diag}(\mathbf{P} - \mathbf{P}_0) \approx \mathrm{diag}\left(\mathbf{V}_0(\widetilde{\mathbf{P}} - \widetilde{\mathbf{P}}_0)\mathbf{V}_0^T\right) = \mathcal{I}$$
- Approximation quality depends **only** on how well columns of $\mathbf{V}\mathbf{V}_0^T$ and $\mathbf{V}_0\mathbf{V}_0^T$ approximate δ-functions
- It appears that $\mathcal{I}$ works well as an **imaging functional** to image the discontinuities of c
- Despite the name "backprojection" our method is **nonlinear** in the data since obtaining $\widetilde{\mathbf{P}}$ from $\mathbf{F}_k$ is a nonlinear procedure (block Lanczos and matrix inversion)

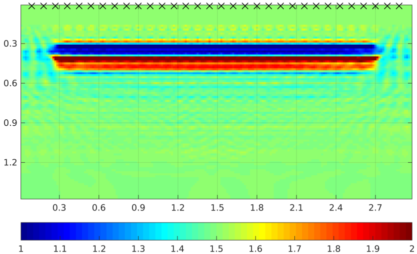
Simple example: layered model

True sound speed c

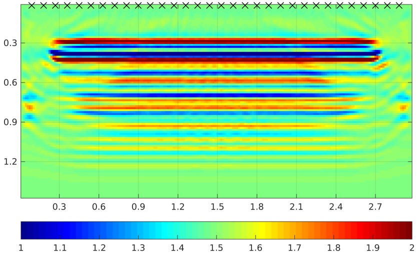


- A simple layered model, $p = 32$ sources/receivers (black $\times$)
- Constant velocity kinematic model $c_0 = 1500 \; m/s$
- Multiple reflections from waves bouncing between layers and surface
- Each multiple creates an RTM artifact below actual layers

Backprojection: $c_0 + \alpha\mathcal{I}$



RTM image



Why ROM backprojection imaging works?

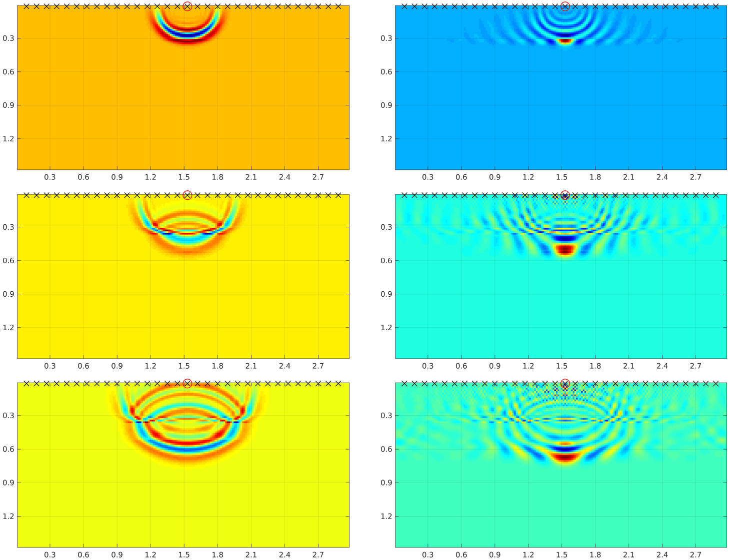
- **Suppression of multiples**: implicit causal orhthogonalization (block Gram-Schmidt) of snapshots **U** removes the "tail" with reflections and produces a focused, localized pulse
- Compare **U** and **V** for various times
- Approximation

$$G(\cdot, \cdot, \tau) - G_0(\cdot, \cdot, \tau) \approx \text{diag}\left(\mathbf{V}_0(\widetilde{\mathbf{P}} - \widetilde{\mathbf{P}}_0)\mathbf{V}_0^T\right)$$

  only works if columns of $\mathbf{V}\mathbf{V}_0^T$ and $\mathbf{V}_0\mathbf{V}_0^T$ are good approximations of δ-functions
- Plot columns of $\mathbf{V}\mathbf{V}_0^T$ and $\mathbf{V}_0\mathbf{V}_0^T$ for various points in the domain
- ROM computation resolves the **dynamics** fully, so image imperfections are mostly due to deficiencies of the **kinematic model**

