Quantifying Nonergodicity via Snapshot Attractors

An Application to Climate Change

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Outline

Introduction: motivation, low-order model setup

Climate change: smooth parameter shift

Snapshot attractors, convergence times

Analyzing nonergodicity

Results in a high-degree-of-freedom GCM (Planet Simulator)

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Motivation

- Climate $\approx$ the statistics of weather.
- We argue that the appropriate measure is the natural measure of the snapshot attractor.
- Only 1 observed realization exists.
- Temporal averages over single realizations are taken in practice, over e.g. 30 years.
- Can the latter yield proper statistics?

Model: Lorenz '84 [Tellus **36A**, 98]

$$\dot{x} = -y^2 - z^2 - \frac{1}{4}x + \frac{1}{4}F$$
$$\dot{y} = -y + xy - 4xz + 1$$
$$\dot{z} = -z + xz + 4xy$$

x: wind speed of the Westerlies
y, z: modes of cyclonic activity
F: temperature contrast parameter (mimics CO_2, i.e. greenhouse gas content), constant

→ Usual attractor

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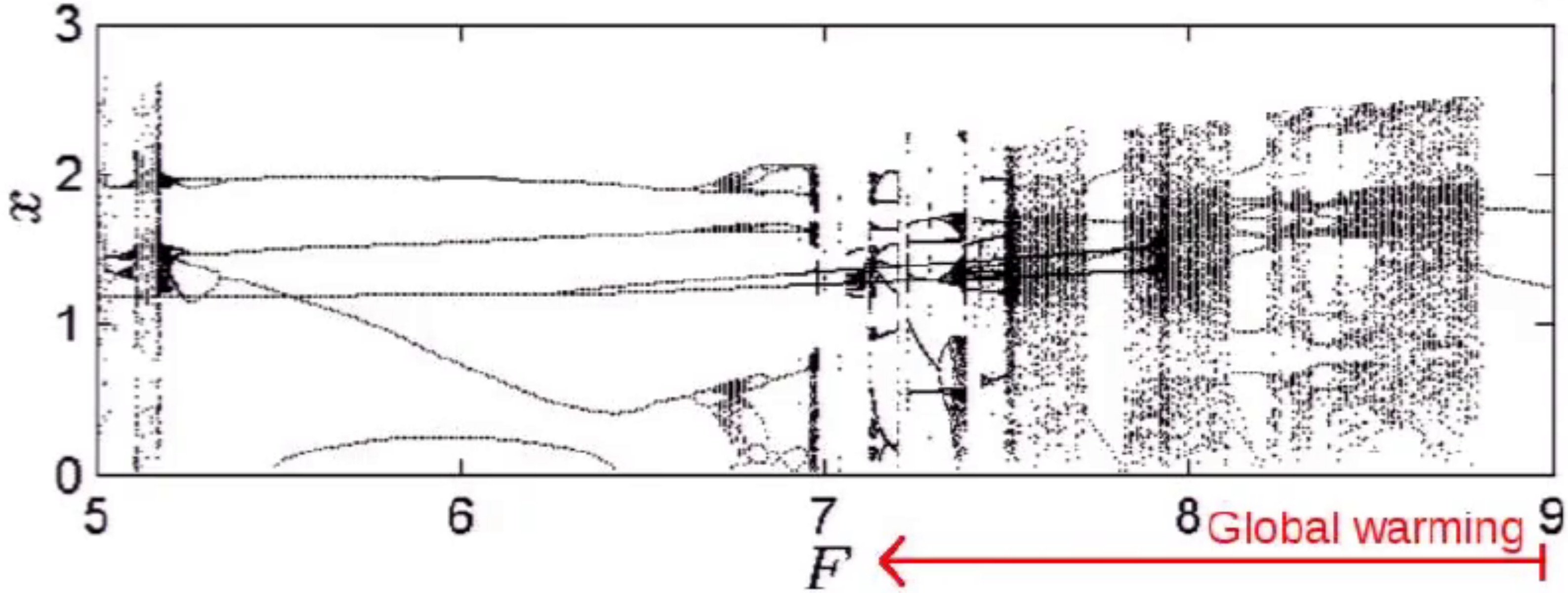
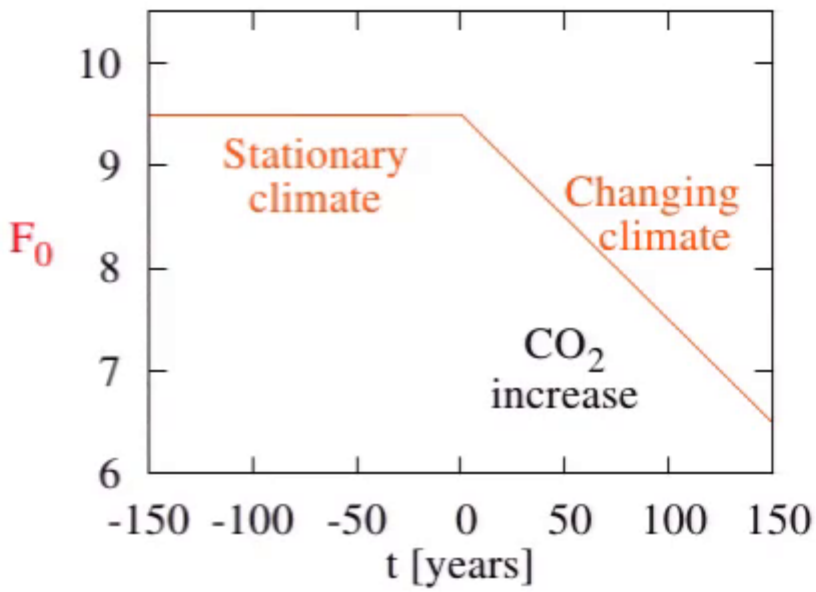


Changing climate model

$$\dot{x} = -y^2 - z^2 - \frac{1}{4}x + \frac{1}{4}F(t)$$
$$\dot{y} = -y + xy - 4xz + 1$$
$$\dot{z} = -z + xz + 4xy$$

$$F(t) = F_0(t) + 2\sin\left(\frac{2\pi}{T}t\right), \qquad T = 1\,\text{year}$$

Seasonality [Lorenz, Tellus **42A**, 378 (1990)]



$$F_0(t) = \begin{cases} 9.5 & \text{for } t \le 0 \\ 9.5 - \dfrac{2t}{100T} & \text{for } t > 0 \end{cases}$$

(we use a "stroboscopic map" even during the climate change in order to filter out seasonality → midwinter time instants)

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Snapshot attractor

- The generalization of the usual attractor for non-autonomous cases

  Romeiras, Grebogi and Ott, Phys. Rev. A **41**, 784 (1990)

  Ghil, Chekroun and Simonnet, Physica D **237**, 2111 (2008) and **240**, 1685 (2011)

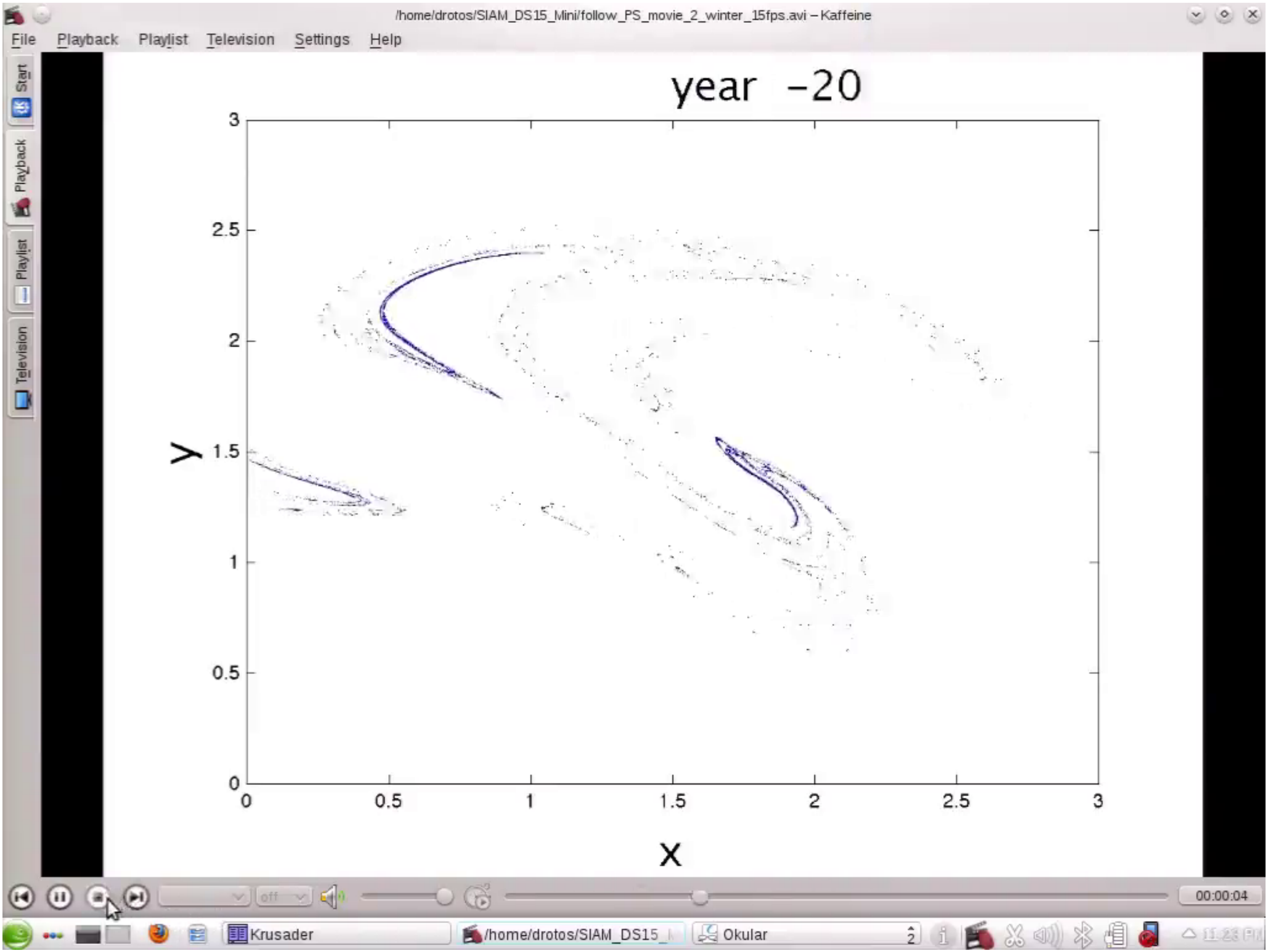
- Apply this for a case of a deterministic smooth parameter shift (most suited for climate changes)

- Initial conditions: randomly distributed in a large box in (x, y, z) much before -150 years ($N \approx 10^5$)

- Monitor all N trajectories up to time $t \rightarrow$ shape: snapshot attractor

