Applications of Koopman Mode Analysis

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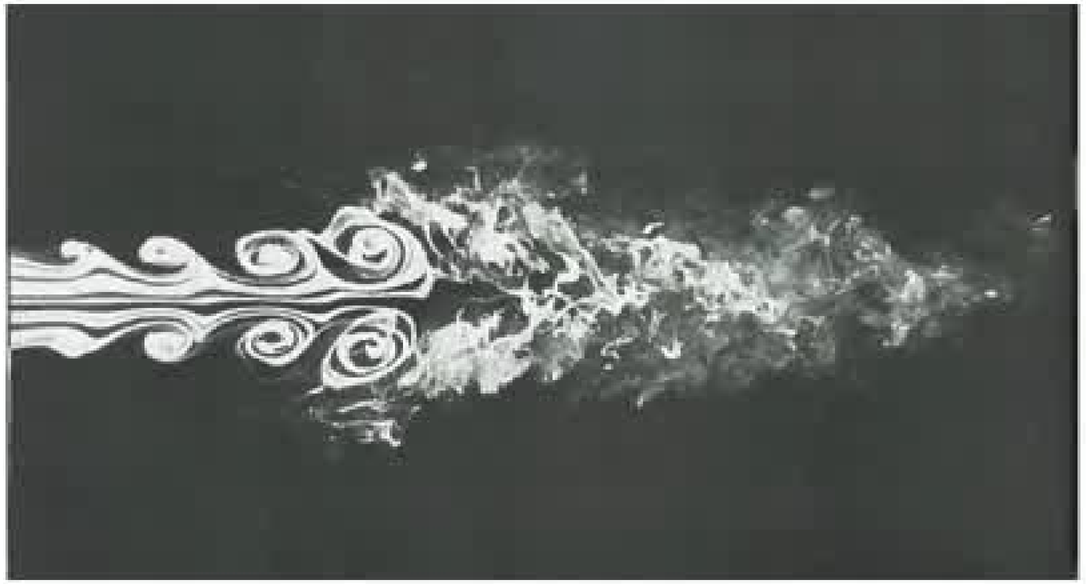
Funding: AFOSR

Questions on Flow Dynamics

Van Dyke:
Album of Fluid Motion



$$\mathbf{z}_0 \Longrightarrow \mathbf{z}_t = \mathbf{F}(\mathbf{z}_0)$$

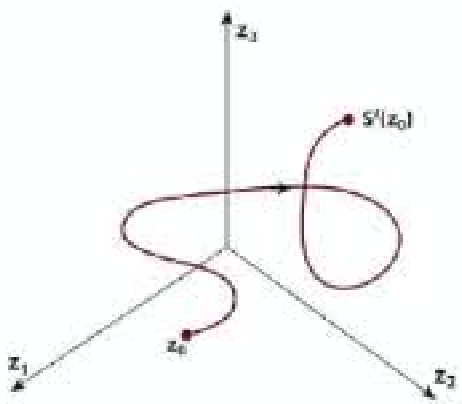
Relevant flow constituents?
Coupling?
Bifurcations?

- **F:** (integral of) Navier-Stokes + Reaction-Diffusion + BC
- Even with a accurate model, difficult to solve **F** under realistic conditions
  Analytical: symmetries → special functions
  Computational: Energy cascading → structures on many scales → large lattices

- Can one use high-resolution, high-frequency experimental data?

Flow dynamics vs. noise?

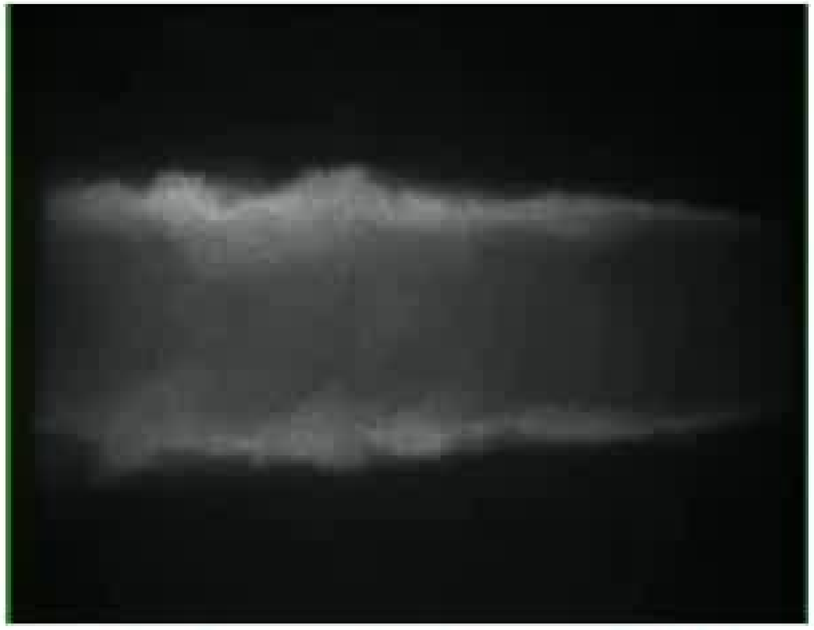
Koopman Operator *[Koopman, Mezic, Rowley,..]*



