On elastic seismic inversion: uniqueness and conditional Lipschitz stability

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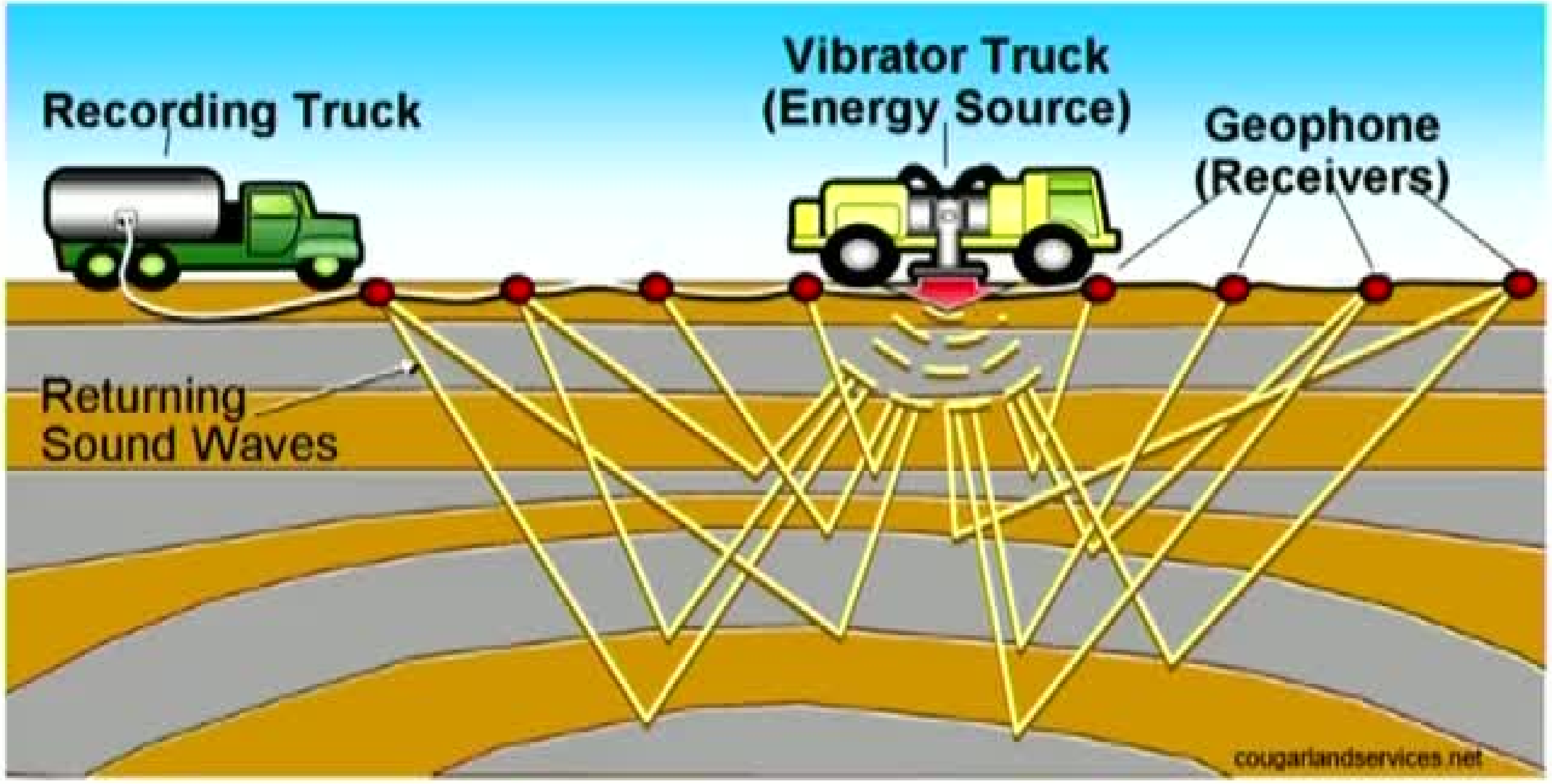
July 13, 2018

Seismic inversion

Recovery of subsurface geological structure



Use seismic waves

The fundamental problem:

Reconstruct the interior structure from ground measurements.

Mathematically formulated as inverse problems

- Uniqueness
- Stability
- Reconstruction

Acoustic model: results abound; not good enough for modeling land exploration

Inverse Problem

Time-harmonic elastic waves:

$$\mathrm{div}(\mathbf{C}\varepsilon(u)) + \rho\omega^2 u = 0 \quad \text{in } \Omega \subset \mathbb{R}^3$$

- $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$;
- ρ: density;
- $\mathbf{C}$: elastic tensor C_{ijkl}.
- u: displacement

Assume $\mathbf{C}$ is isotropic:

$$C_{ijkl}(x) = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Inverse boundary value problem

Determine λ, μ, ρ from boundary measurements (data)

Neumann-to-Dirichlet map

Data: local Neumann-to-Dirichlet map $\Lambda^{\Sigma}_{\mathbf{C},\rho}$:

$$\Lambda^{\Sigma}_{\mathbf{C},\rho} : (\mathbf{C}\varepsilon(u))\nu \to u|_{\Sigma}$$

with $(\mathbf{C}\varepsilon(u))\nu$ supported in $\Sigma \subset \partial\Omega$