Snapshot attractor of midwinters, $z = 0$

[video] by T. Bódai



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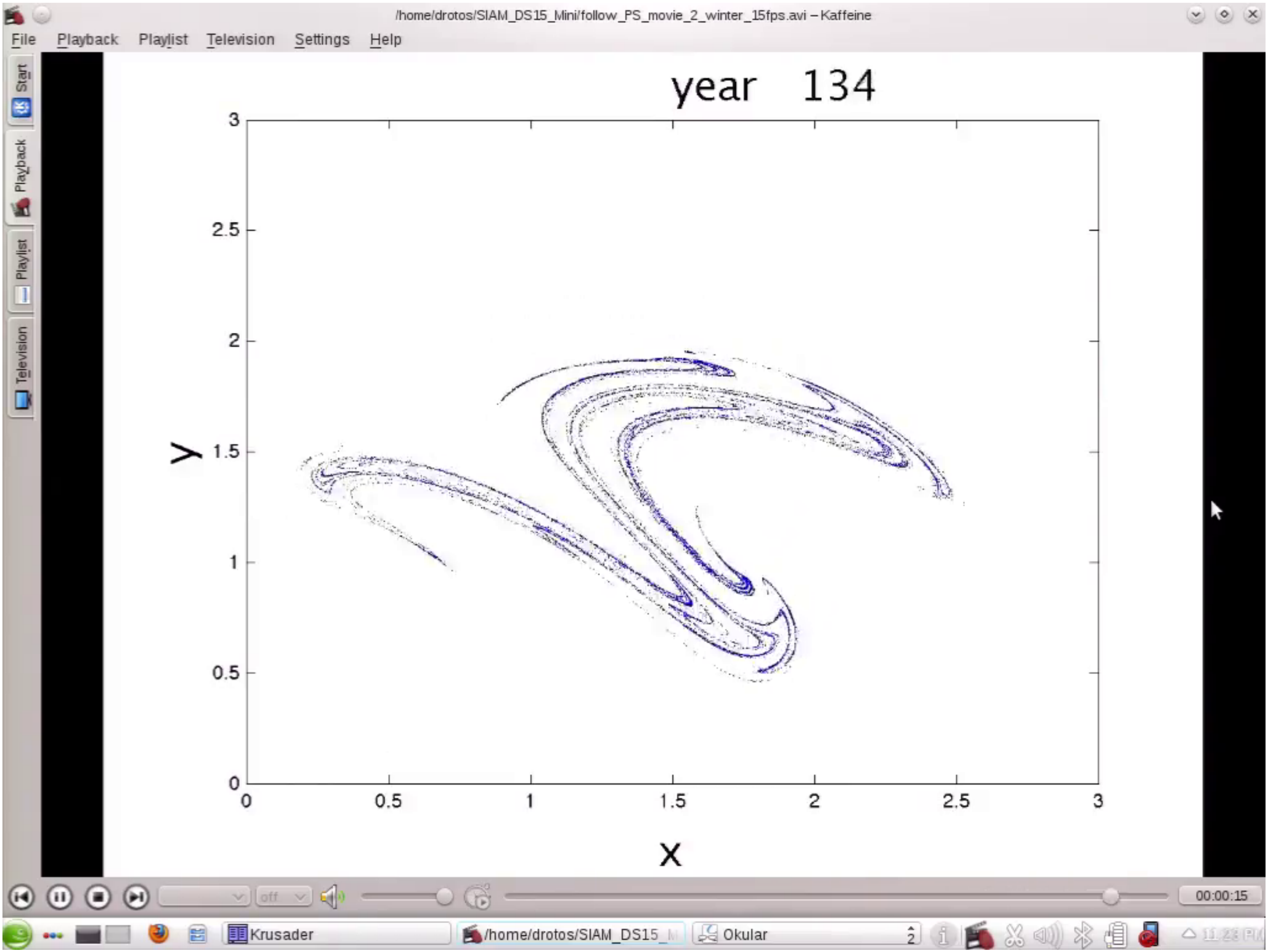
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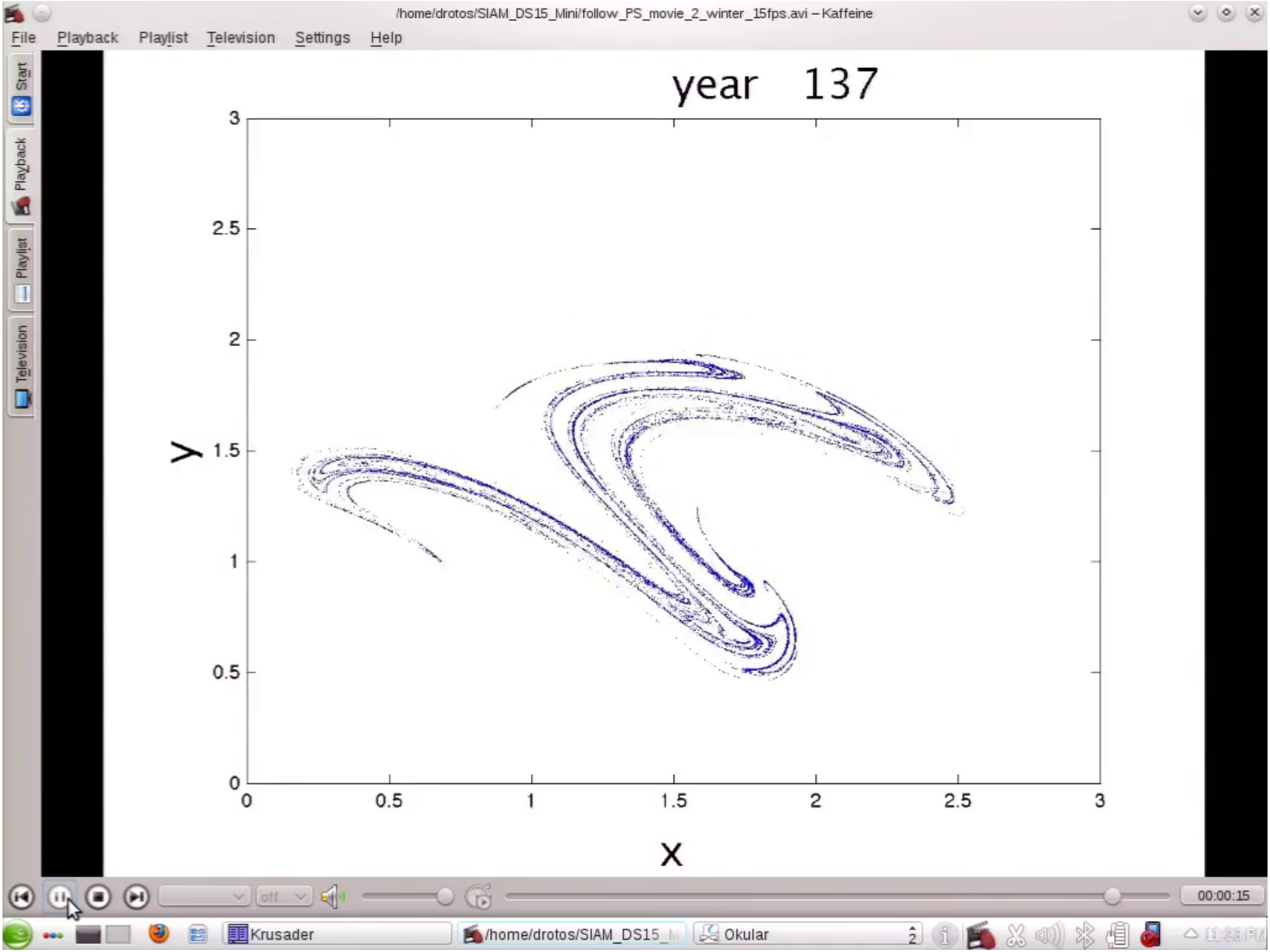
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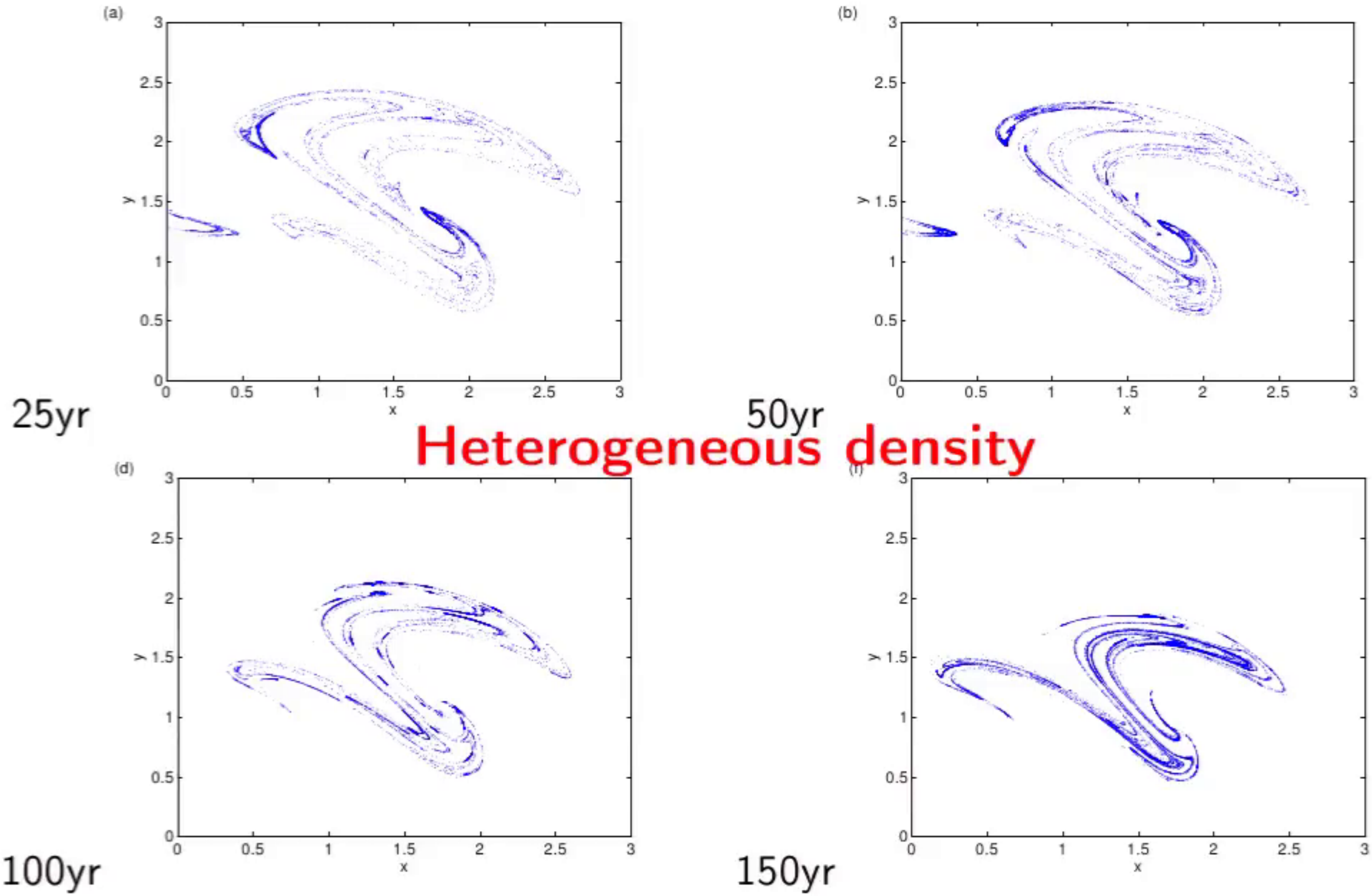
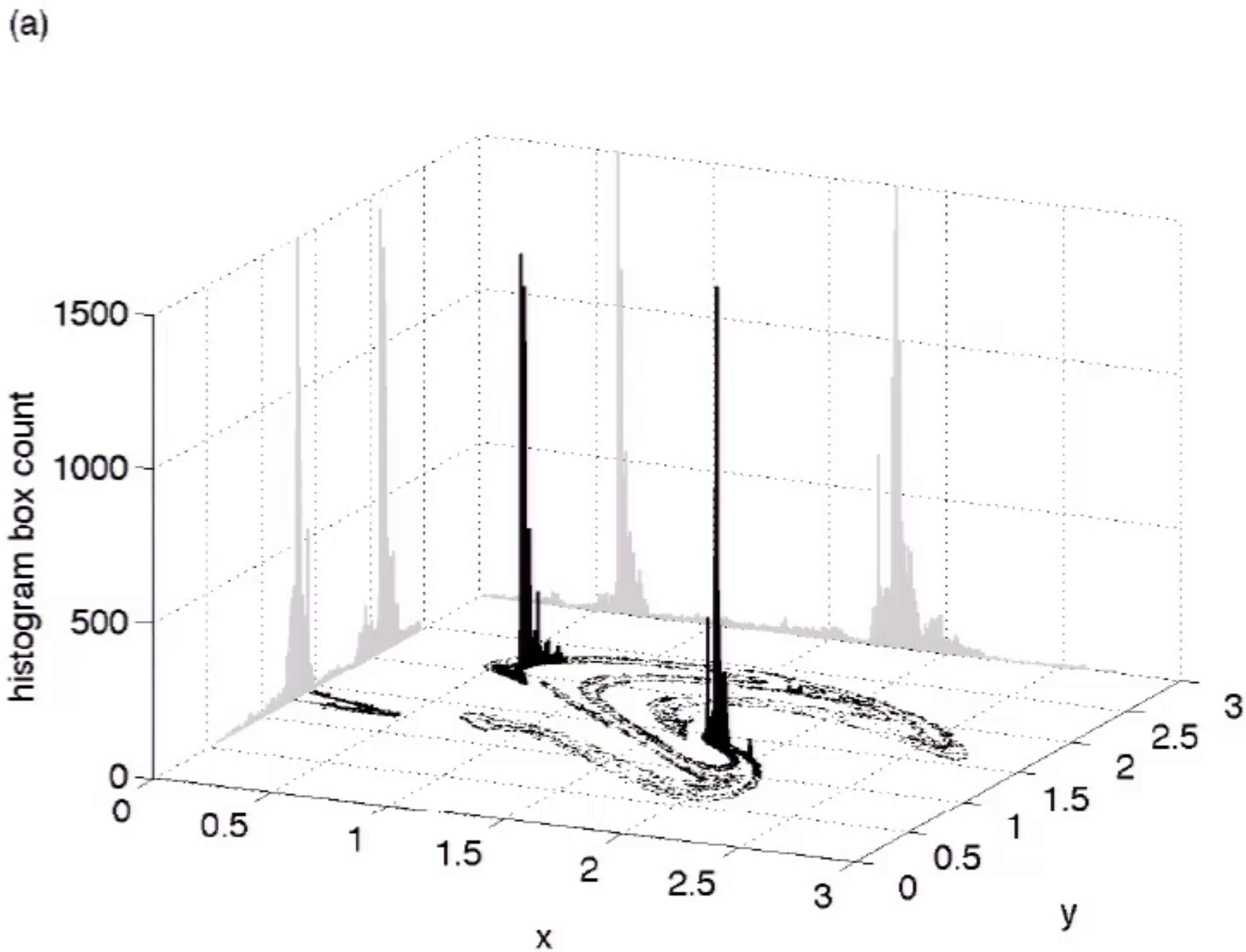
Snapshot attractor of midwinters, $z = 0$