Snapshot orthogonalization

Snapshots **U**	Orthogonalized snapshots **V**	



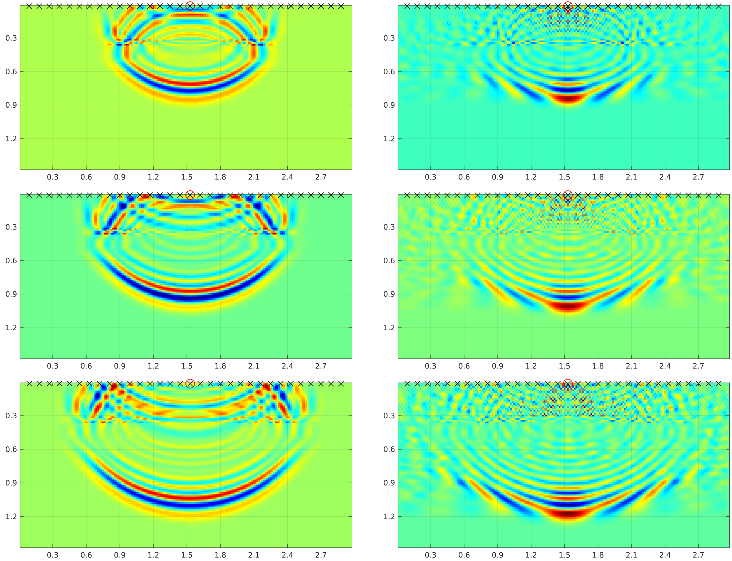
$t = 10\tau$

$t = 15\tau$

$t = 20\tau$

Snapshot orthogonalization

Snapshots **U** | Orthogonalized snapshots **V**



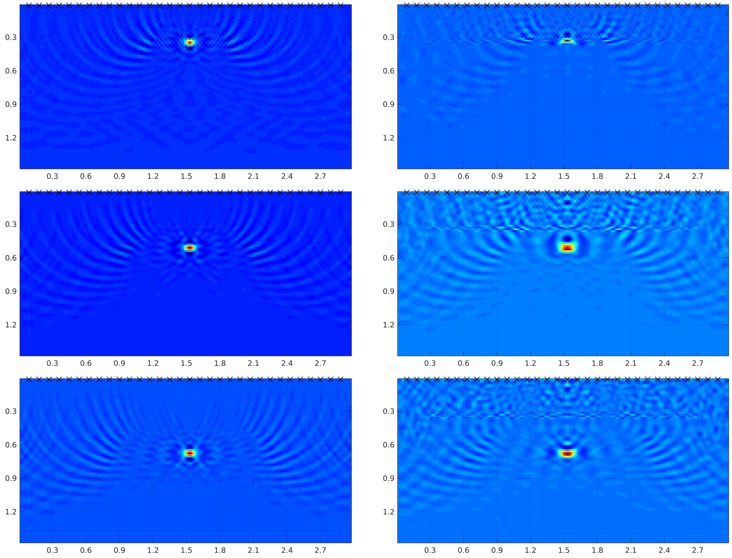
$t = 25\tau$

$t = 30\tau$

$t = 35\tau$

Approximation of δ-functions

Columns of $\mathbf{V}_0\mathbf{V}_0^T$ | Columns of $\mathbf{V}\mathbf{V}_0^T$



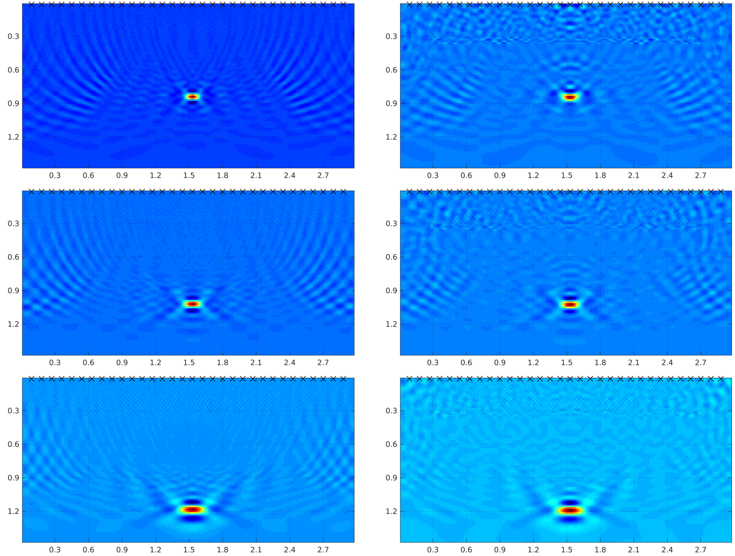
$y = 345\,m$

$y = 510\,m$

$y = 675\,m$

Approximation of δ-functions