$$\mathbf{z}_0 \Longrightarrow \mathbf{z}_t = \mathbf{F}(\mathbf{z}_0)$$

$$\psi(\mathbf{z}) \in \mathcal{F} \qquad \text{(Observable)}$$

$$\mathcal{K}^t : \psi(\mathbf{z}_0) \Longrightarrow \psi(\mathbf{z}_t) = \psi \circ \mathbf{F}(\mathbf{z}_0)$$



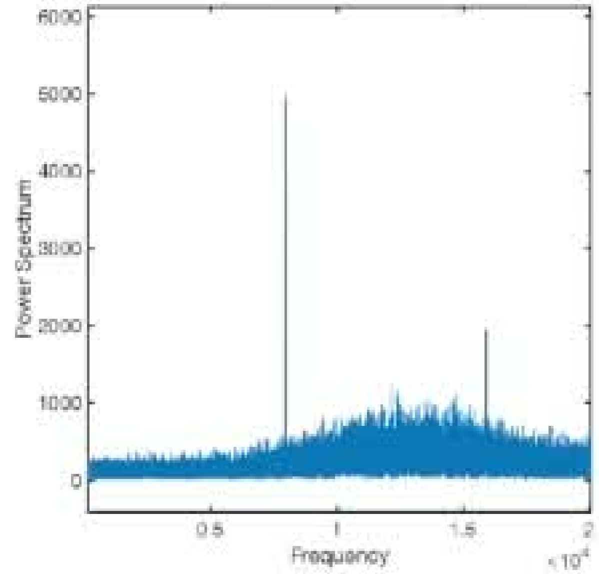
Combustion flow behind a V-shaped bluff body

State: {P(z), v(z), T(z), c(z)}

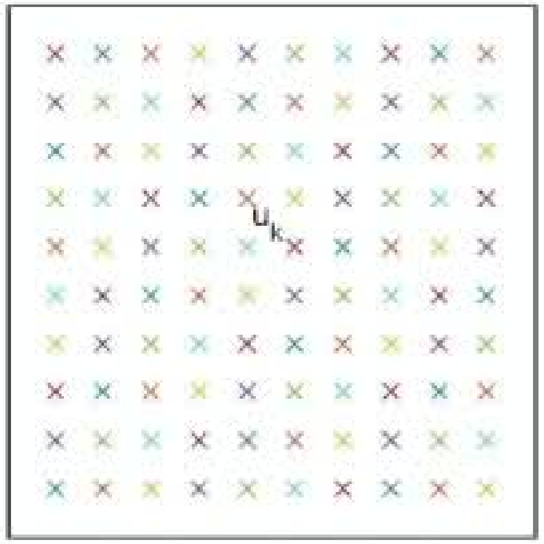
- $\mathcal{K}^t$: Infinite dimensional linear operator
- Independent of $\psi(\mathbf{z})$
- What can we learn about **F** from $\mathcal{K}^t$?

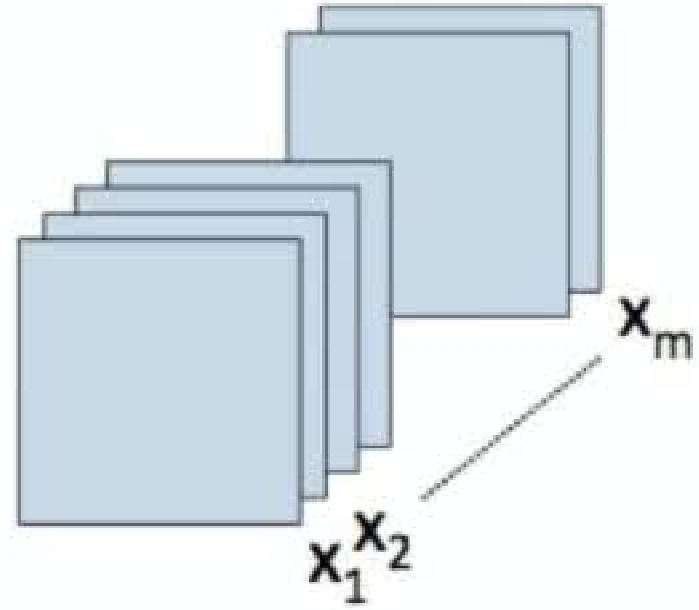
Theorem: Under general conditions, spectral properties of **F** are contained in the eigenspectrum of $\mathcal{K}^t$ [Koopman & vonNeumann, Rowley et al.]