Heterogeneous density

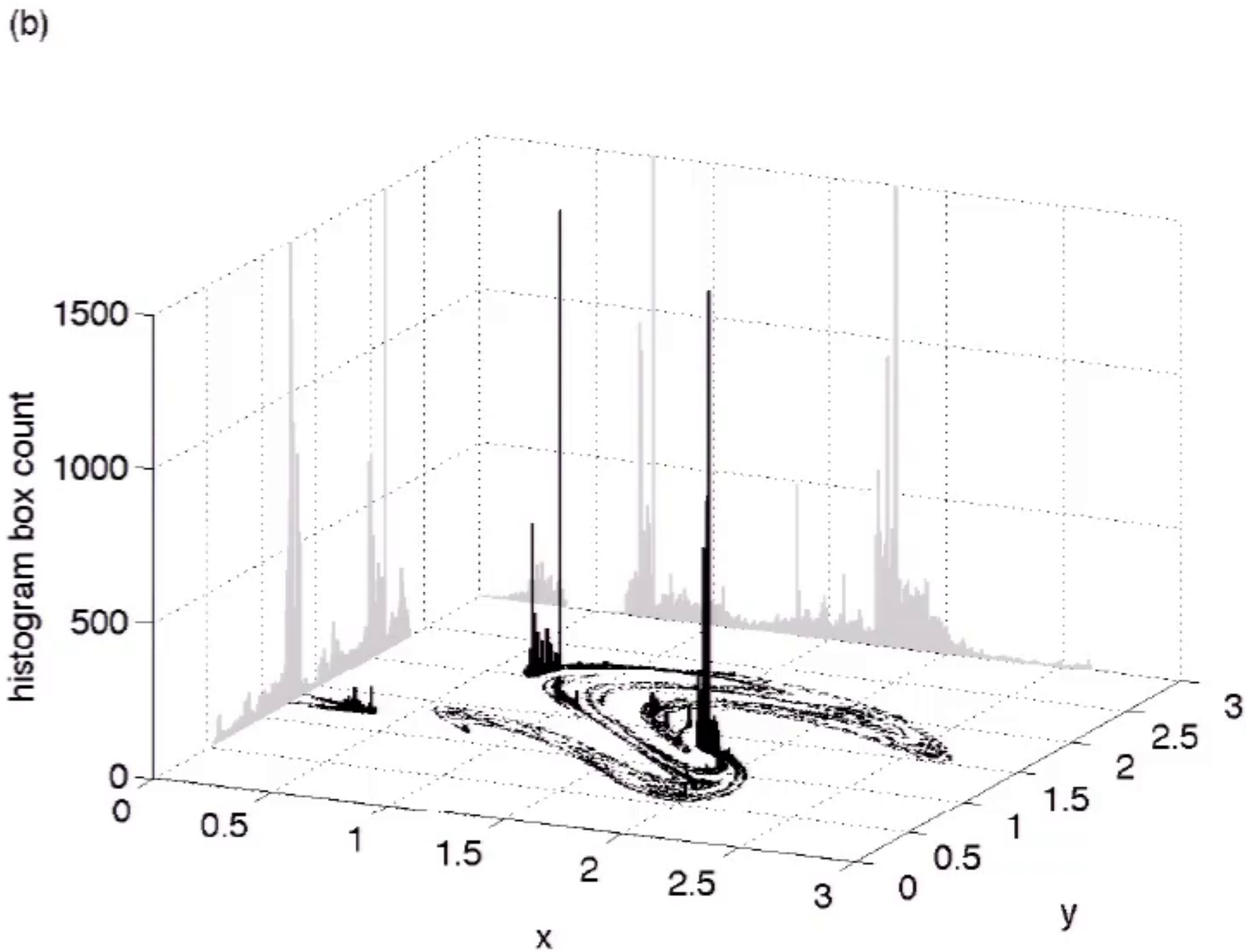
Natural measure (sample measure) of the snapshot attractor

Year 25



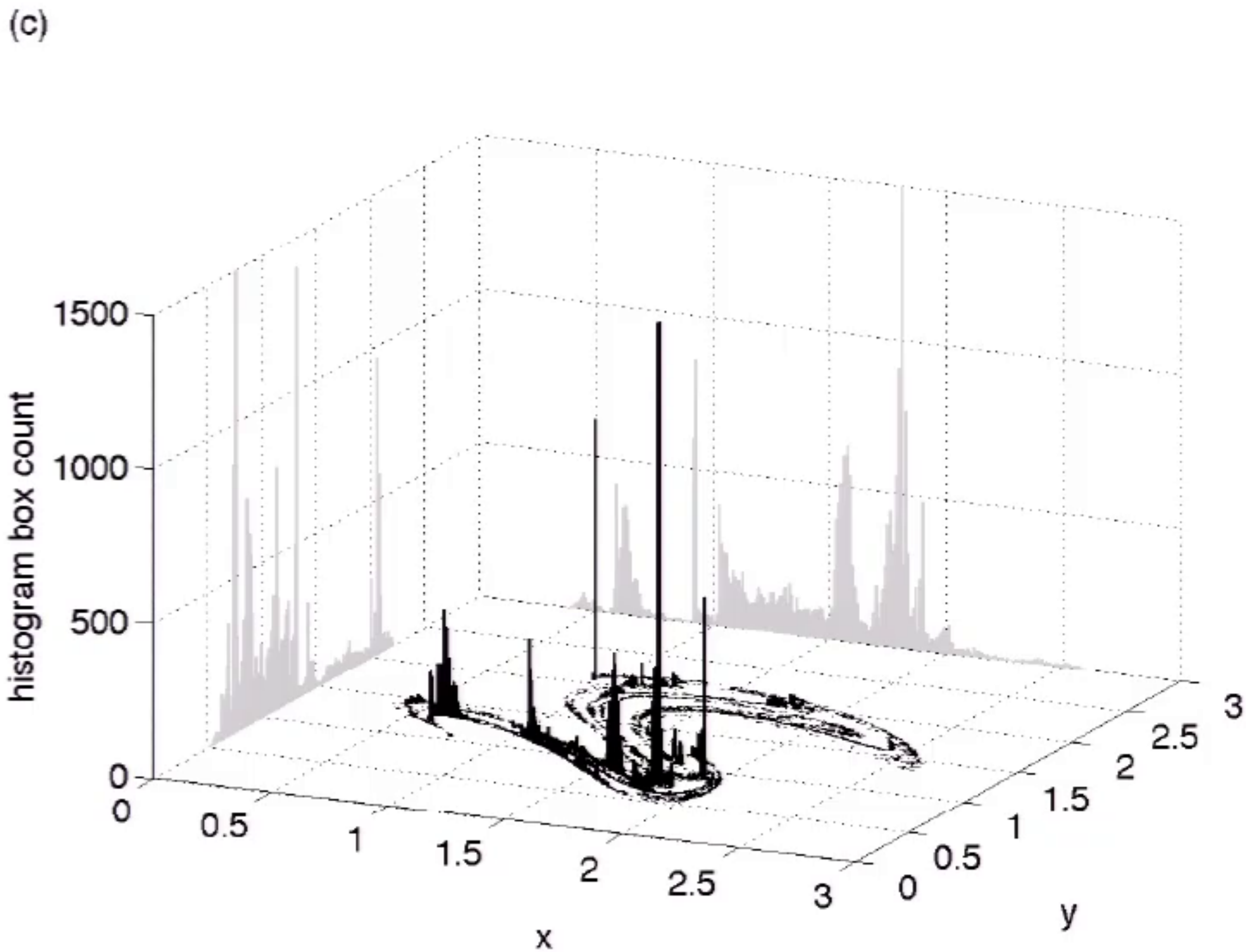
Natural measure (sample measure) of the snapshot attractor

Year 50



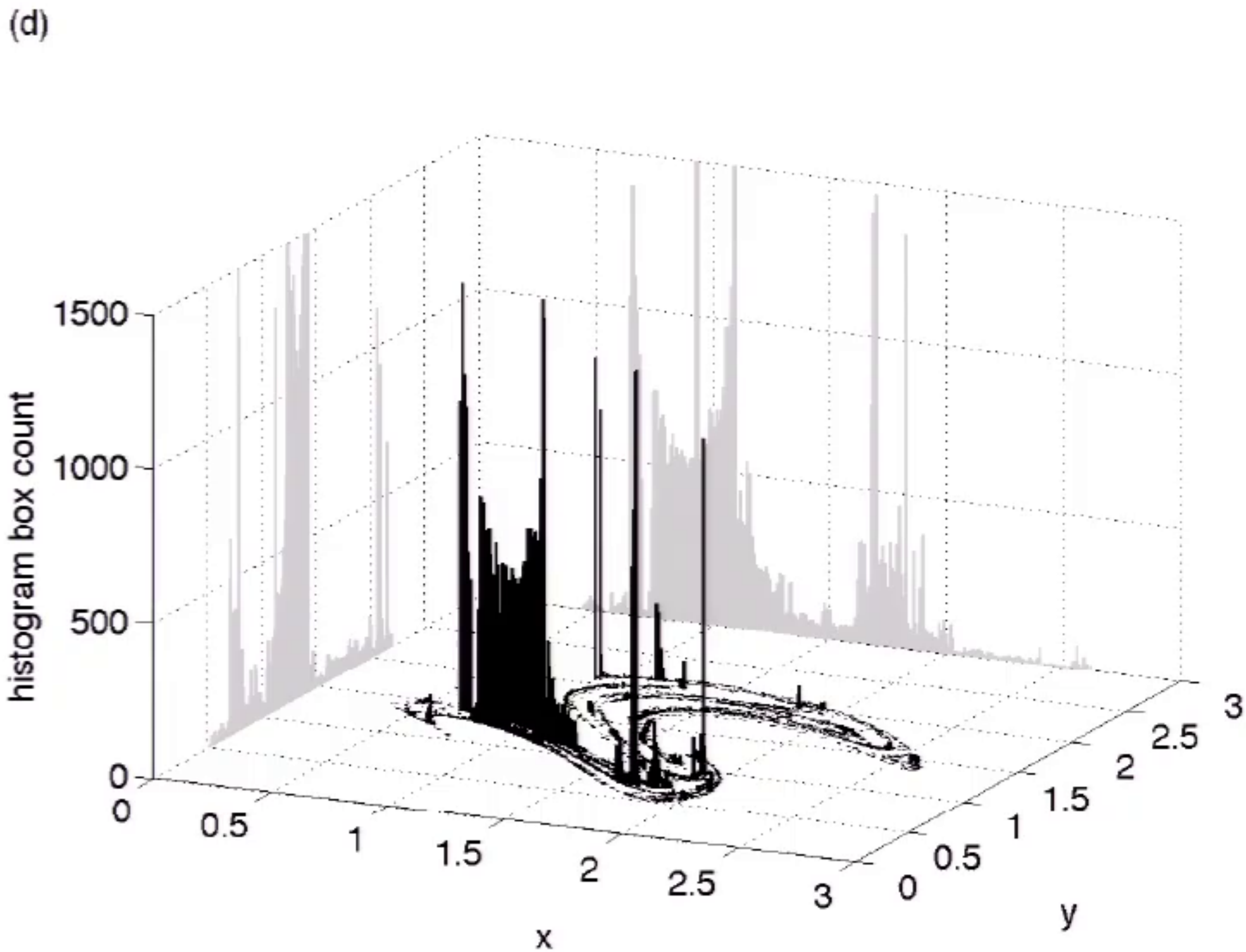
Natural measure (sample measure) of the snapshot attractor

Year 88



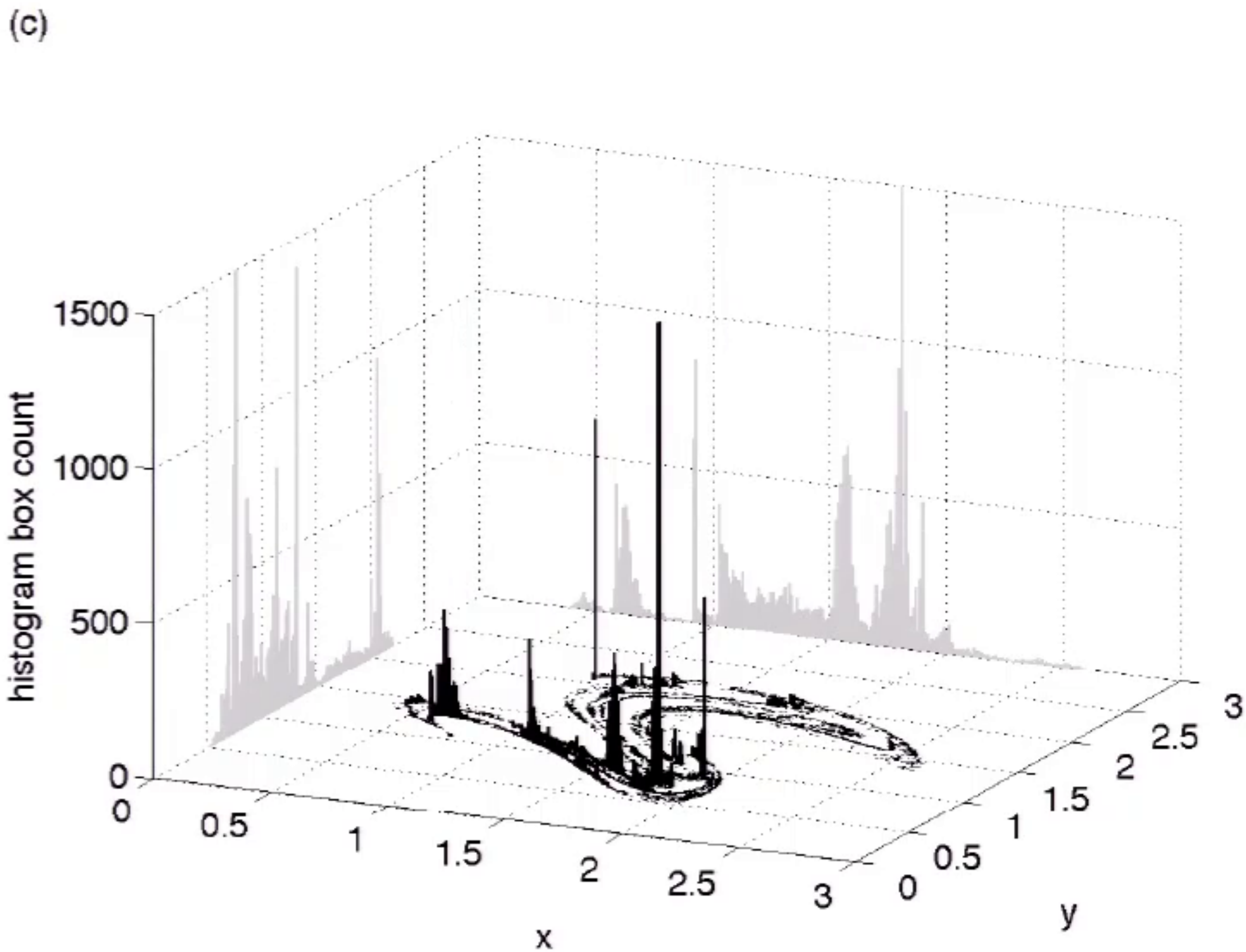
Natural measure (sample measure) of the snapshot attractor