For acoustic waves

$$\Delta u + \omega^2 c^{-2} u = 0$$

- Data 1: Green's function $G(x, y)$ given at all $x \in \partial\Omega$ and $y \in \partial\Omega$;
- Data 2: NtD (DtN) map $\Lambda_{\omega^2 c^{-2}}$

Equivalent! (Nachman 88')

Alessandrini's identity

Assume that u_1 and u_2 are solutions to

$$\operatorname{div}(\mathbf{C}^k \varepsilon(u_k)) + \rho^k \omega^2 u_k = 0 \text{ in } \Omega$$

for $k = 1, 2$. Then we have the following Alessandrini's identity

$$\int_\Omega ((\mathbf{C}^1 - \mathbf{C}^2)\varepsilon(u_1) : \varepsilon(u_2) - (\rho^1 - \rho^2)\omega^2 u_1 \cdot u_2)$$
$$= \langle (\Lambda_{\mathbf{C}^1,\rho^1} - \Lambda_{\mathbf{C}^2,\rho^2})(\mathbf{C}\varepsilon(u_1))\nu, (\mathbf{C}\varepsilon(u_2))\nu \rangle$$

- Relate the data with the parameters;
- General strategy: test a class of special solutions, and extract uniqueness and stability.

Nonuniqueness

Time-harmonic acoustic waves

$$\nabla \cdot (\gamma \nabla u) + \omega^2 \rho u = 0$$

γ, ρ smooth.

- Uniqueness from DtN maps with two frequencies (Nachman 88')
- Nonuniqueness with single frequency data (Arridge, Lionheart 98')

Stability with smoothness assumptions

Inverse conductivity

Identify γ in $\mathrm{div}(\gamma \nabla u) = 0$ from Dirichlet-to-Neumann map Λ_γ

$n \geq 3$, Alessandrini 88', 90'

$$\|\gamma\|_{H^{s+2}} \leq E, s > n/2$$

then

$$\|\gamma_1 - \gamma_2\| \leq C\omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|)$$

where

$$\omega(t) = |\log t|^{-\eta}$$

Optimal! (Mandache 01')

Increasing fequency? still optimal in general; believed to be true with more/other assumptions (non-trapping?)

Lipschitz stability with piecewise constant assumptions

Theorem (Beretta-de Hoop-Francini-Vessella-Z, 2017)

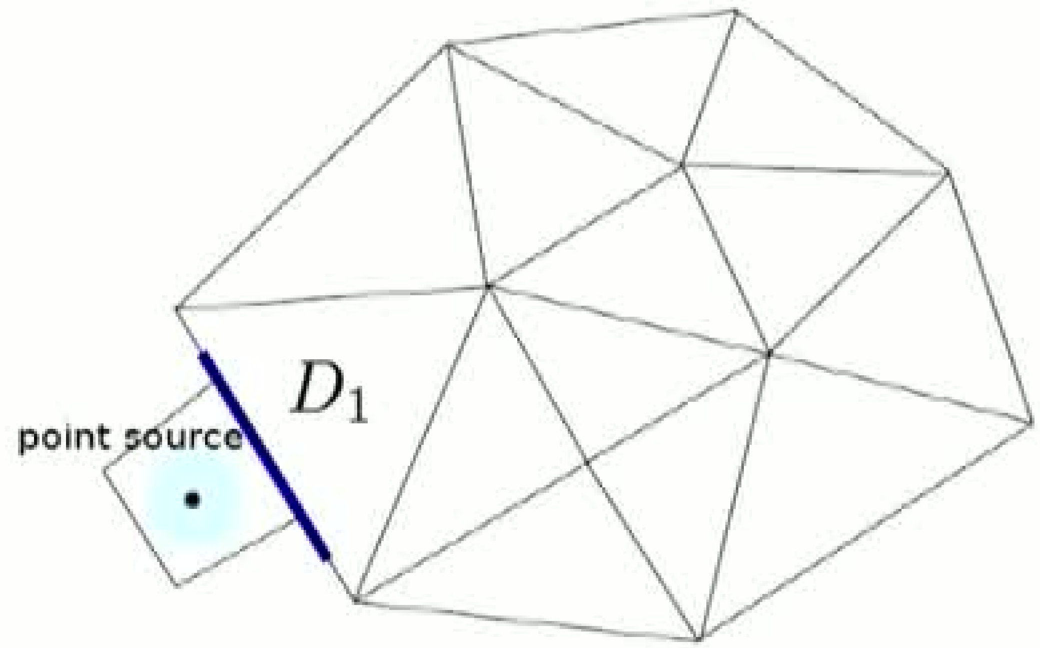
Assume $\mathbf{C}^1, \mathbf{C}^2$ are two isotropic elasticity tensors, ρ^1, ρ^2 are two densities, and $\mathbf{C}^i, \rho^i, i = 1, 2$ are piecewise constant on a given domain partitioning. There exists a positive constant C such that,

$$\|\lambda^1 - \lambda^2\| + \|\mu^1 - \mu^2\| + \|\rho^1 - \rho^2\| \leq C\|\Lambda^{\Sigma}_{\mathbf{C}^1,\rho^1} - \Lambda^{\Sigma}_{\mathbf{C}^2,\rho^2}\|.$$

de Hoop, Qiu, Scherzer (2012): Hölder stability $\Rightarrow$ local convergence of iterative reconstruction methods

Stability for piecewise constant coefficients

Test singular solutions in Alessandrini's identity.