- Model-free estimates for the flow; may be used to
  - validate models
  - search for flow constituents, locate bifurcations



Dynamic Mode Decomposition





(Kutz, Fu, et al.: 2015)

- Measurements & data: $\mathbf{u} = [\psi_1, \psi_2, ..., \psi_N]^T$
- For equally spaced snapshots: $\mathbf{u}_{k+1} \approx A\mathbf{u}_k$;
  $\boldsymbol{u}_k$: "data" from the k^{th} snapshot
- With N successive equally-spaced snapshots:

$$[\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{M+1}] = A\,[\mathbf{u}_0, \mathbf{u}_1, ..., \mathbf{u}_M]$$

- For sufficiently large M, N → compute spectrum of A (SVD)

DMD and Koopman Operator

- $\mathcal{K}^t$ acts on functions: $\mathcal{K}^t : \psi \to \psi \circ F$
- A acts on data: $A \cdot \boldsymbol{u}_n \approx \boldsymbol{u}_{n+1}$

<u>**Theorem**</u> (*Tu, Rowley, …*): Suppose Θ is an eigenfunction of $\mathcal{K}^t$ with eigenvalue Λ, and Θ lies in the span of the set of "measurements" $\Psi = \{\psi_1, \psi_2, \ldots, \psi_N\}$, so that

$$\Theta(\boldsymbol{z}) = \boldsymbol{w}^* . \Psi(\boldsymbol{z}) .$$

Suppose further that $\boldsymbol{w}$ lies in the range of the data matrix $\boldsymbol{U}$. Then Λ is an eigenvalue of A with a left eigenvector $\boldsymbol{w}^*$.

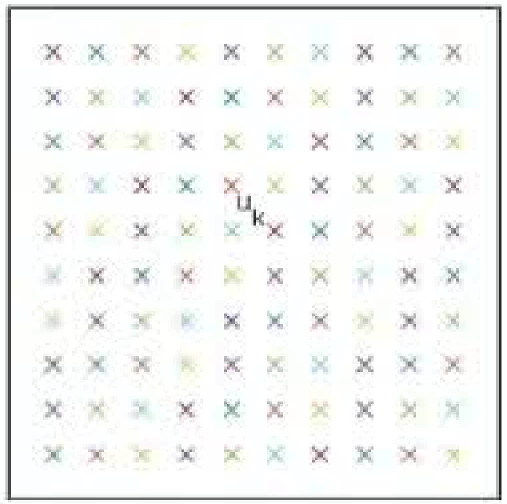
N.B.: data gives $\Theta(\Psi)$.

- The right eigenvectors of A are <u>Koopman Modes</u> $\boldsymbol{v}_k$
- Evolution of a state is given by

$$u_0 = \sum a_k \, \boldsymbol{v}_k \Longrightarrow u_n = \sum \Lambda_k^n a_k \, \boldsymbol{v}_k$$

- Koopman eigenfunctions can be computed. If space is not large enough

  → add measurements; extended DMD, KB DMD, etc.