Year 89



Natural measure (sample measure) of the snapshot attractor

Year 88



Natural measure (sample measure) of the snapshot attractor

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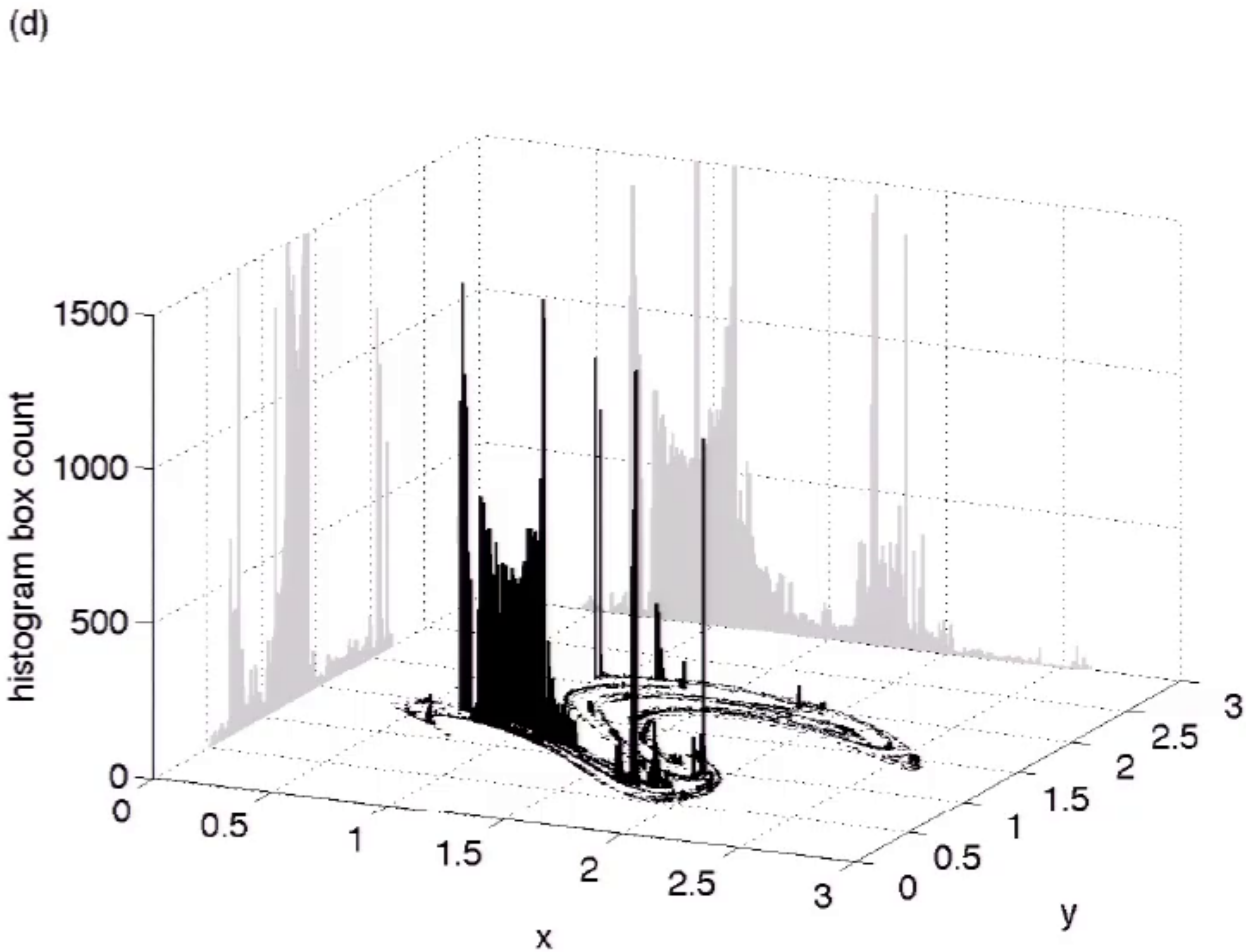
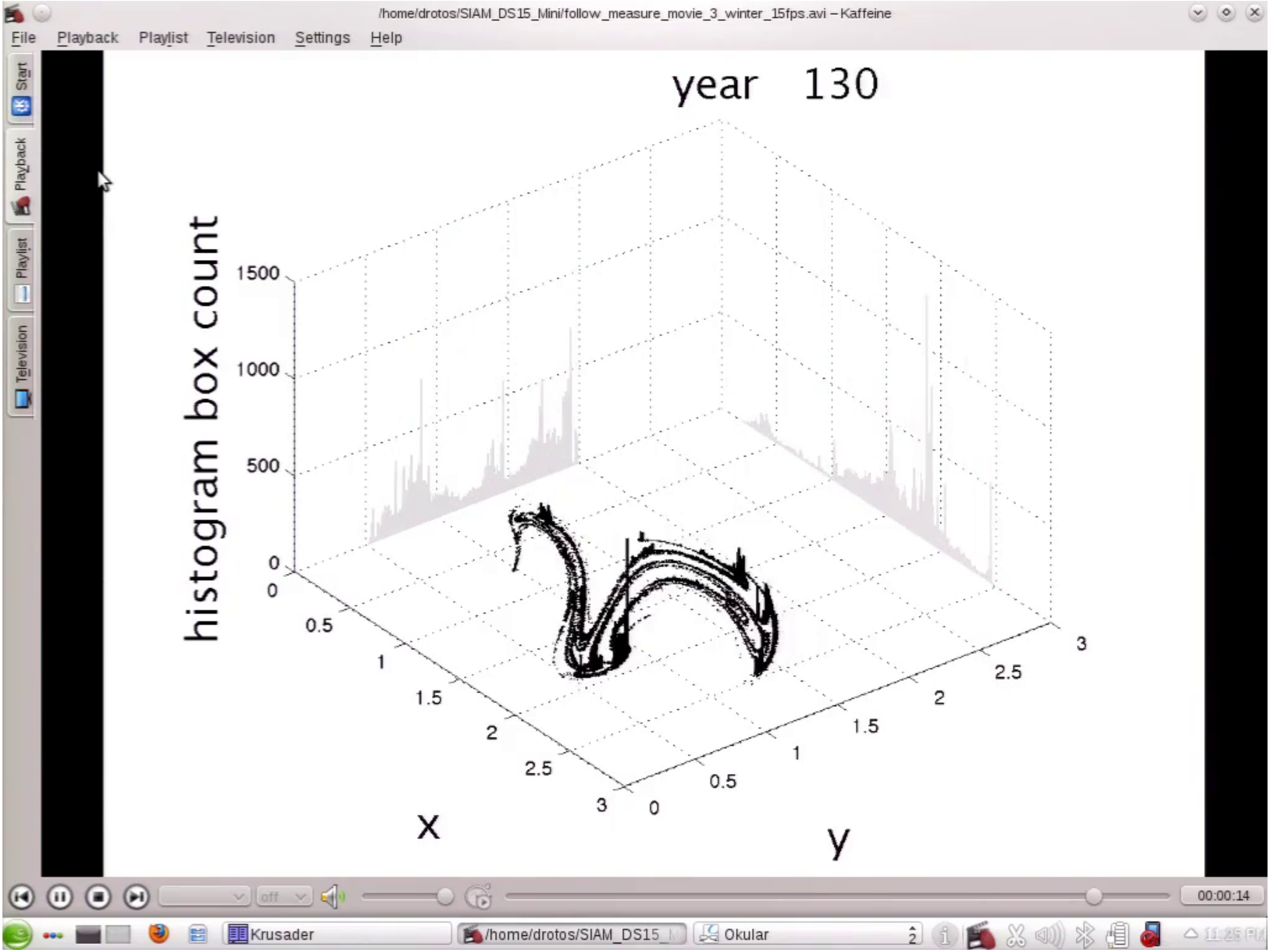


Natural measure (sample measure) of the snapshot attractor

[video] by T. Bódai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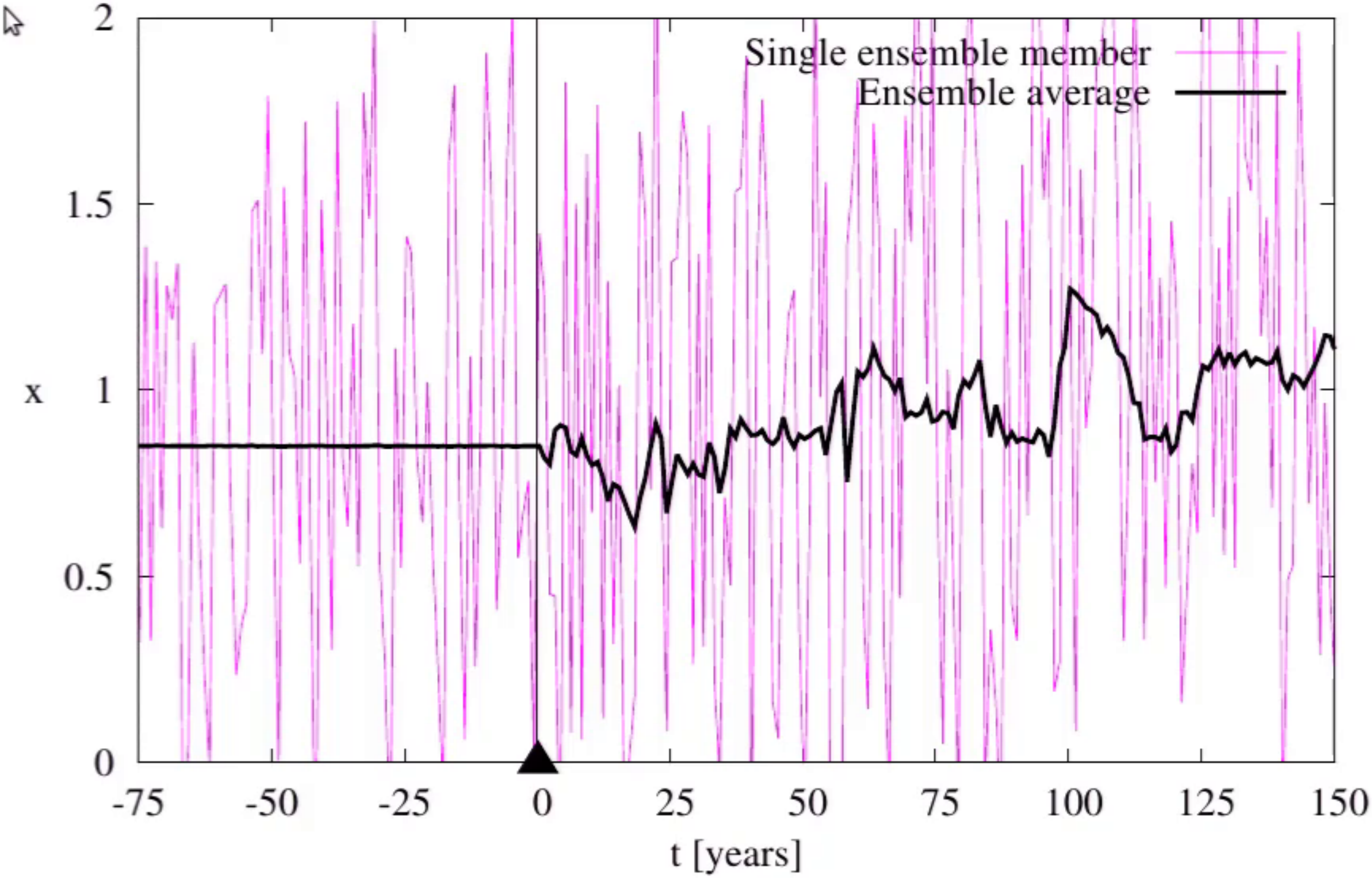
Natural measure (sample measure) of the snapshot attractor

[video] by T. Bódai

Time evolution

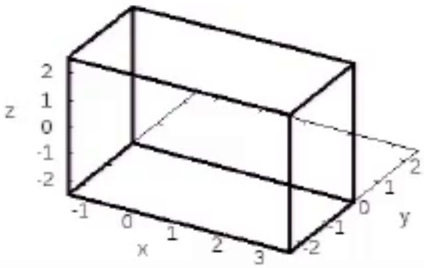


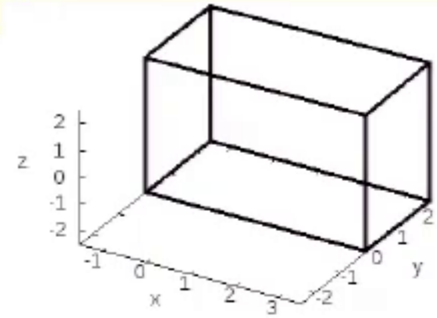
Ensemble average: constant over a stationary climate

When are initial conditions forgotten?

