Stochastic homogenization of quasilinear Hamilton-Jacobi equations and geometric motions

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We study homogenization of non-convex viscous Hamilton-Jacobi equations in random environment

$$\partial_t u^\varepsilon - \varepsilon \operatorname{tr}\left(A\left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\right) D^2 u^\varepsilon\right) + H\left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\right) = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty).$$

where

- A, H are stationary in an ergodic environment
- and $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is coercive but not necessarily convex.

Problem : show that there exists *a homogenized Hamiltonian* $\overline{H} : \mathbb{R}^d \to \mathbb{R}$ such that $u^\varepsilon \to u$ where u solves

$$\partial_t u + \overline{H}(Du) = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty).$$

Typical examples :

- Viscous Hamilton-Jacobi equations :

$$\partial_t u^\varepsilon - \varepsilon \Delta u^\varepsilon + H\left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty),$$

(homogenization=open problem in random environment when $H = H(\xi, x)$ not convex in ξ)

- Forced mean curvature motion :

$$\partial_t u^\varepsilon - \varepsilon \, \mathrm{tr}\left(\left(I_d - \frac{Du^\varepsilon \otimes Du^\varepsilon}{|Du^\varepsilon|^2}\right) D^2 u^\varepsilon\right) + a\left(\frac{x}{\varepsilon}, \omega\right) |Du^\varepsilon| = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty),$$

where a is a forcing term : corresponds to a front propagation problem with normal velocity

$$V_{t,x} = \varepsilon \, \mathrm{curv} + a(\frac{x}{\varepsilon}, \omega).$$

(homogenization=open problem in random environment)

General remarks and references :

- In periodic environment, homogenization of HJ equations relies on the existence of a corrector.

  Lions-Papanicolau-Varadhan (unpublished), Evans ('92), Arisawa-Lions ('98), Capuzzo Dolcetta-Ishii ('01), Lions-Souganidis ('05)

- In random environment, existing proofs rely on the *subadditive ergodic Theorem* or on *duality techniques*, requiring the "convexity properties" of the equation.

  Souganidis ('99), Rezakhanlou-Tarver ('00), Lions-Souganidis ('05 and '10), Kosygina-Rezakhanlou-Varadhan ('06), Schwab (2009), Davini-Siconolfi ('11), Armstrong-Souganidis ('12),...

- For nonconvex HJ eq's, such subadditive quantity is not known...
  ...except in few particular settings.

  Armstrong-Tran ('14), Armstrong-Tran-Yu ('14), Gao ('15).

- $\longrightarrow$ Key idea : rely on a quantitative approach...
  ... developed so far for convex HJ eq's.

  Matic-Nolen ('12), Armstrong-C.-Souganidis ('14), Armstrong-C. ('15)

Some references for the homogenization of the MCM equation :

$$\partial_t u^\varepsilon - \varepsilon\, \mathrm{tr}\left(\left(I_d - \frac{Du^\varepsilon \otimes Du^\varepsilon}{|Du^\varepsilon|^2}\right) D^2 u^\varepsilon\right) + a\left(\frac{x}{\varepsilon},\omega\right)|Du^\varepsilon| = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty),$$

- In periodic environment :
  - Plane-like solutions : *Caffarelli-de la Llave ('01), Chambolle-Thouroude ('09).*
  - Homogenization : *Lions-Souganidis ('05), Caffarelli-Monneau ('14).*
  - Sign changing velocities : *Dirr-Karali-Yip. ('08), C.-Lions-S-Souganidis ('09), Barles-Cesaroni-Novaga ('11), ...*
  - Properties of the effective Hamiltonian : *Chambolle-Goldman-Novaga ('14).*
- In random environment :
  - Pinning phenomena in random media : *Dirr-Dondl-Scheutzow. ('11).*

Assumptions

We consider the problem

$$\partial_t u^\varepsilon - \varepsilon \,\mathrm{tr}\left(A\left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\right) D^2 u^\varepsilon\right) + H\left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega\right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty).$$

where

1. (A, H) is stationary ergodic satisfying a finite range condition,
2. $A = A(\xi, x, \omega)$ is $0-$homogeneous in ξ and $A = \dfrac{1}{2}\sigma\sigma^T$ with,
   $|\sigma(e,x,\omega)| + |D_x\sigma(e,x,\omega)| + \left|D_\xi\sigma(e,x,\omega)\right| \le C_0 \qquad \text{in } \partial B_1 \times \mathbb{R}^d \times \Omega,$
3. $H(\cdot,\cdot,\omega) \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ is such that, for every $t > 0$ and $\xi, x \in \mathbb{R}^d$,

$$H(t\xi, x, \omega) = t^p H(\xi, x, \omega) \qquad \text{and} \qquad c_0 |\xi|^p \le H(\xi, x, \omega) \le C_0 |\xi|^p$$

   and

$$|D_x H(\xi, x, \omega)| + |\xi| \left|D_\xi H(\xi, x, \omega)\right| \le C_0 |\xi|^p \qquad \text{in } \left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{R}^d \times \Omega.$$

4. "Lions-Souganidis (LS)" coercivity condition holds.