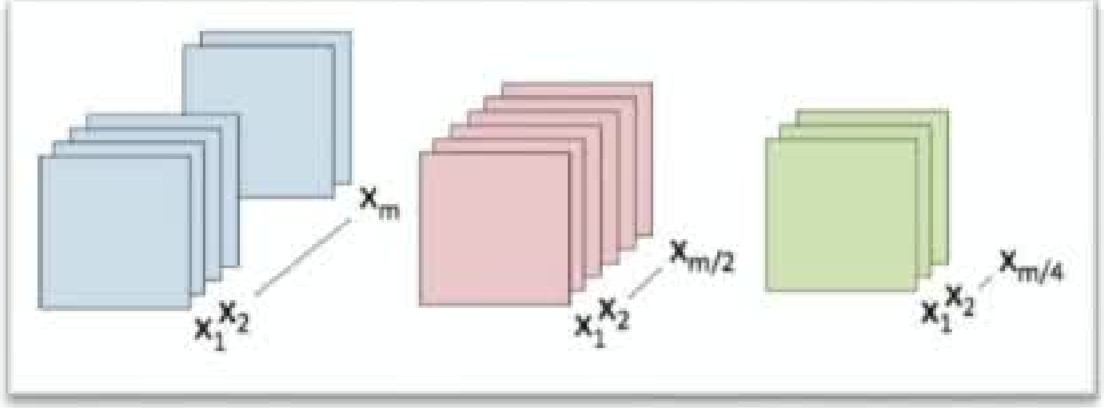


Robust Modes

Look for common
Koopman Eigenvalues/Modes in

Several nominally
identical experiments

or



- Real parts of eigenvalues from sub-groups are close

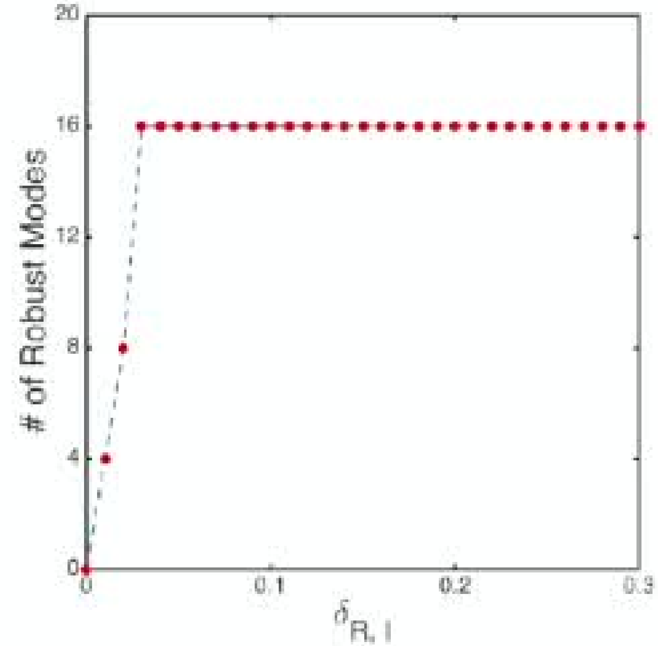
$$\max_g(|\,\mathrm{Im}(\Lambda_i) - \mathrm{Im}(\Lambda_j^{(g)})|) \le \delta_I$$

- Imaginary parts are close

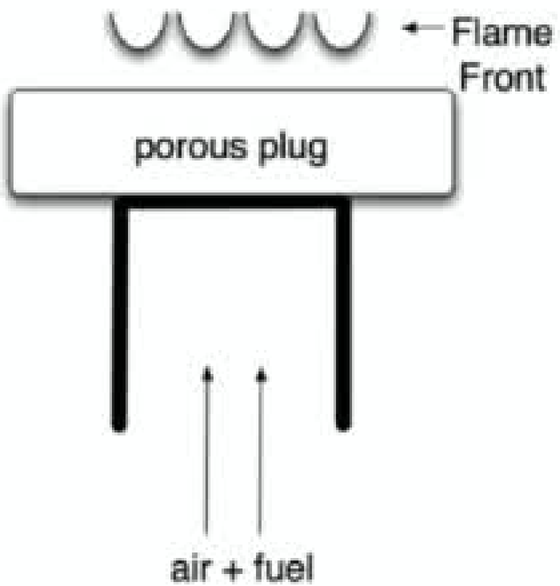
$$\max_g(|\,\mathrm{Re}(\Lambda_i) - \mathrm{Re}(\Lambda_j^{(g)})|) \le \delta_R$$

- Eigen-functions are close (modulo phase)

$$\max_g\{\min_\theta |\exp(i\theta)\Phi_i(\mathbf{x}) - \Phi_j^{(g)}(\mathbf{x})|\} \le \Delta$$



Cellular Flame Patterns *[Gorman]*



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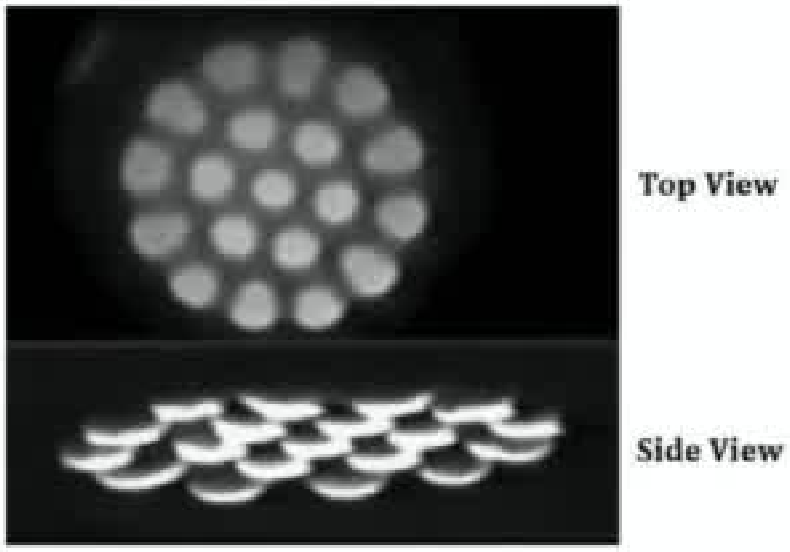