Columns of $\mathbf{V}_0\mathbf{V}_0^T$ | Columns of $\mathbf{V}\mathbf{V}_0^T$

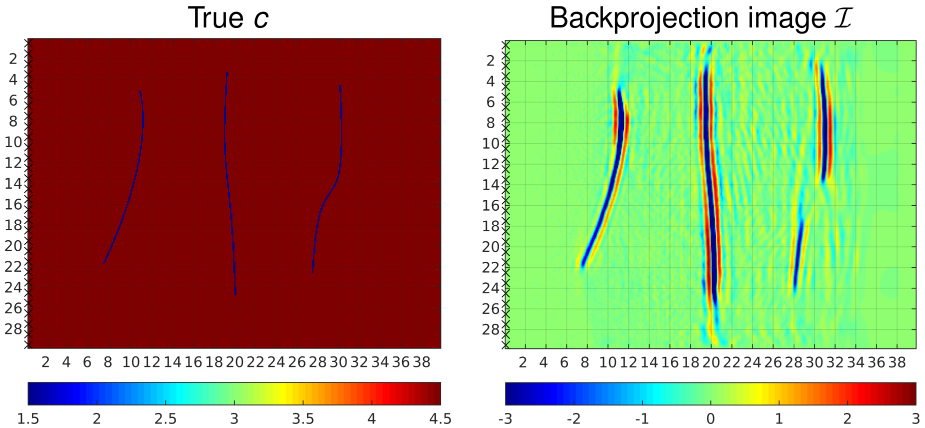


$y = 840\,m$

$y = 1020\,m$

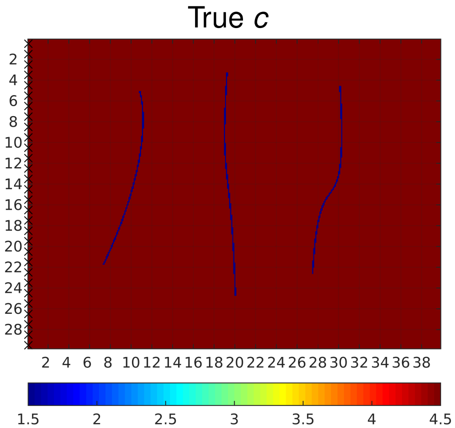
$y = 1185\,m$

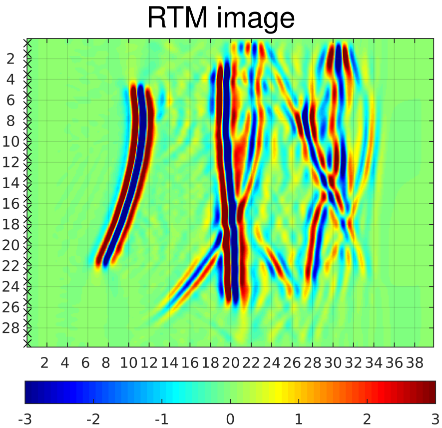
High contrast example: hydraulic fractures



- Important application: acoustic monitoring of hydraulic fracturing
- Multiple thin fractures (down to 1*cm* in width, here 10*cm*)
- Very high contrasts: $c = 4500 m/s$ in the surrounding rock, $c = 1500 m/s$ in the fluid inside fractures