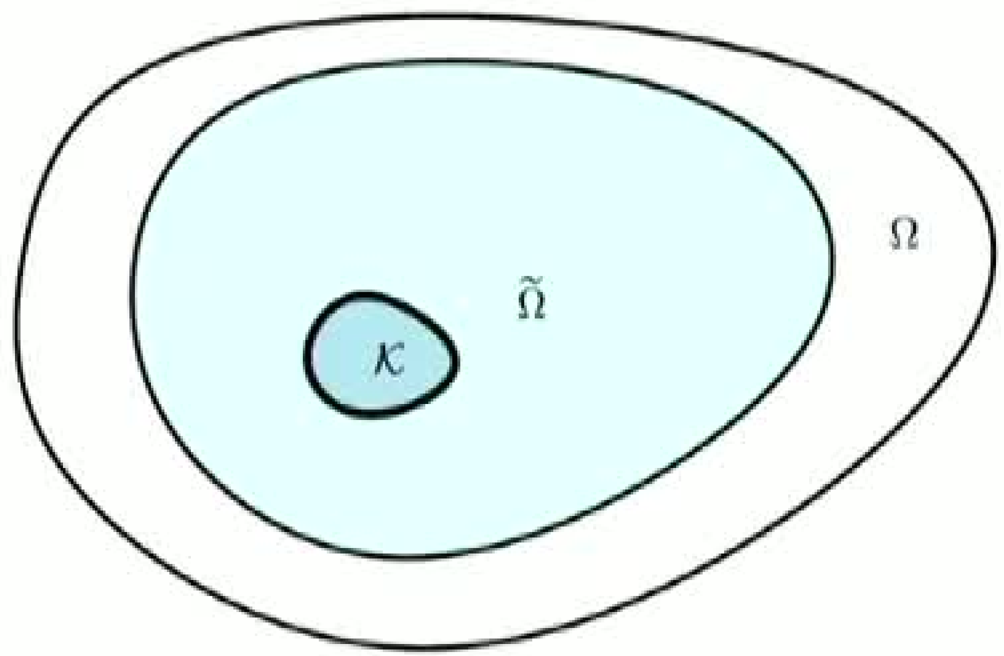


- Extend the domain;
- Point sources outside the domain, Green's function $G(\cdot, y)$ with y outside the domain;
- Alessandrini-Vessella 05' for conductivity equation;
- For different IBVPs, Beretta, de Hoop, Francini, Morassi, Qiu, Rosset, ...

Key ingredient: Unique continuation principle

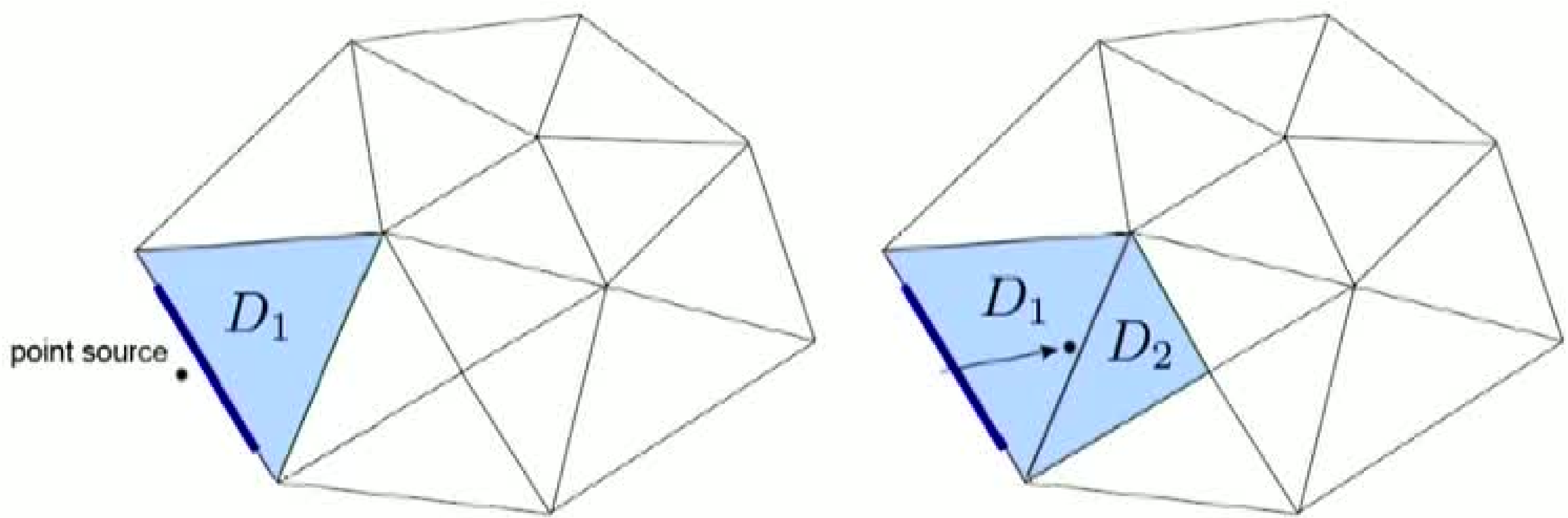
Suppose u is a solution of some elliptic PDE in Ω, $\|u\|$ is small in $\mathcal{K} \subset\subset \Omega$, then $\|u\|$ is also small in $\widetilde{\Omega} \subset\subset \Omega$, with