$N = 10^6$

$t_0 = 10$yr

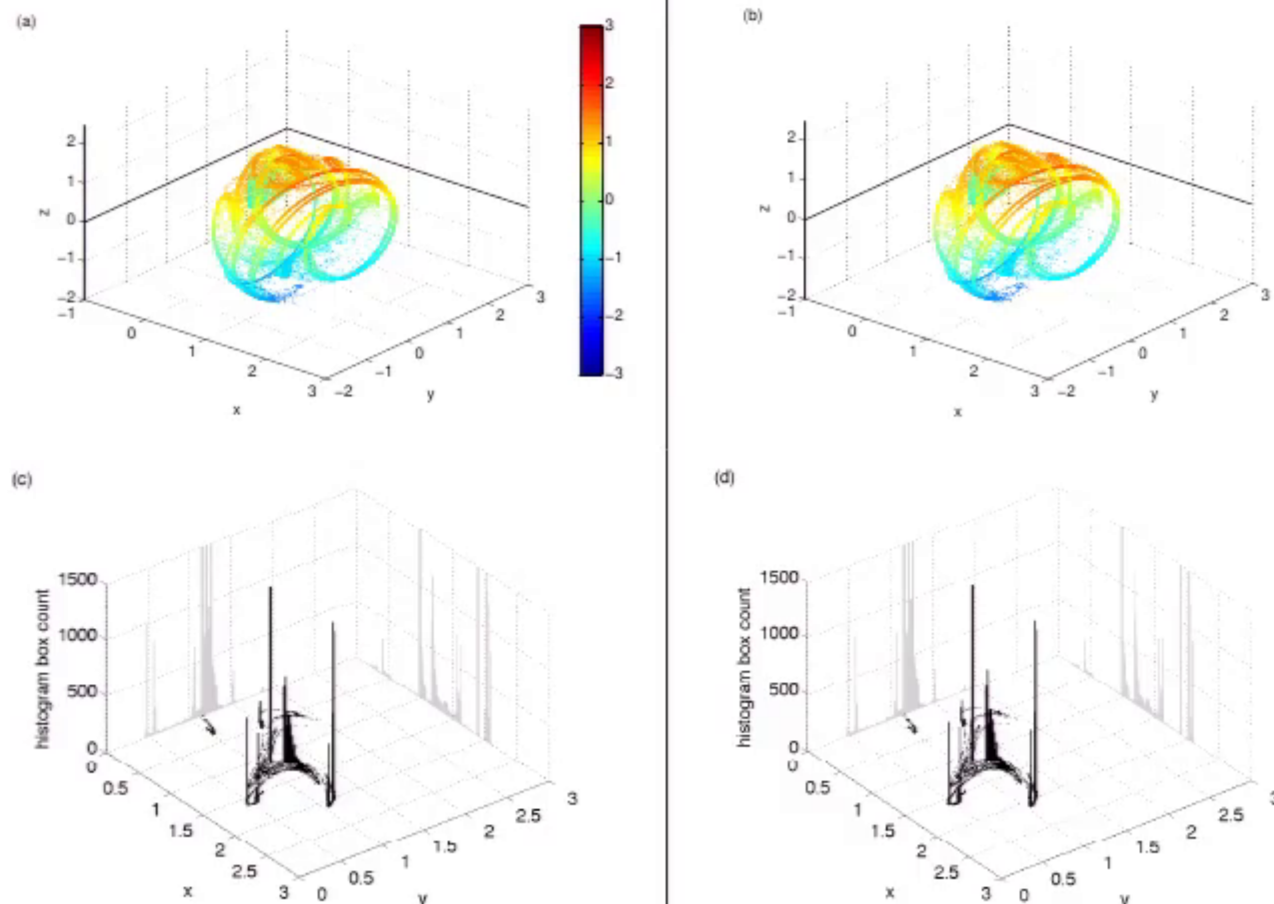




$N = 10^6$

$t_0 = 30$yr

$t = 50$yr



$t = 50$yr

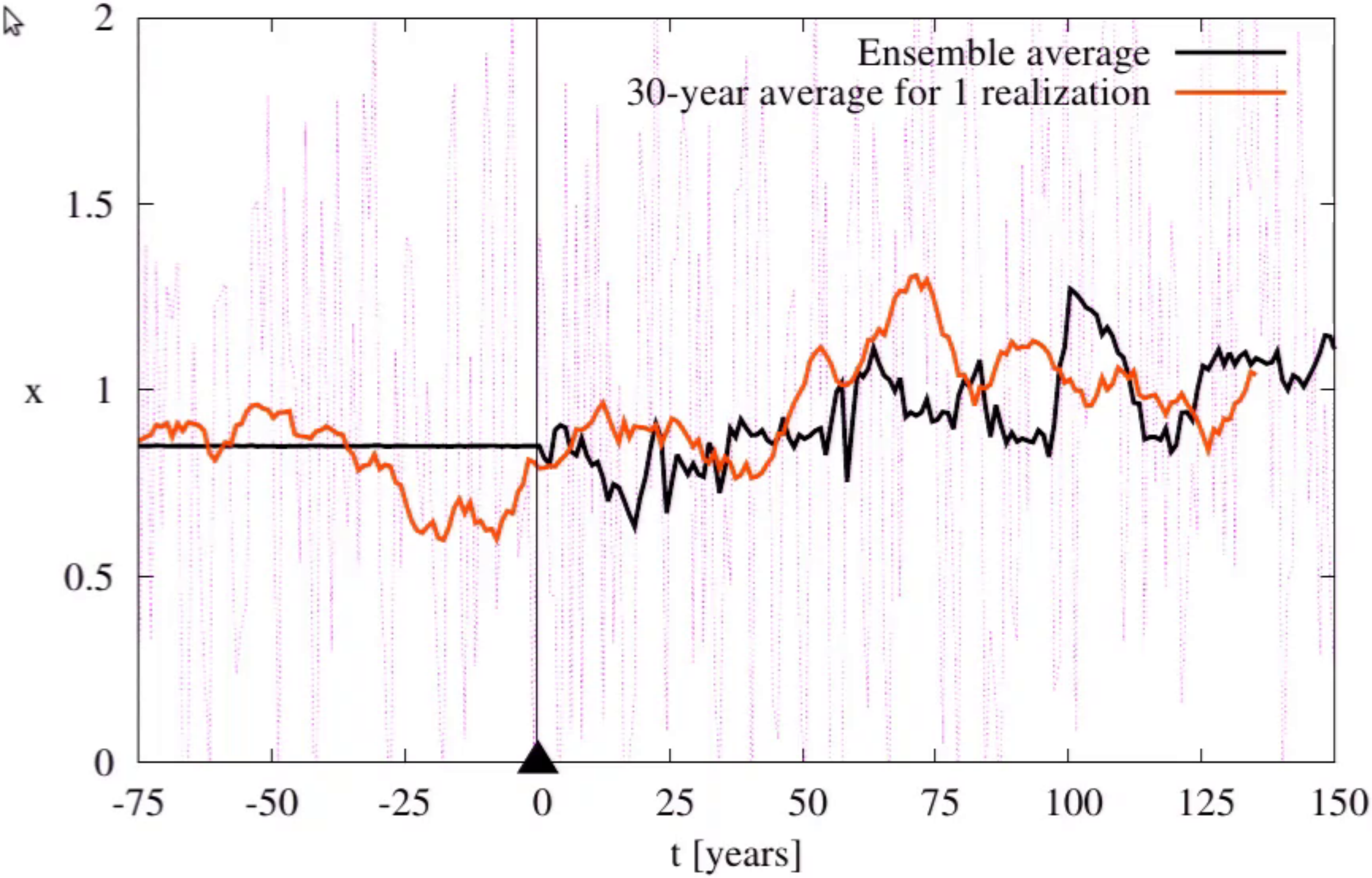
Finite convergence time

Detailed analysis:

The convergence time is only $t_c \approx 5$ years
→ practically, *there is no need to go back to the infinite past*

Drótos, Bódai and Tél, J. Climate **28**, 3275 (2015)

Single-realization 30-year average vs. ensemble average



Strong deviation from each other, different trends

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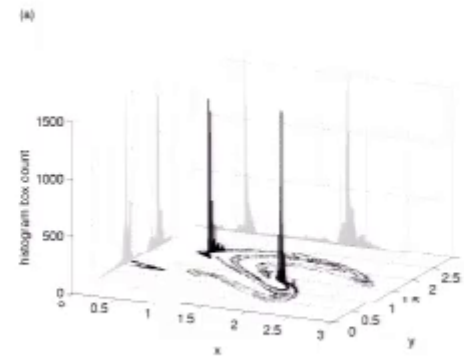
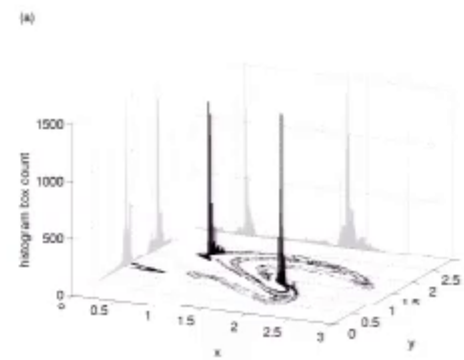
Ensemble and single-realization temporal statistics

- Natural measure of the snapshot attractor: $\mu(t)$

$$E(A(t)) = \int A \, \mathrm{d}\mu(t)$$



Ensemble and single-realization temporal statistics



- Natural measure of the snapshot attractor: $\mu(t)$

$$E(A(t)) = \int A \, \mathrm{d}\mu(t)$$

- Along a single realization:

$$E_\tau(A(t)) = \frac{1}{\tau} \int_{t-\tau/2}^{t+\tau/2} A(t') \, \mathrm{d}t'$$

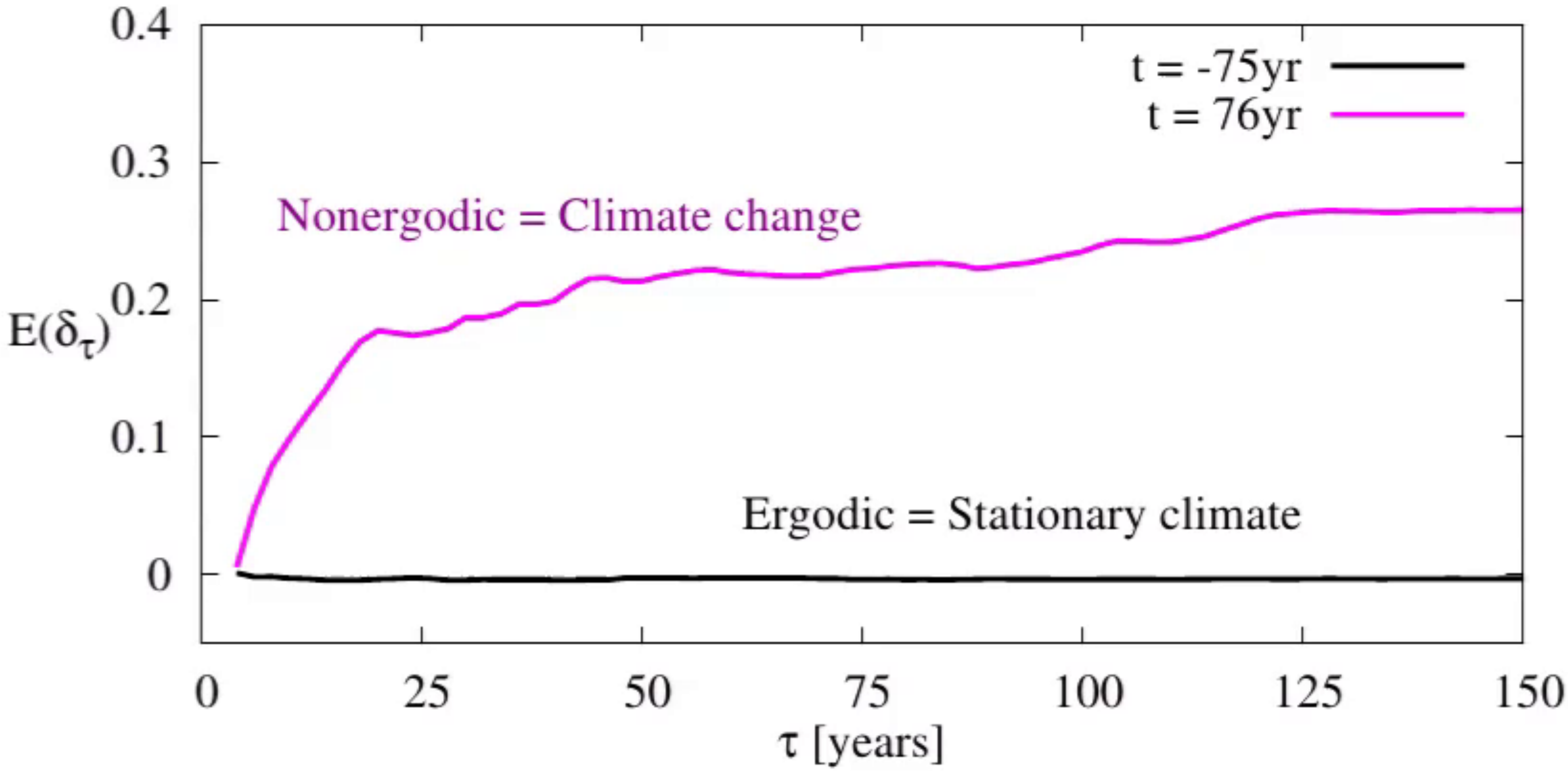
  on a finite window length τ (an infinite length is unrealistic)