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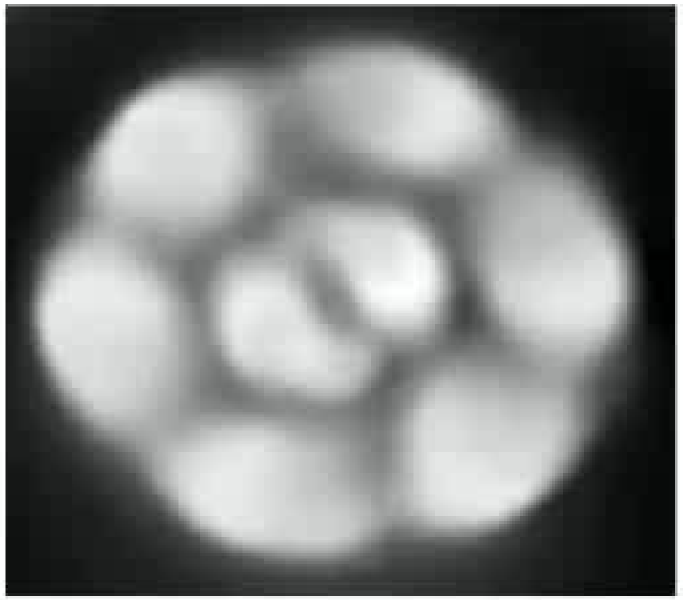
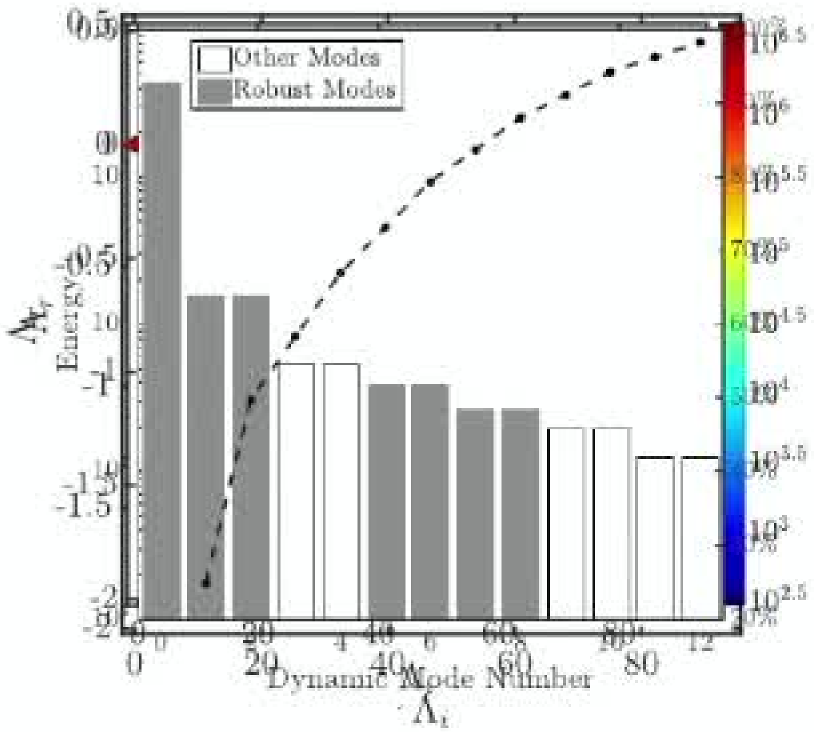
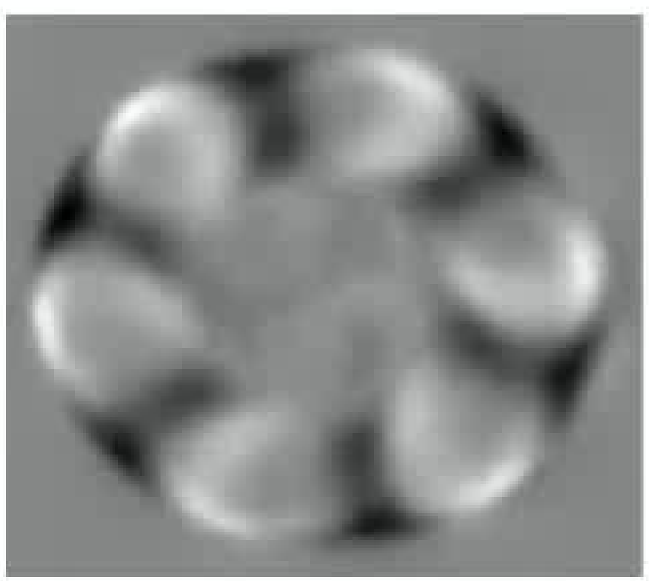


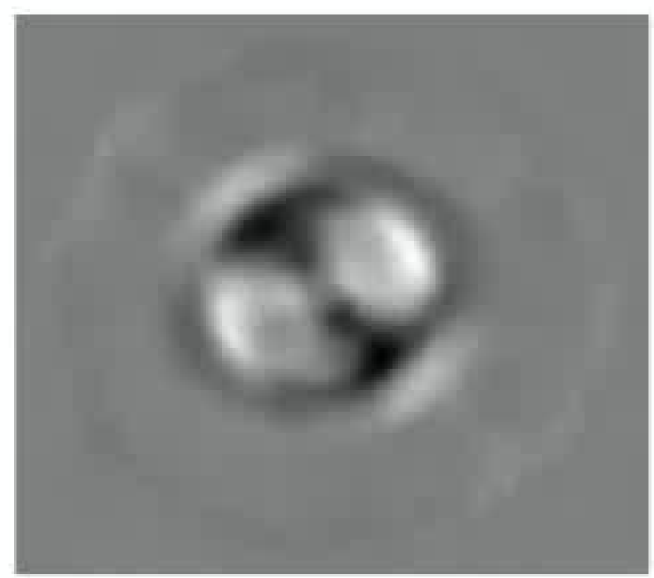
- Spontaneous symmetry breaking → a wide range of stationary and dynamic states
- *e.g.*, "Double Rotating State"