$$\|u\|_{\widetilde{\Omega}} \leq C\|u\|_{\mathcal{K}}^{\alpha}$$



Idea of proof

- Point sources outside the domain;
- Approach the boundary, recover coefficients on D_1;
- Propagation of the point sources, approach the interface of D_1 and D_2; (UCP plays a role)
- Iterate this process.



Comments on the Lipschitz constant

C: Lipschitz constant, N: the number of subdomains, ω: frequency

- C grows as N grows;
- C decreases as ω grows (conjecture).

An improved Unique Continuation Principle (almost Lipschitz) for

$$\Delta u + k^2 u = 0$$

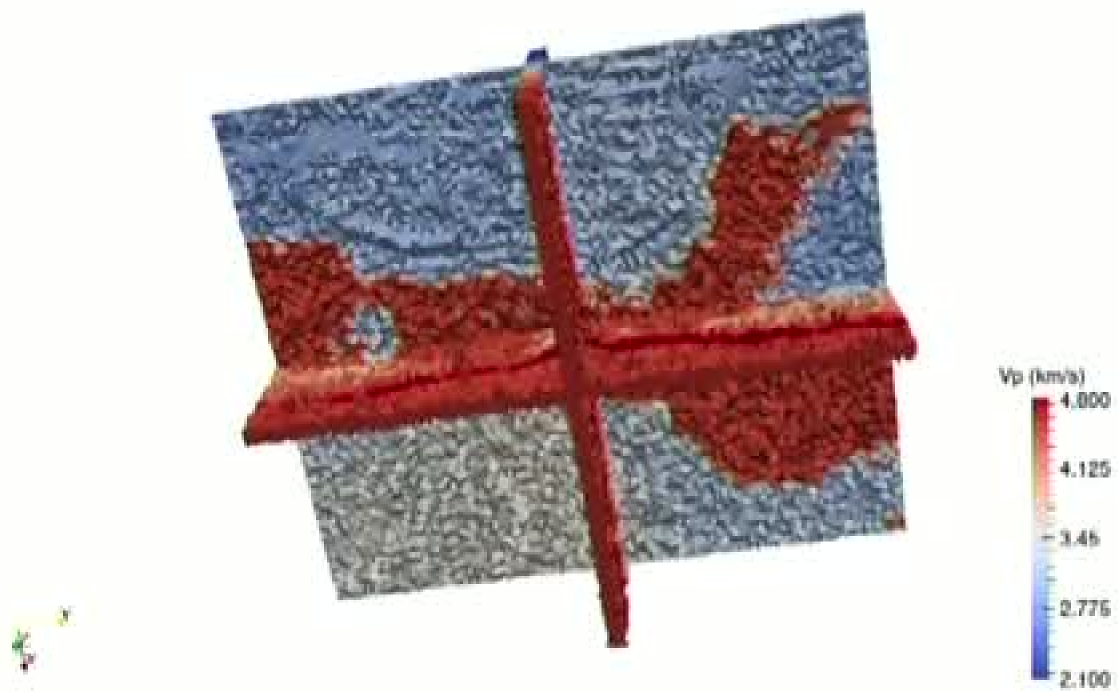
by Hrycak-Isakov 04',

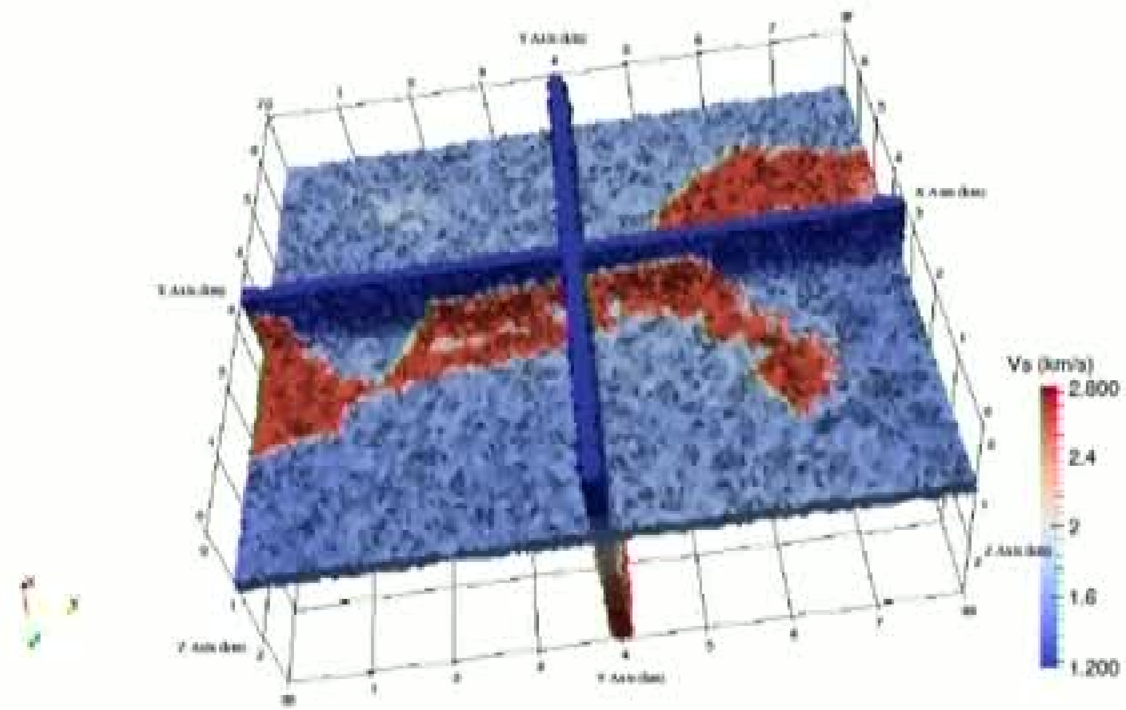
$$\|u\|_{\tilde{\Omega}} \leq C\|u\|_{\mathcal{K}} + \mathcal{O}(\frac{1}{k})$$

Reconstruction scheme

- Start from a low frequency and a coarse domain partition;
- Iterate until convergence;
- Increase the frequency and refine the partition, iterate until convergence, and use it as a new initial guess; Continue this step.

Numerical experiment (Shi *et al*)





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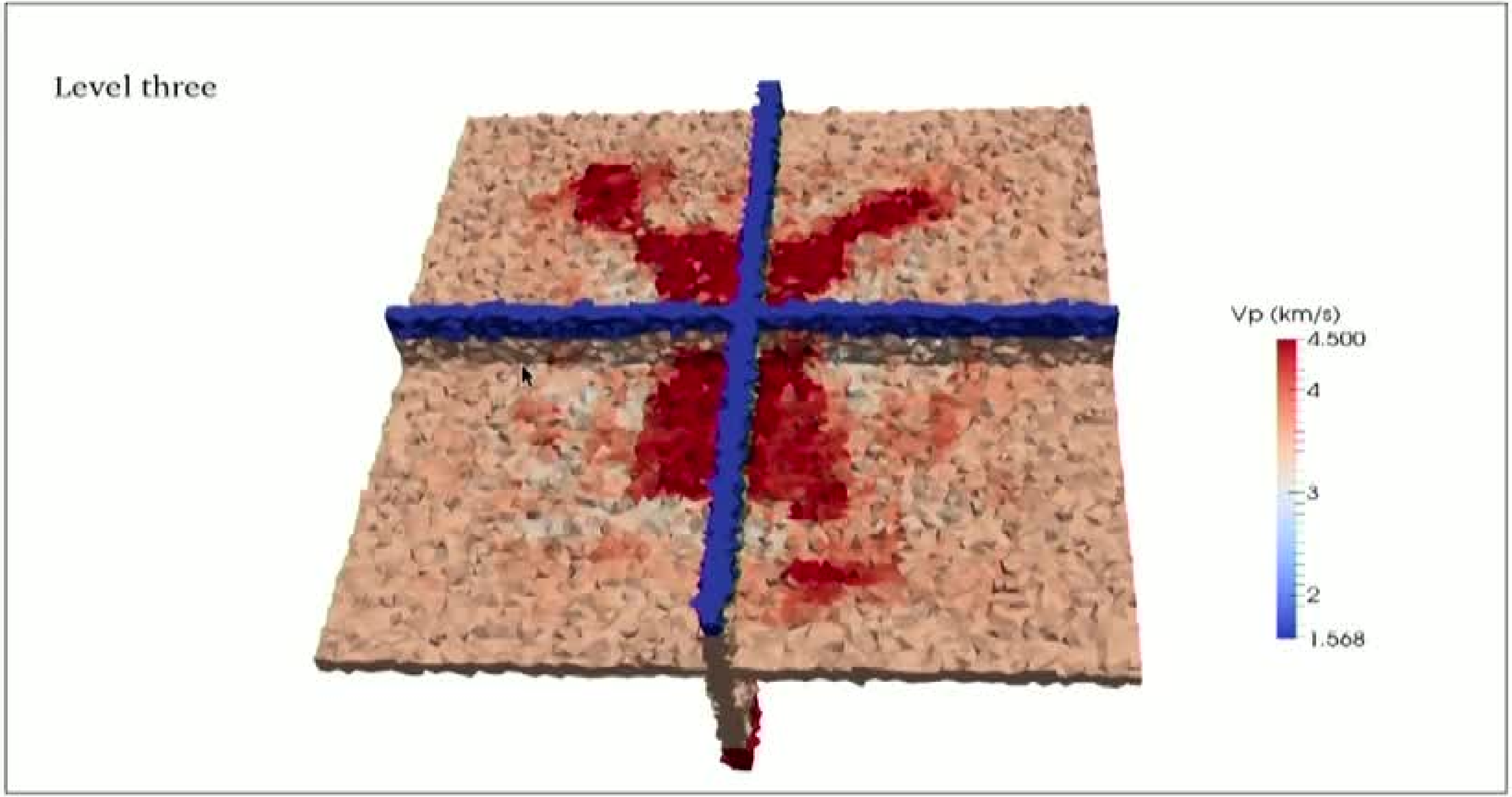