High contrast example: hydraulic fractures



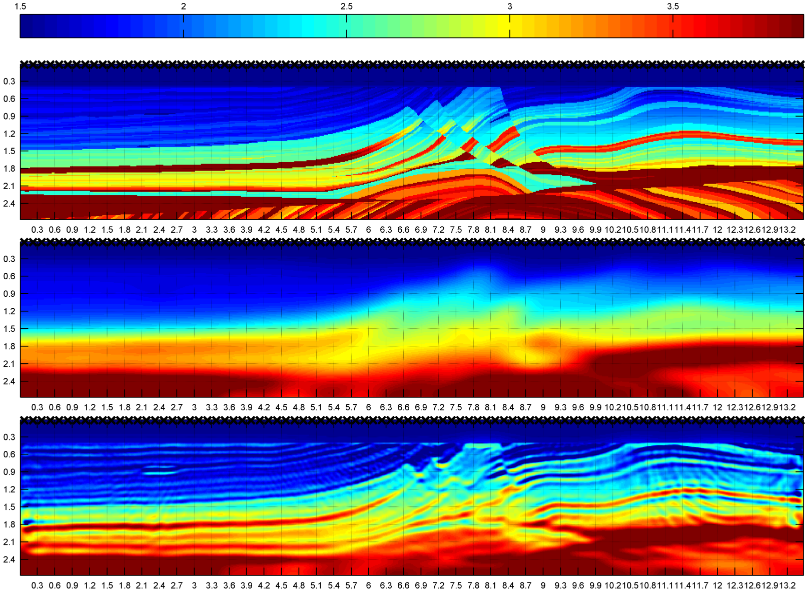


- Important application: acoustic monitoring of hydraulic fracturing
- Multiple thin fractures (down to $1cm$ in width, here $10cm$)
- Very high contrasts: $c = 4500m/s$ in the surrounding rock, $c = 1500m/s$ in the fluid inside fractures

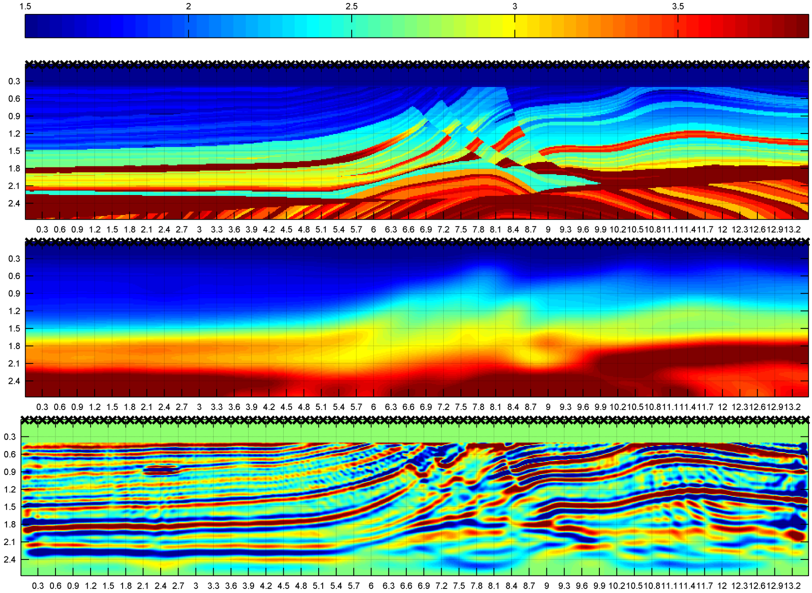
Large scale example: Marmousi model

- Standard Marmousi model, $13.5km \times 2.7km$
- Forward problem is discretized on a $15m$ grid with $N = 900 \times 180 = 162,000$ nodes
- Kinematic model c_0: smoothed out true c ($465m$ horizontally, $315m$ vertically)
- Time domain data sample rate $\tau = 33.5ms$, source frequency about $15Hz$, $n = 35$ data samples measured
- Number of sources/receivers $p = 90$ uniformly distributed with spacing $150m$
- Data is split into 17 overlapping windows of 10 sources/receivers each ($1.5km$ max offset)
- Reflecting boundary conditions
- No data filtering, everything used as is (surface wave, reflections from the boundaries, multiples)