Dynamic Mode Decomposition



- DMD → many modes, including periodic expansions of irregular constituents)
- Robust modes: multiple sub-groupings of snapshots
- Mode “energies;” robust modes are not necessarily those with largest energies
- Flow constituents can be established





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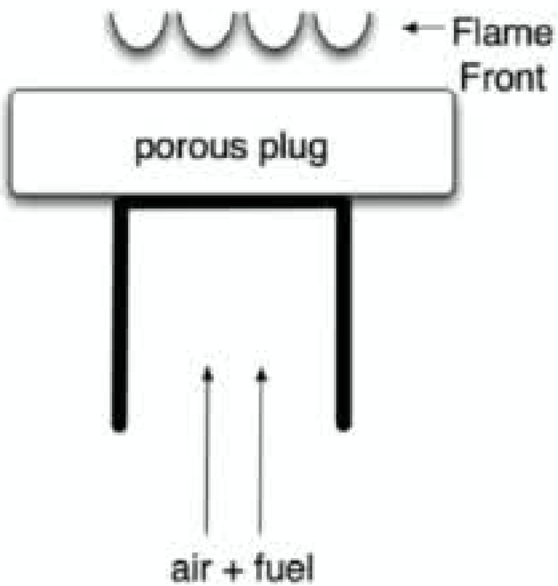

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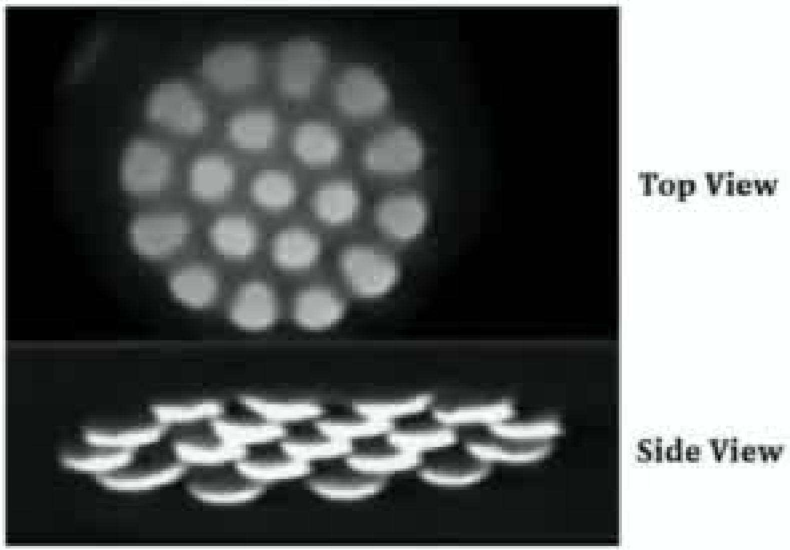

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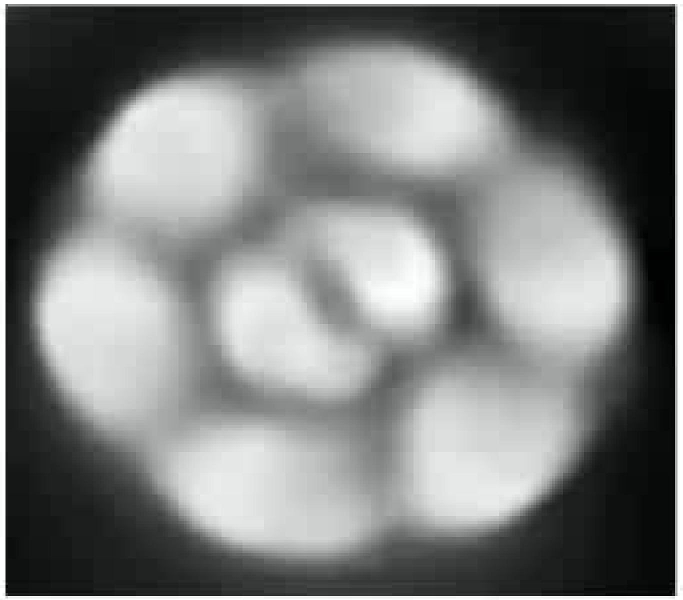