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(Reconstruction of v_p)

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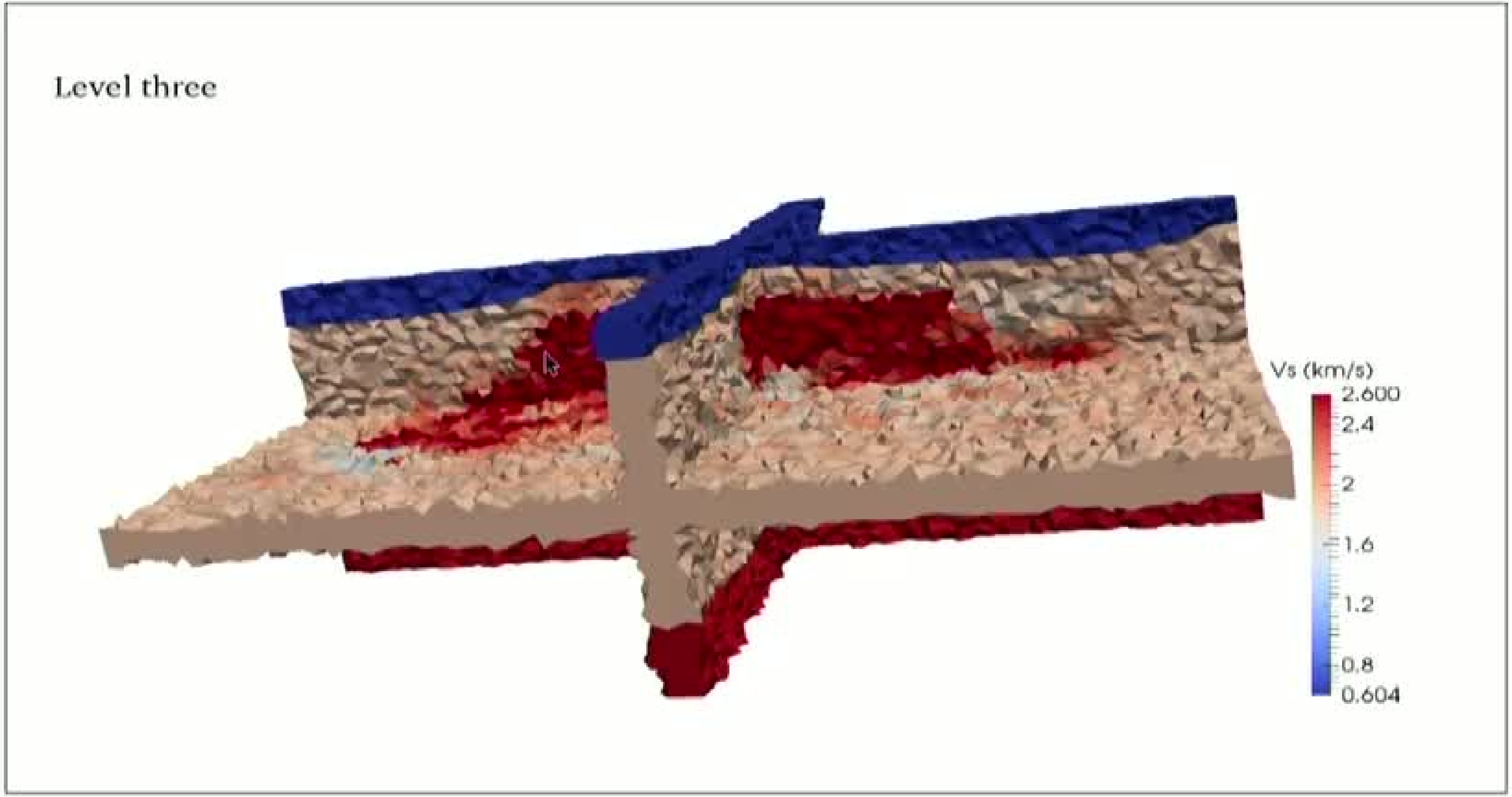

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(Reconstruction of v_s)