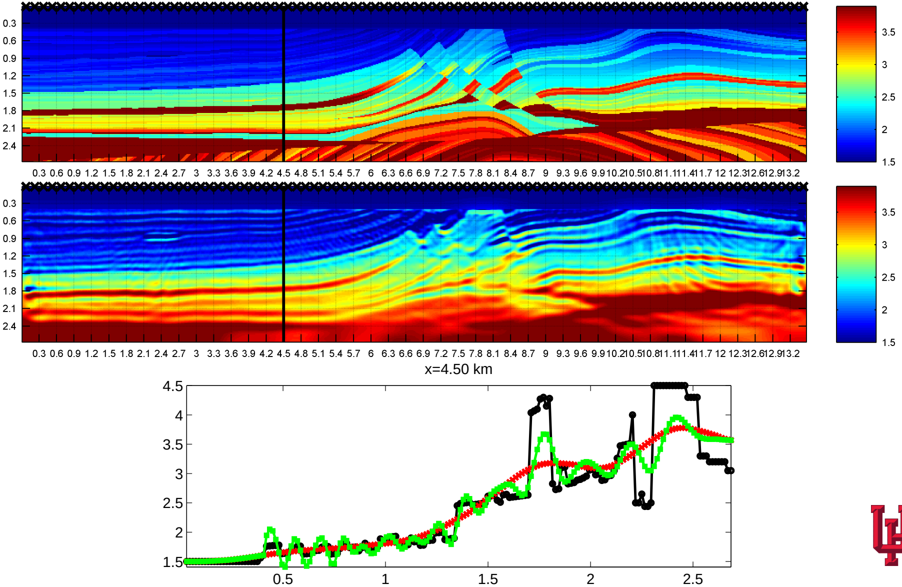
Backprojection imaging: Marmousi model



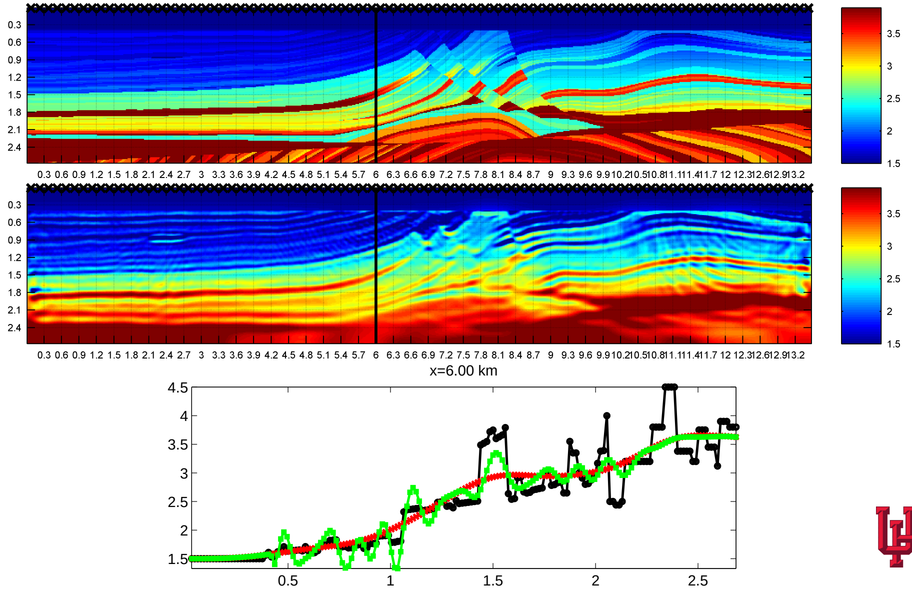
Backprojection imaging: Marmousi model



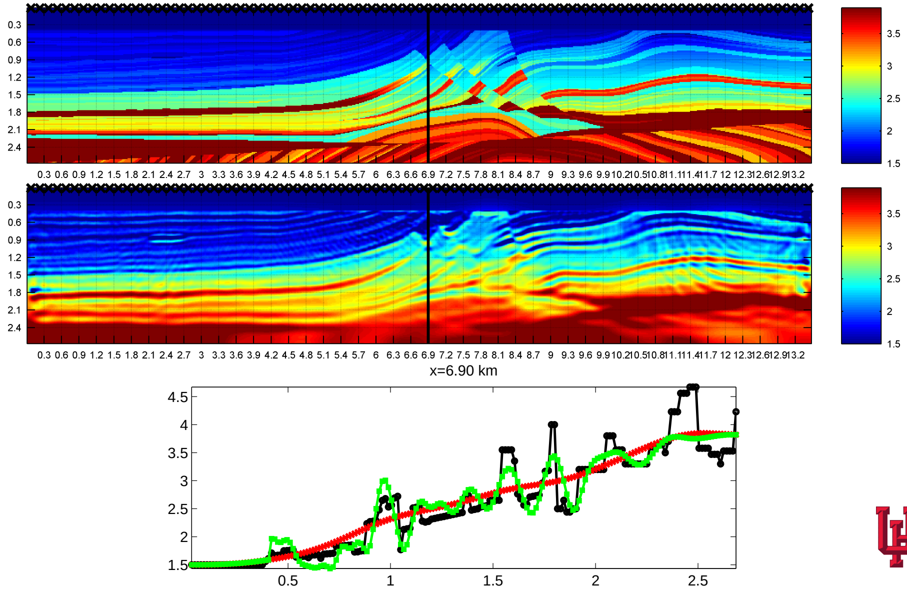
Marmousi backprojection image: well log



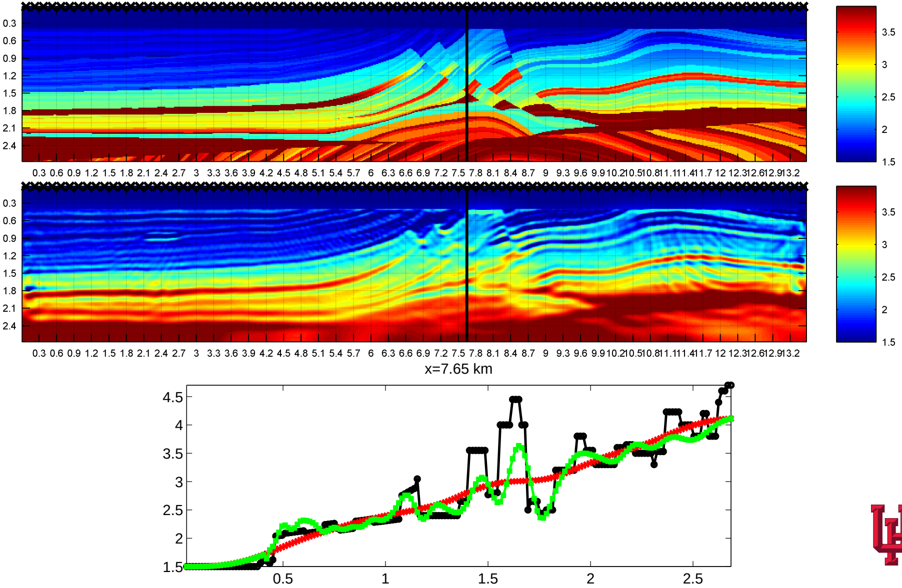
Marmousi backprojection image: well log



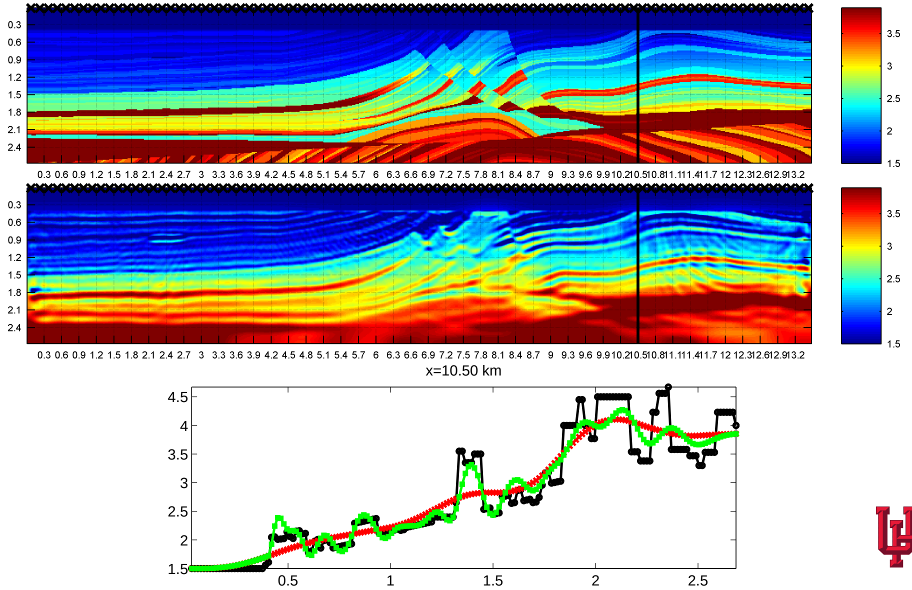
Marmousi backprojection image: well log



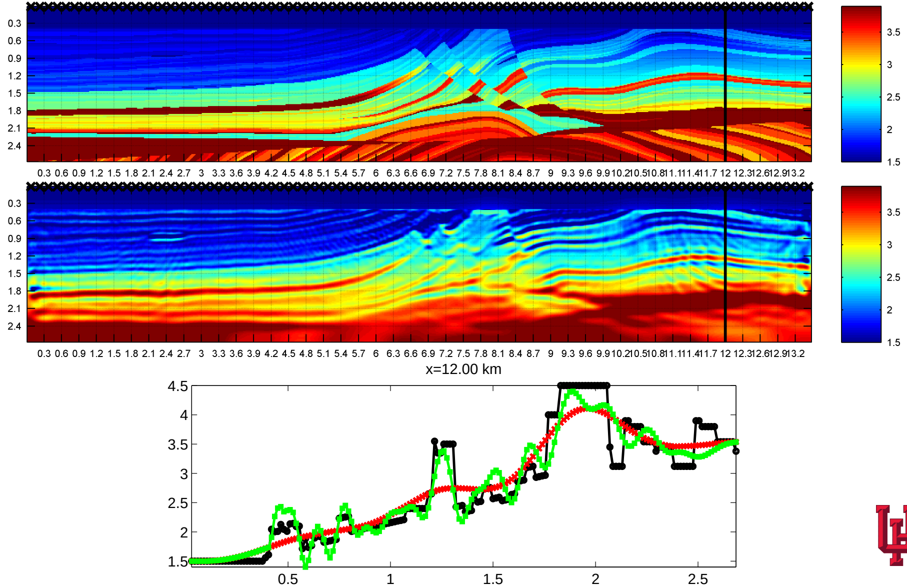
Marmousi backprojection image: well log



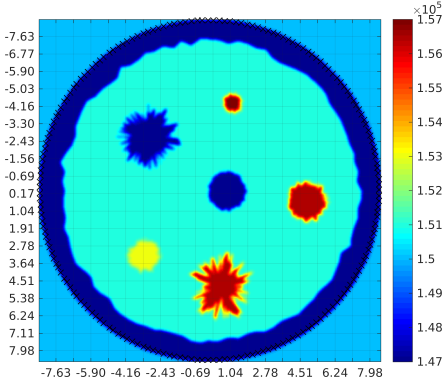
Marmousi backprojection image: well log