- Define: a deviation from ergodicitiy for a single realization:

$$\delta_\tau(t) = E_\tau(A(t)) - E(A(t))$$

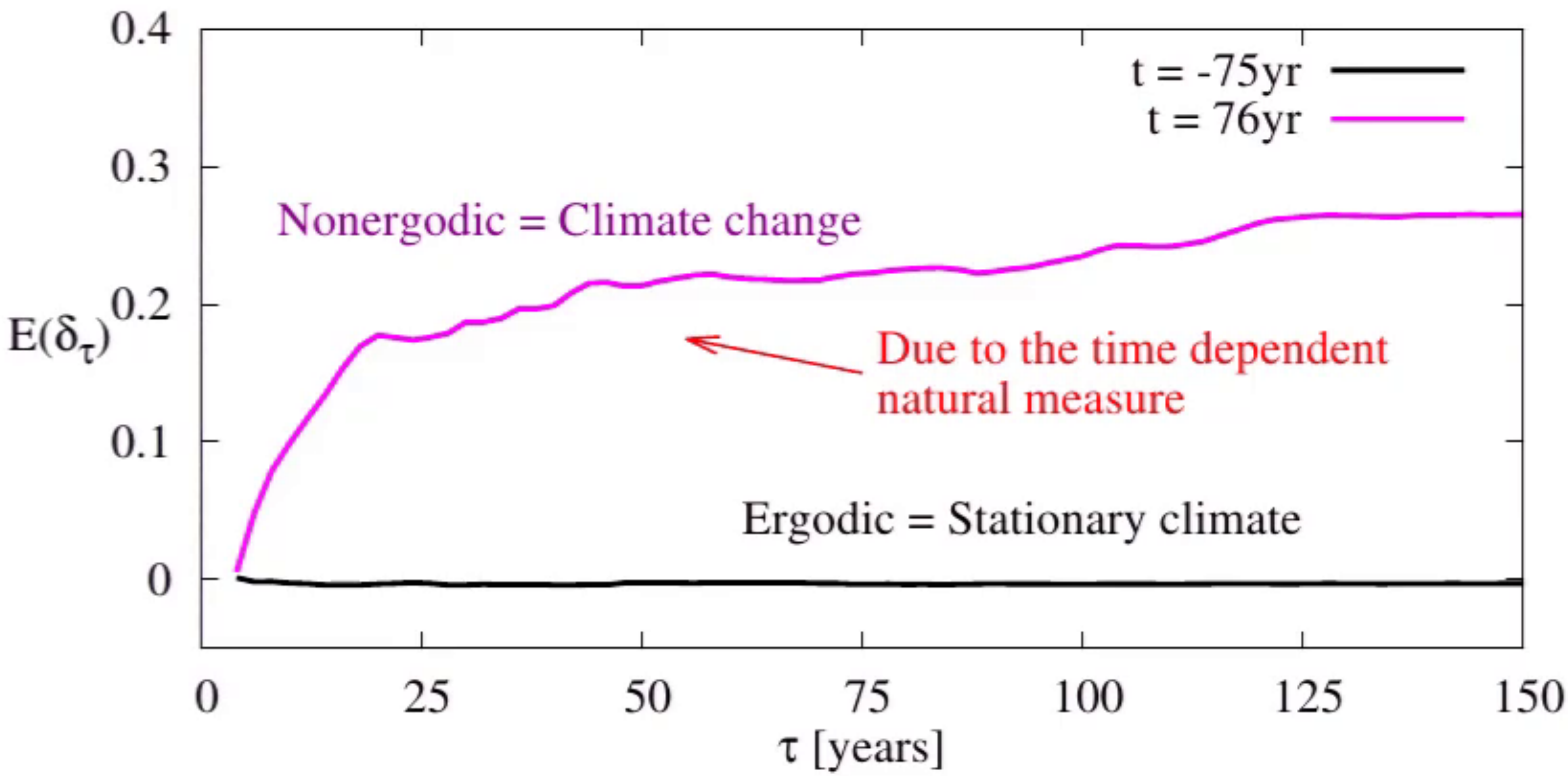
- Generate the pdf of δ_τ for $A = y$
  (Initialization in $t_0 = -250$ years with 10 000 trajectories)

The dependence of the average $E(\delta_\tau)$ on the window length



- Ergodic case: zero
- Nonergodic case: increases with τ

The dependence of the average $E(\delta_\tau)$ on the window length

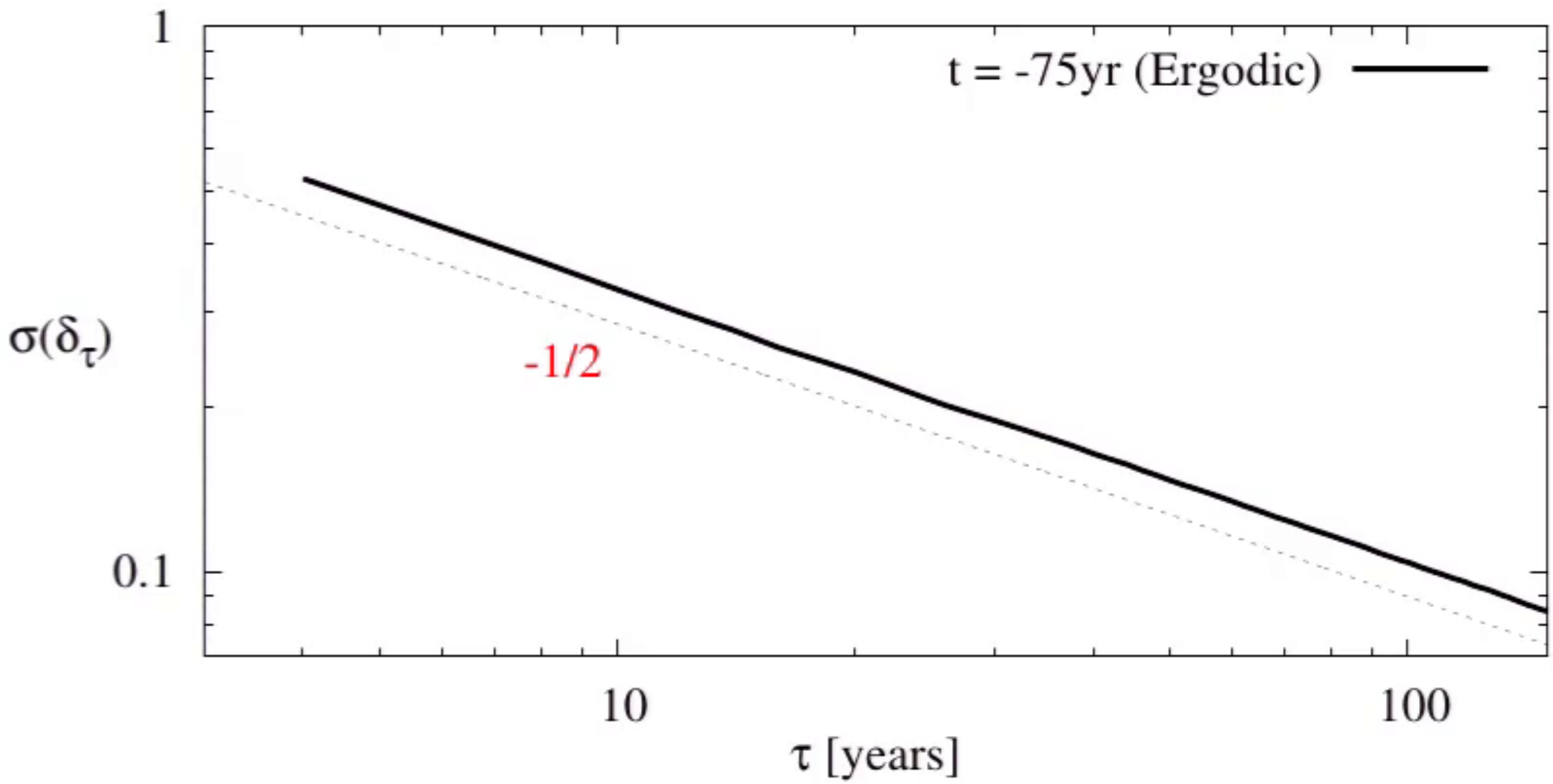


- Ergodic case: zero
- Nonergodic case: increases with τ

We suggest $E(\delta_\tau)$ to use as an indicator of nonergodicity (and of climate change), even for finite τ

The dependence of the standard deviation $\sigma(\delta_\tau)$ on τ

Ergodic case:

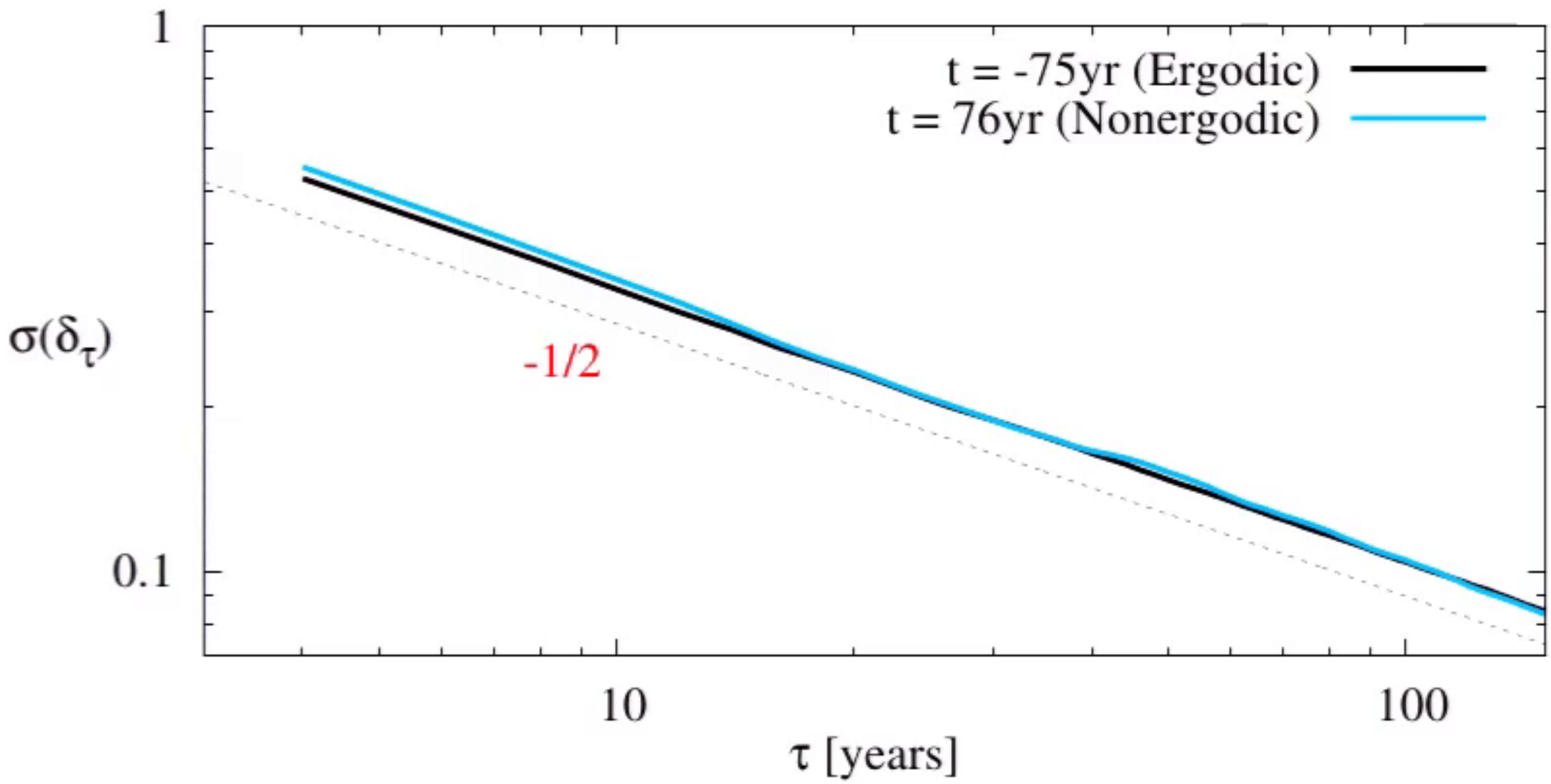


$\sigma(\delta_\tau) \sim 1/\sqrt{\tau}$, slow convergence to the single-trajectory ergodic behavior:

- a factor of 10 / 200 years
- without any characteristic time $\rightarrow \tau \rightarrow \infty \rightarrow$ unfeasible in practice

The dependence of the standard deviation $\sigma(\delta_\tau)$ on τ

Nonergodic case:



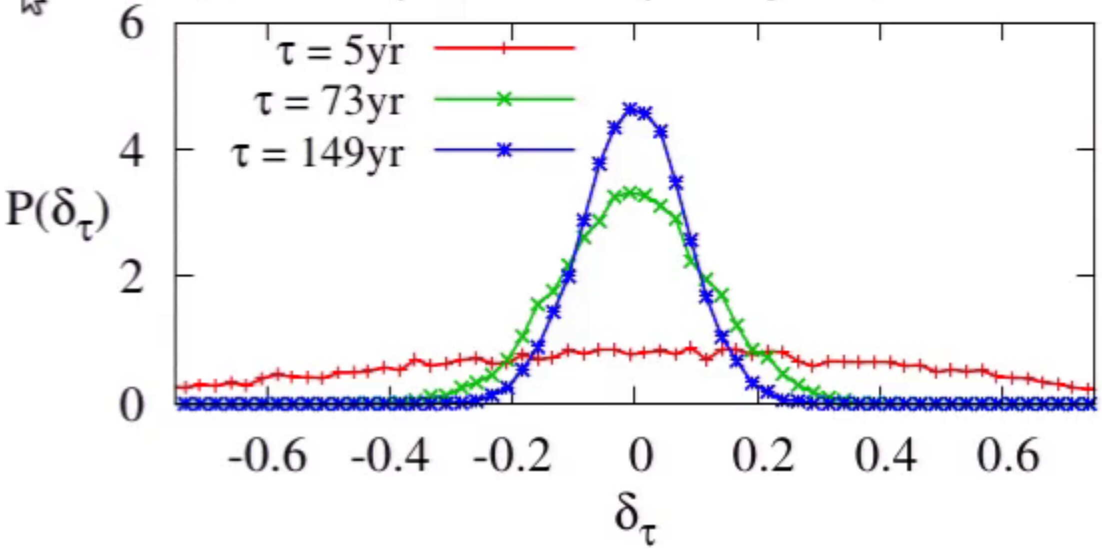
Again: $\sigma(\delta_\tau) \sim 1/\sqrt{\tau}$, slow convergence to $\delta_\tau = 0$:

- a factor of 10 / 200 years
- without any characteristic time $\rightarrow \tau \rightarrow \infty \rightarrow$ unfeasible in practice

Compare with the exponential convergence to the snapshot attractor

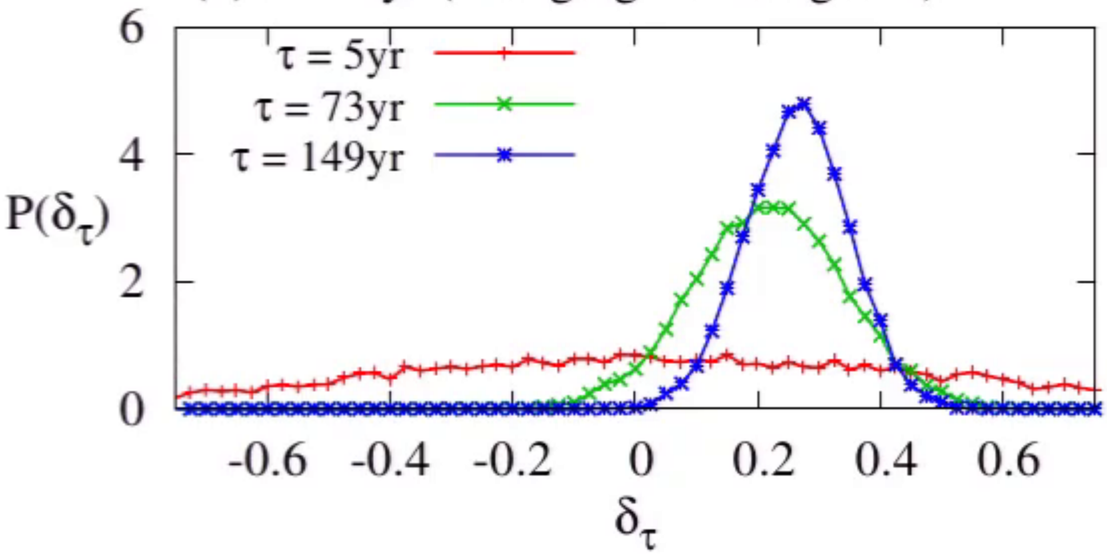
The pdf of the deviation δ_τ

(a) t = -75yr (stationary = ergodic)



Shrinking width
with increasing
window length τ

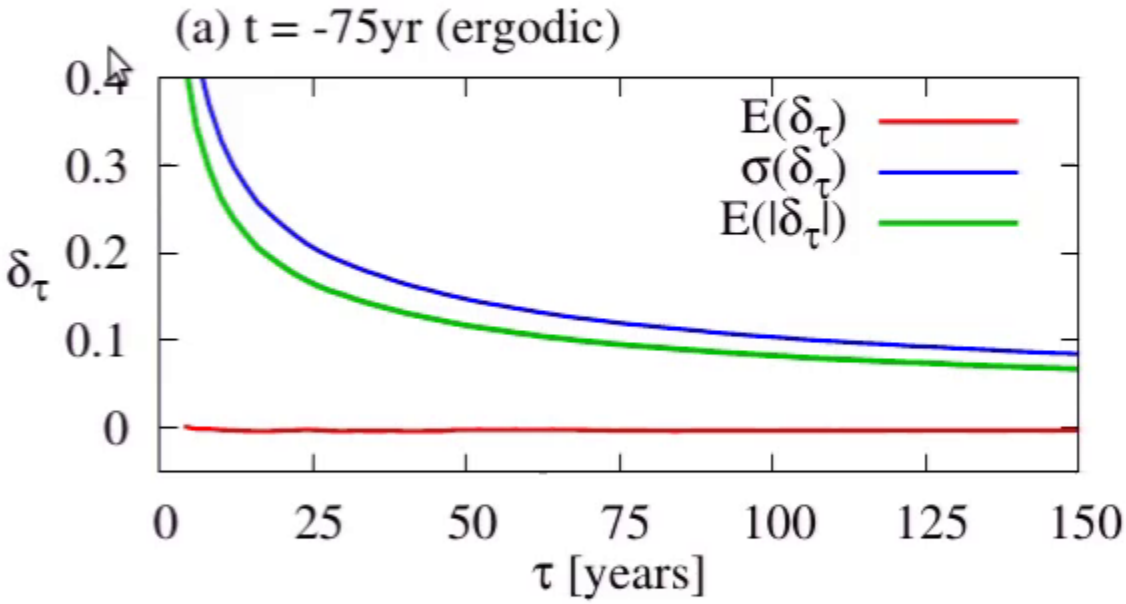
(b) t = 76yr (changing = nonergodic)



Expected deviation $E(\delta_\tau)$

shifting
in the nonergodic case

The dependence of $E(|\delta_\tau|)$ on τ

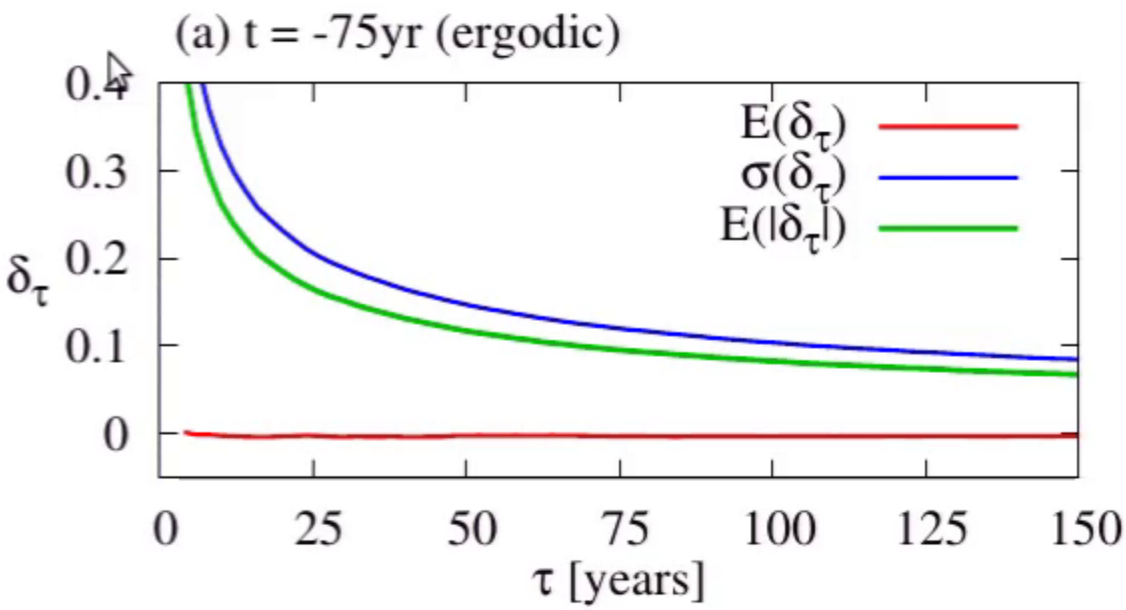


$E(|\delta_\tau|)$:

Expected absolute deviation

recall: $\delta_\tau = E_\tau(y) - E(y)$

The dependence of $E(|\delta_\tau|)$ on τ



$E(|\delta_\tau|)$:

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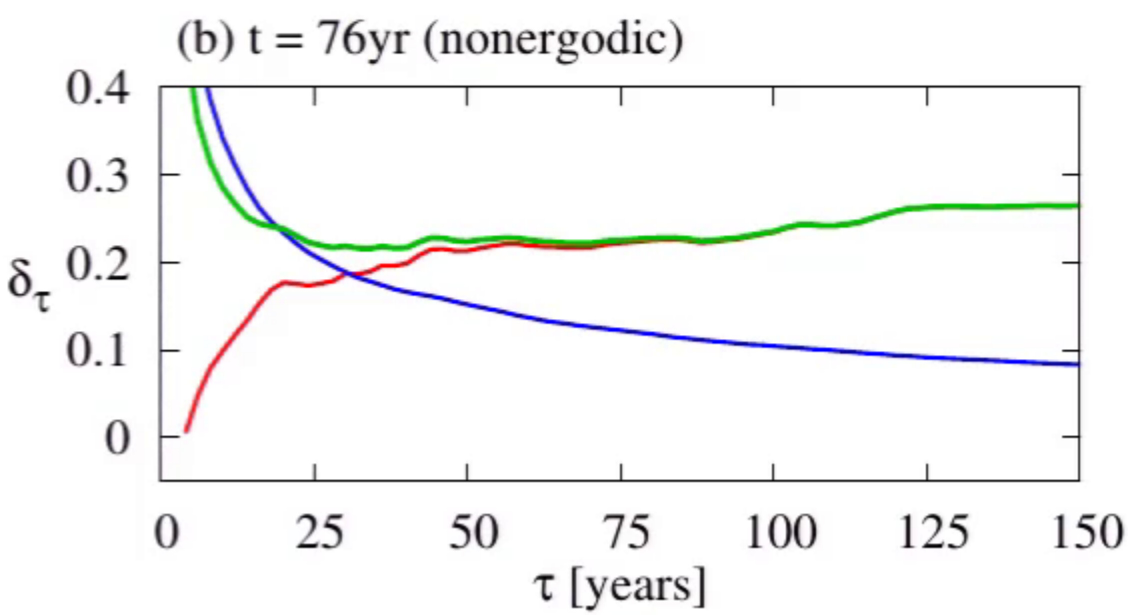
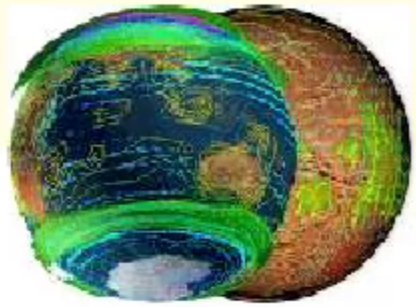