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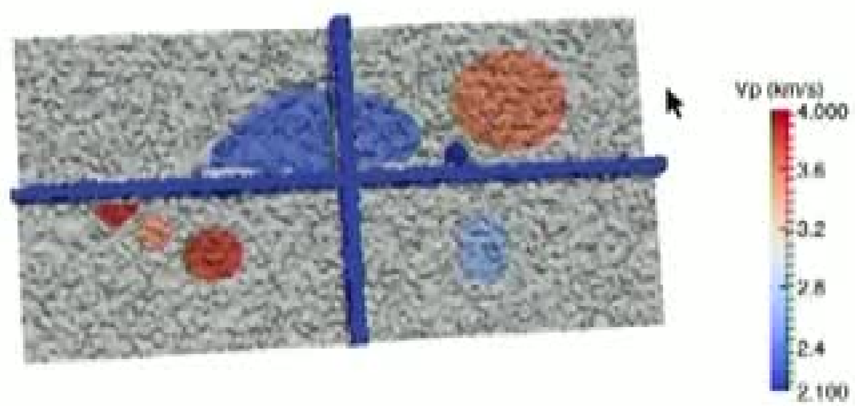

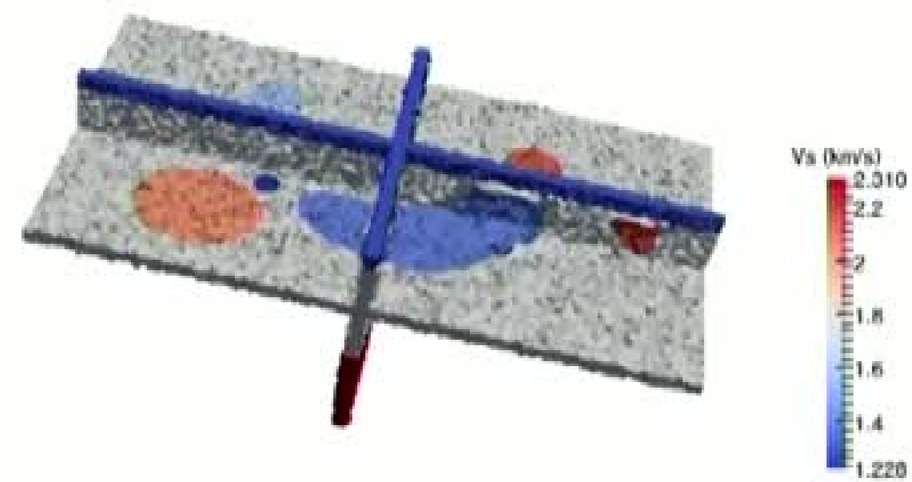

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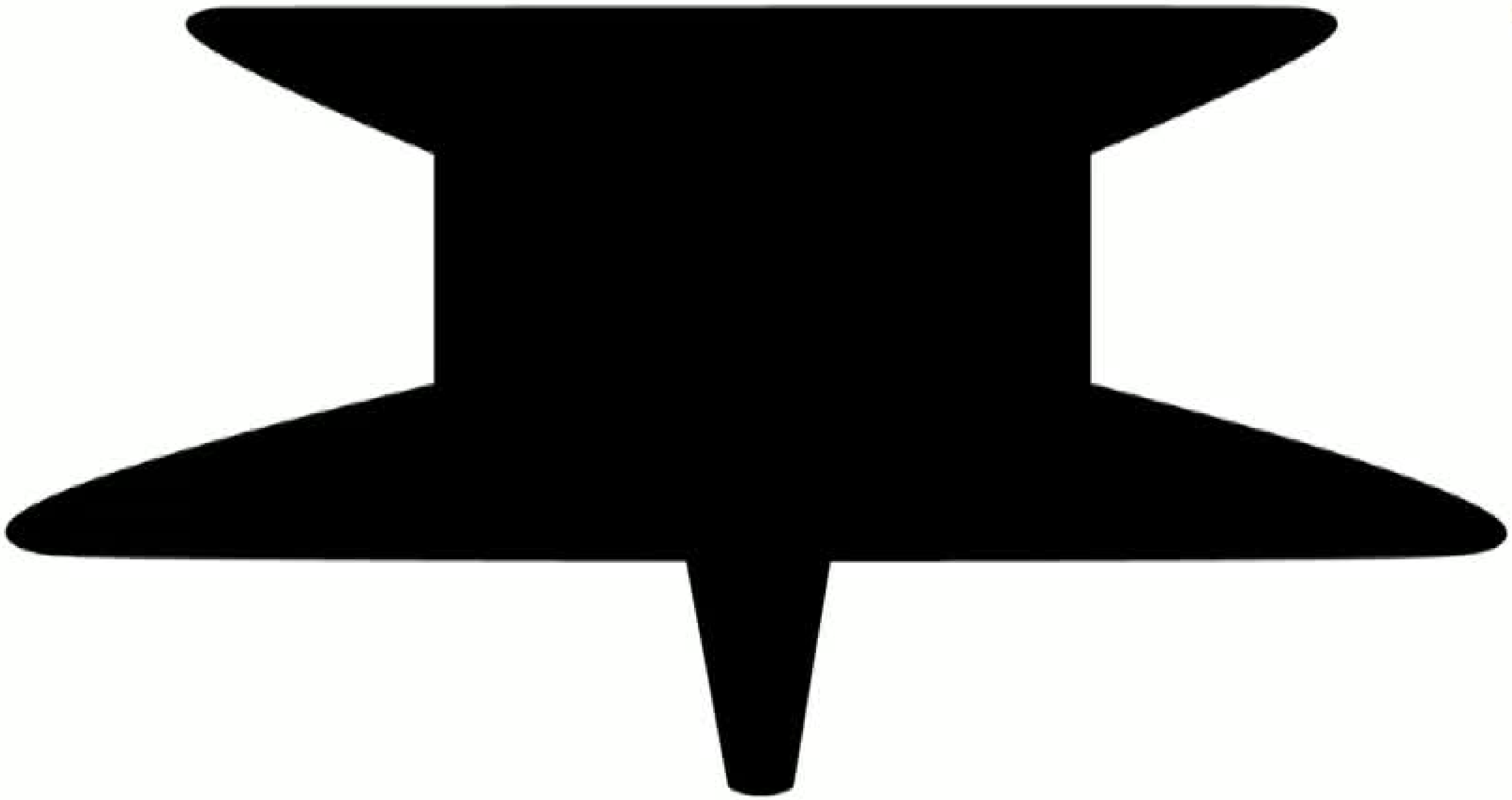