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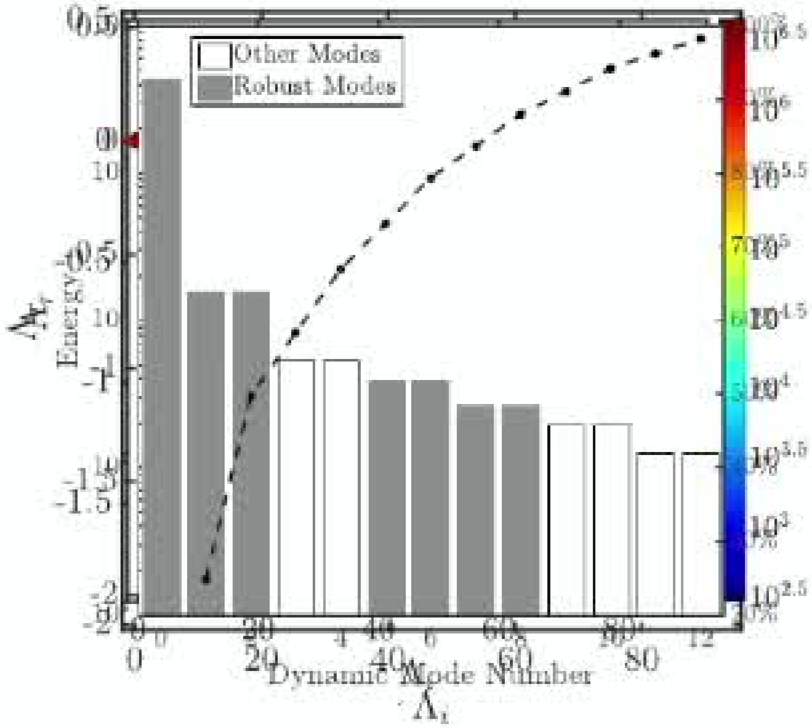