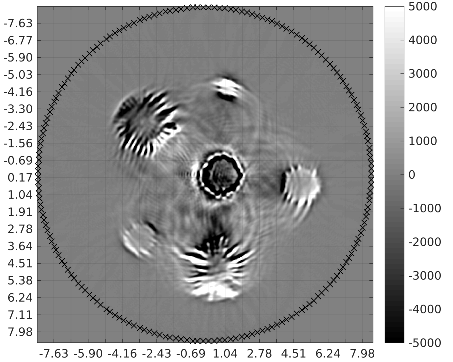
Marmousi backprojection image: well log



True c



Backprojection image



- Ultrasound screening for early detection of breast cancer
- Conventional ultrasound imaging techniques are rather crude, advanced methods originating in geophysics are in demand

Conclusions and future work

- Novel approach to acoustic imaging using reduced order models
- Time domain formulation is essential, makes use of causality (linear algebraic analogues - Gram-Schmidt, Cholesky decomposition)
- Nonlinear imaging: strong suppression of multiple reflection artifacts; improved resolution compared to RTM

Future work:

- Non-symmetric forward model and ROM for non-collocated sources/receivers
- Better theoretical understanding, relation of $\mathcal{I}$ to c
- Use for ROMs for full waveform inversion

References:

[1] A.V. Mamonov, V. Druskin, M. Zaslavsky, *Nonlinear seismic imaging via reduced order model backprojection*, SEG Technical Program Expanded Abstracts 2015: pp. 4375–4379.

[2] V. Druskin, A. Mamonov, A.E. Thaler and M. Zaslavsky, *Direct, nonlinear inversion algorithm for hyperbolic problems via projection-based model reduction*. arXiv:1509.06603 [math.NA], 2015.