$E(|\delta_\tau|) \approx \max\left(E(\delta_\tau), \sigma(\delta_\tau)\right)$

Single-trajectory statistics are meaningless in a changing climate

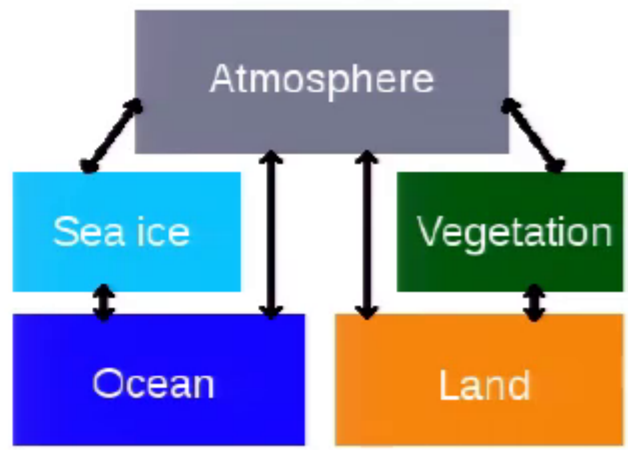
Drótos, Bódai and Tél, in preparation

Planet Simulator



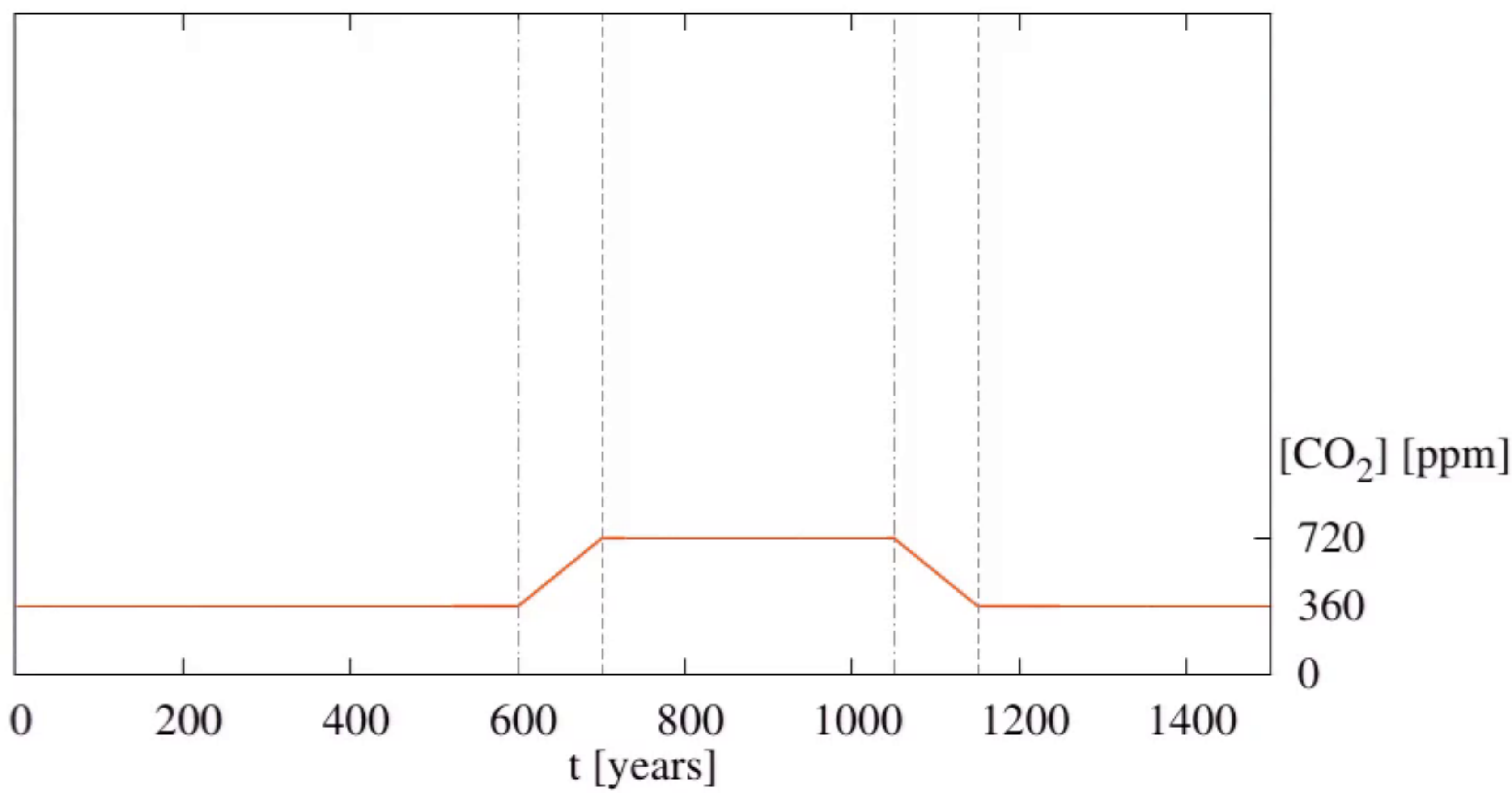
University of Hamburg

- Intermediate-complexity GCM (General Circulation Model)
- Conservation laws (momentum, mass, heat, water)
- Parameterizations
- Horizontal resolution: a few 100 km (spectral representation)
- 10 atmospheric layers
- Ocean: heat and water reservoir, no dynamics
- Degrees of freedom: $\approx 10^5$
- Open-source, free to download at `http://www.mi.uni-hamburg.de/plasim`

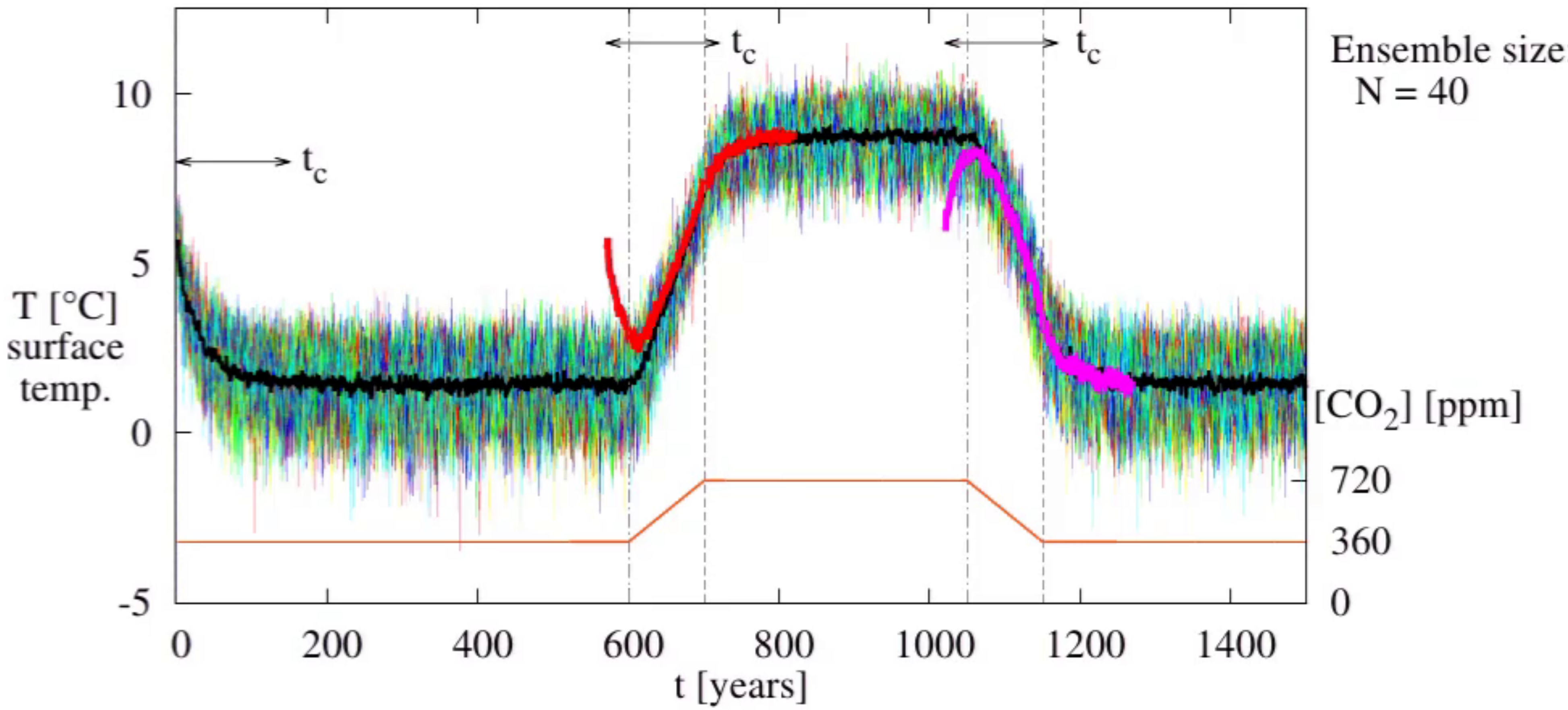


No ambition for realistic climate projections

Direct CO_2 forcing taken



Response in the temperature of a grid point in Central Europe



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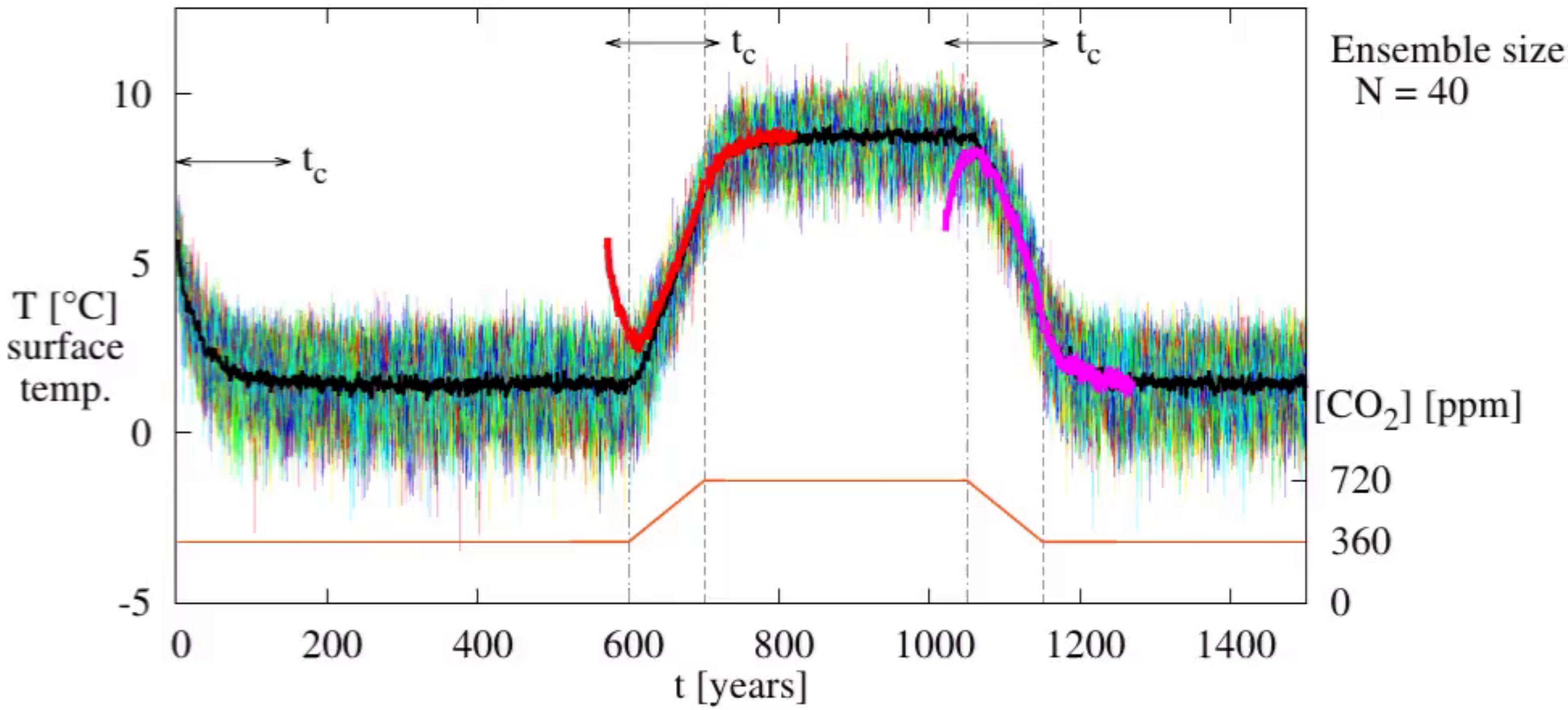
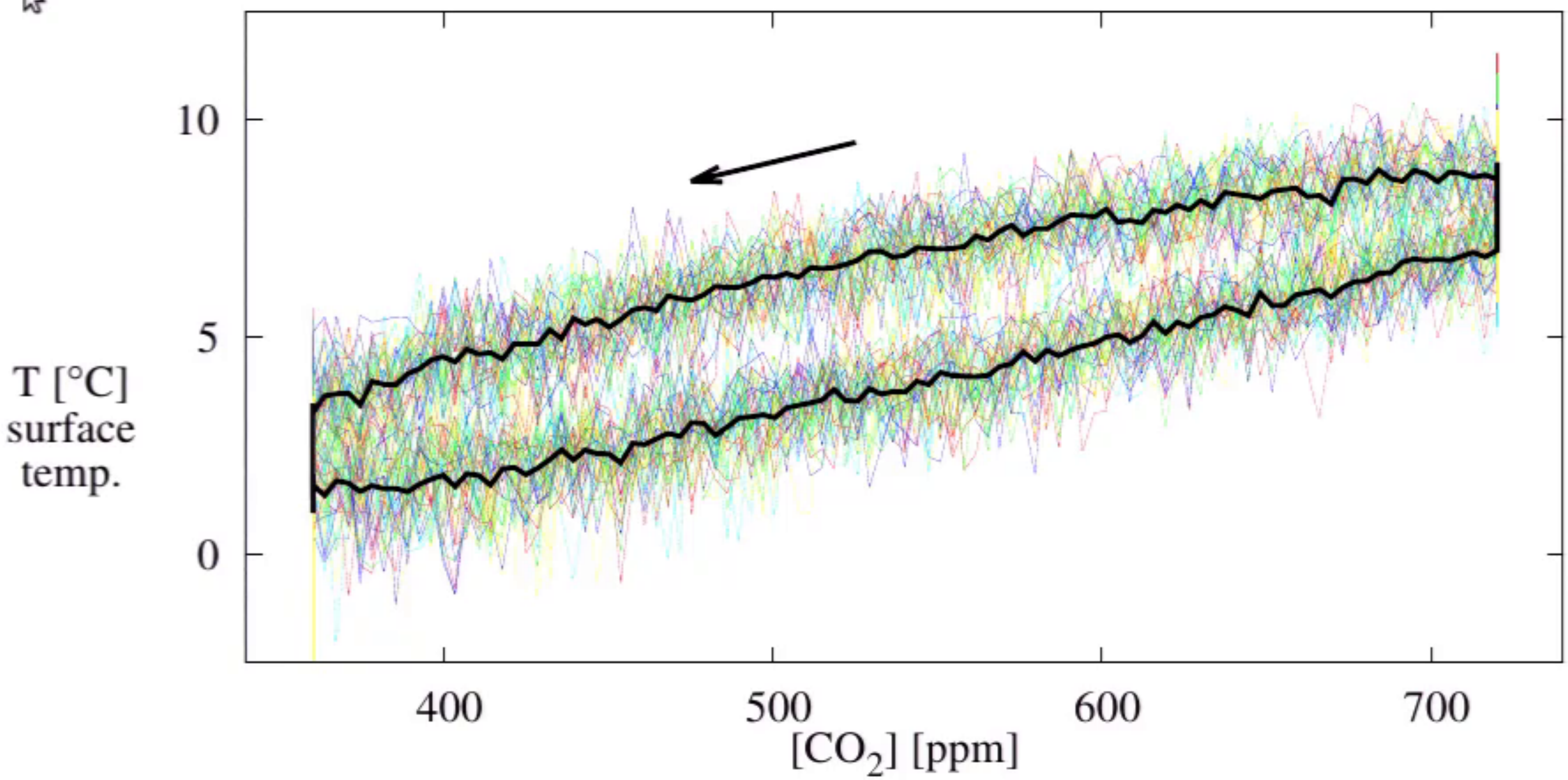


- Attracting property demonstrated
- Exponential convergence is found in a GCM
- Convergence time $t_c \approx 150$ years for any initialization
- Deviation from the shape of the CO_2 scenario

A different representation: T vs. $[CO_2]$



Dynamical hysteresis

The snapshot attractor approach is useful here

Herein, Márfy, Drótos and Tél, submitted