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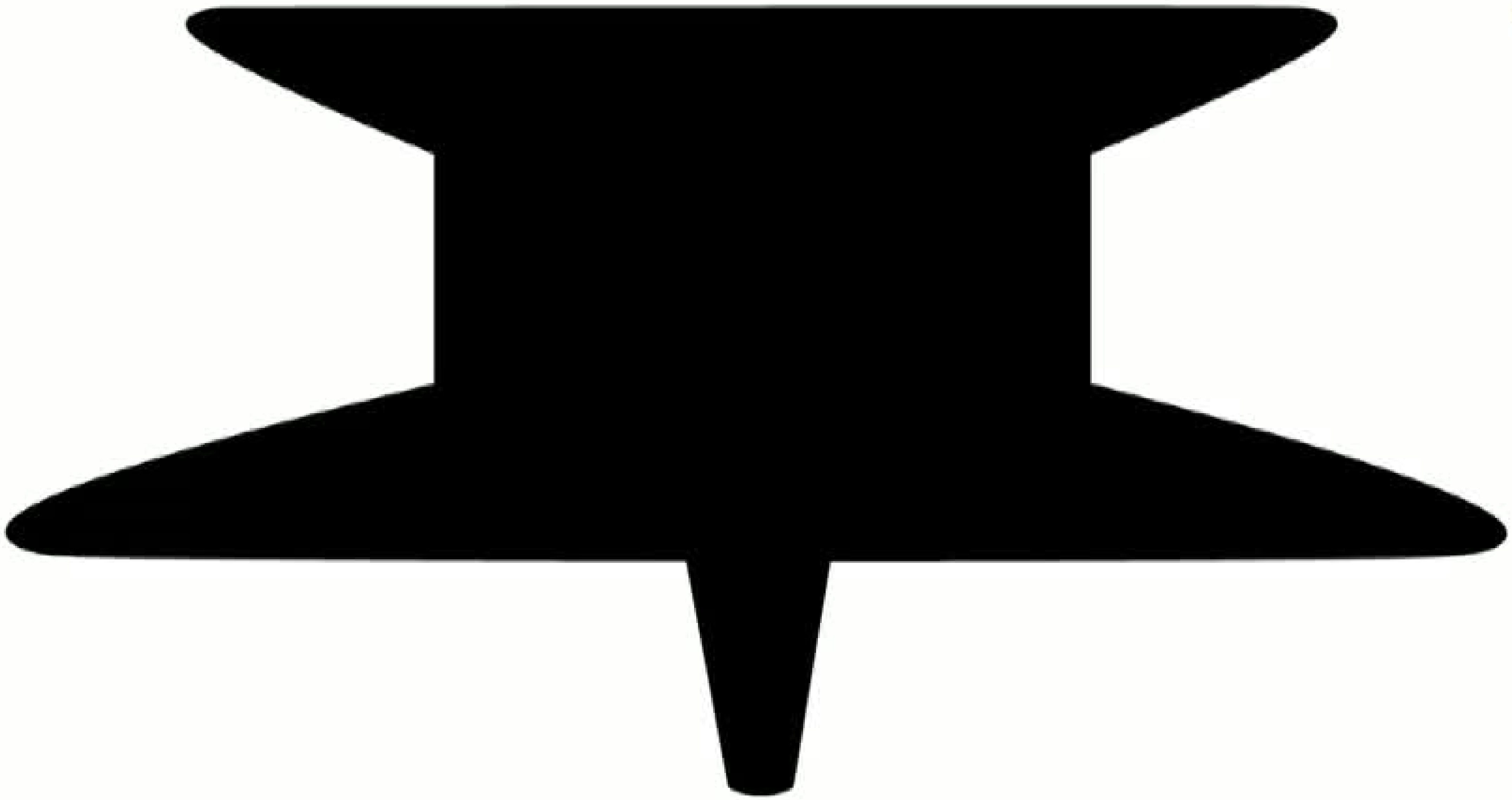

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