- Viscous Hamilton-Jacobi equations :

$$\partial_t u^\varepsilon - \varepsilon\Delta u^\varepsilon + H\left(Du^\varepsilon, \frac{x}{\varepsilon}\right) = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty),$$

where $H(\xi, x)$ is homogeneous and coercive in ξ : $\exists p > 1$ with

$$H(t\xi, x) = t^p H(\xi, x) \qquad \text{and} \qquad c_0 |\xi|^p \leq H(\xi, x) \leq C_0 |\xi|^p$$

but not necessarily convex.

- Forced mean curvature motion :

$$\partial_t u^\varepsilon - \varepsilon \,\mathrm{tr}\left(\left(I_d - \frac{Du^\varepsilon \otimes Du^\varepsilon}{|Du^\varepsilon|^2}\right) D^2 u^\varepsilon\right) + a\left(\frac{x}{\varepsilon}\right)|Du^\varepsilon| = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty),$$

where the forcing field a is positive, Lipschitz, bounded and satisfies the Lions-Souganidis (LS) condition

$$\inf_{x\in\mathbb{R}^d} \left(a^2(x) - (d-1)\,|Da(x)|\right) > 0 \quad \mathbb{P}\text{–a.s.}$$

(needed for homogenization in periodic setting (Caffarelli-Monneau (14')))

- Anisotropic forced mean curvature motion :

$$\partial_t u^\varepsilon - \varepsilon \operatorname{tr}\left(A\left(\frac{Du^\varepsilon}{|Du^\varepsilon|}, \frac{x}{\varepsilon}\right) D^2 u^\varepsilon\right) + \left|B\left(\frac{Du^\varepsilon}{|Du^\varepsilon|}, \frac{x}{\varepsilon}\right) Du^\varepsilon\right| = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty),$$

  where $A = \sigma\sigma^T$ with $\sigma(p/|p|, x)p = 0$ and the (LS) condition holds :

$$\inf_{(e,x)\in\partial B_1\times\mathbb{R}^d} \left[|B(e,x)e|^2 - |\sigma(e,x)|^2\left(|\sigma_x(e,x)|^2 + |B_x(e,x)|\right)\right] > 0.$$

- Examples of randomness
  - Sum of i.i.d. r.v. :

$$a(x) = 1 + \sum_{k\in\mathbb{Z}^d} a_0(x-k, Z_k)$$

    where $a_0 = a_0(x,z)$ is deterministic with compact support and $(Z_k)_{k\in\mathbb{Z}^d}$ are i.i.d.

  - Poisson point process :

$$a(x) = 1 + \left(\sum_k a_0(x - Y_k, Z_k)\right) \wedge M$$

    where $a_0 = a_0(x,z)$ is deterministic with compact support, (Y_k) is a stationary Poisson point process and $(Z_k)_{k\in\mathbb{Z}^d}$ are i.i.d.

Theorem (Homogenization)

Under the above assumptions, there exists universal exponents $\alpha, \beta \in (0,1)$ and a function $\overline{H} \in C^{0,\beta}_{\rm loc}(\mathbb{R}^d)$ satisfying, for every $\xi \in \mathbb{R}^d$,

$$c_0\,|\xi|^p \leq \overline{H}(\xi) \leq C_0\,|\xi|^p$$

such that, for every $T \geq 1$ and $u, u^\varepsilon \in W^{1,\infty}(\mathbb{R}^d \times [0,T])$ satisfying

$$\begin{cases} \partial_t u^\varepsilon - \varepsilon\,\mathrm{tr}\left(A\left(Du^\varepsilon, \frac{x}{\varepsilon}\right) D^2 u^\varepsilon\right) + H\left(Du^\varepsilon, \frac{x}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^d \times (0,T], \\ \partial_t u + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^d \times (0,T], \\ u^\varepsilon(\cdot,0) = u(\cdot,0) & \text{on } \mathbb{R}^d, \end{cases}$$

we have

$$\mathbb{P}\left[\sup_{R\geq 1}\limsup_{\varepsilon\to 0}\sup_{(x,t)\in B_R\times[0,T]} \varepsilon^{-\alpha}\,|u^\varepsilon(x,t) - u(x,t)| = 0\right] = 1.$$

Note : one can choose $\beta \approx 2/7$.

Previous approaches based on the metric problem rely on

- sub-additive ergodic theorem
- applied to the solution $m_\mu := m_\mu(x, y)$ of the point-to-point metric problem :

$$\begin{cases} -\operatorname{tr}\left(A(x)D^2 m_\mu\right) + H(Dm_\mu, x) = \mu & \text{in } \mathbb{R}^d \backslash B_1(y), \\ m_\mu = 0 & \text{on } \partial B_1(y). \end{cases}$$

- Then $\overline{H}$ is defined as a kind of Fenchel conjugate of $\overline{m}_\mu(z) := \lim_{t\to+\infty} m_\mu(tz, 0)/t$.

$\longrightarrow$ Requires the convexity of the equation.

Our main idea : use a quantitative approach.

- Analysis of the *point-to-plane* metric problem.
- Obtain variance estimates for its solutions
- Derive the convergence of its solution by
  - the variance estimate
  - and a finite speed of propagation property

Conclusion and open problems

- We have obtained the homogenization and the convergence rate for viscous HJ equations in random media under the structural conditions
  - homogeneity and coercivity of the Hamiltonian,
  - $0-$homogeneity of $A = A(\xi, x)$ w.r.t. ξ,
  - Finite range condition.
- Can one get rid of (one of) these conditions ?
- Other classes of problems (time-dependent ?).
- Properties of the homogenized Hamiltonian ?