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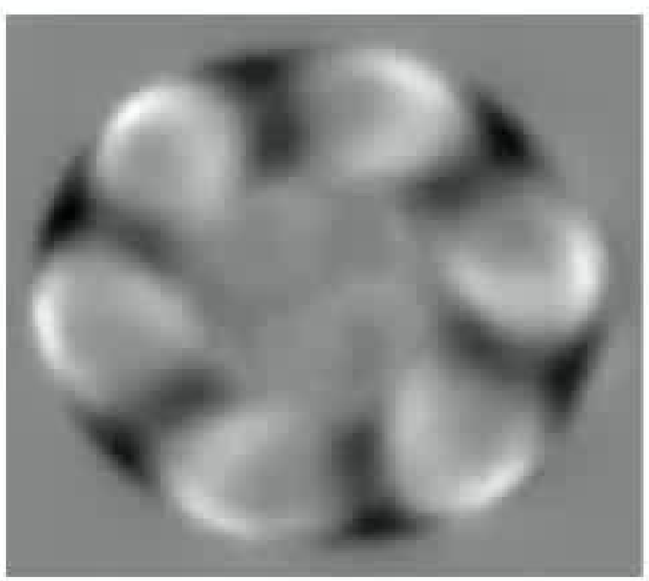
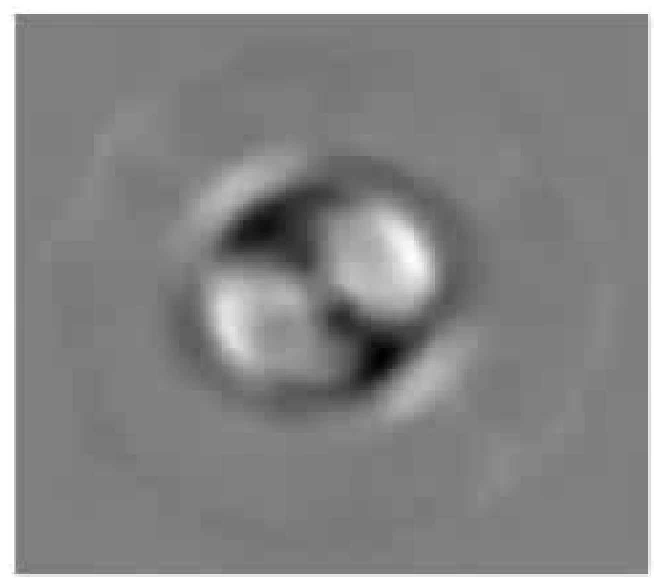
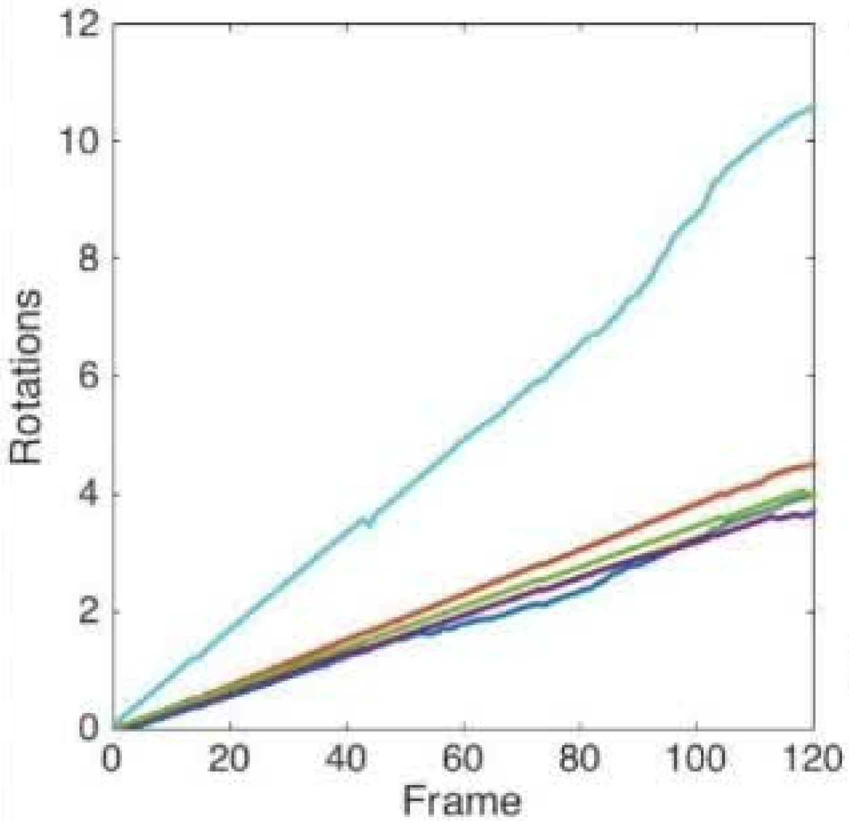
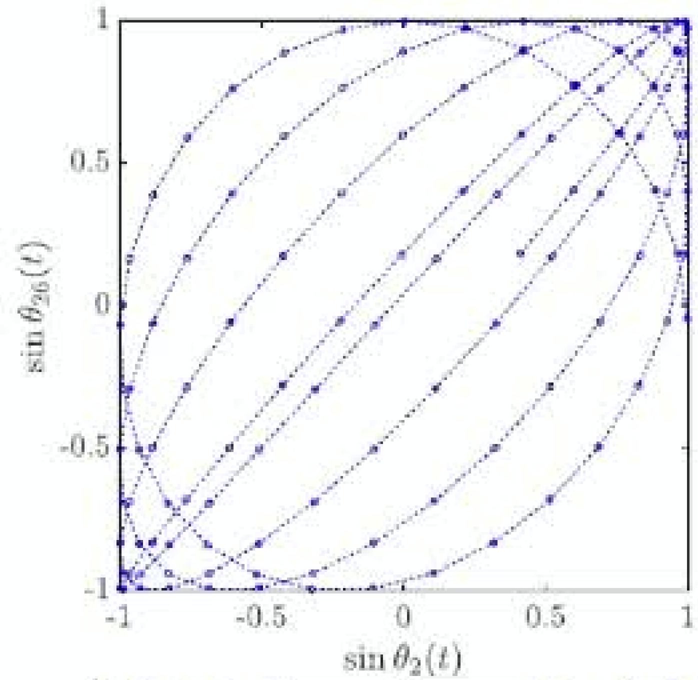
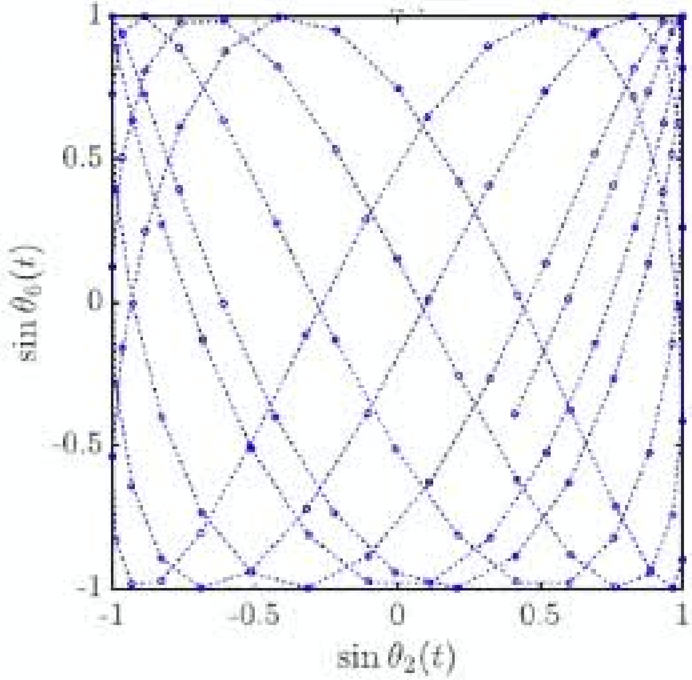




Flow Constituents





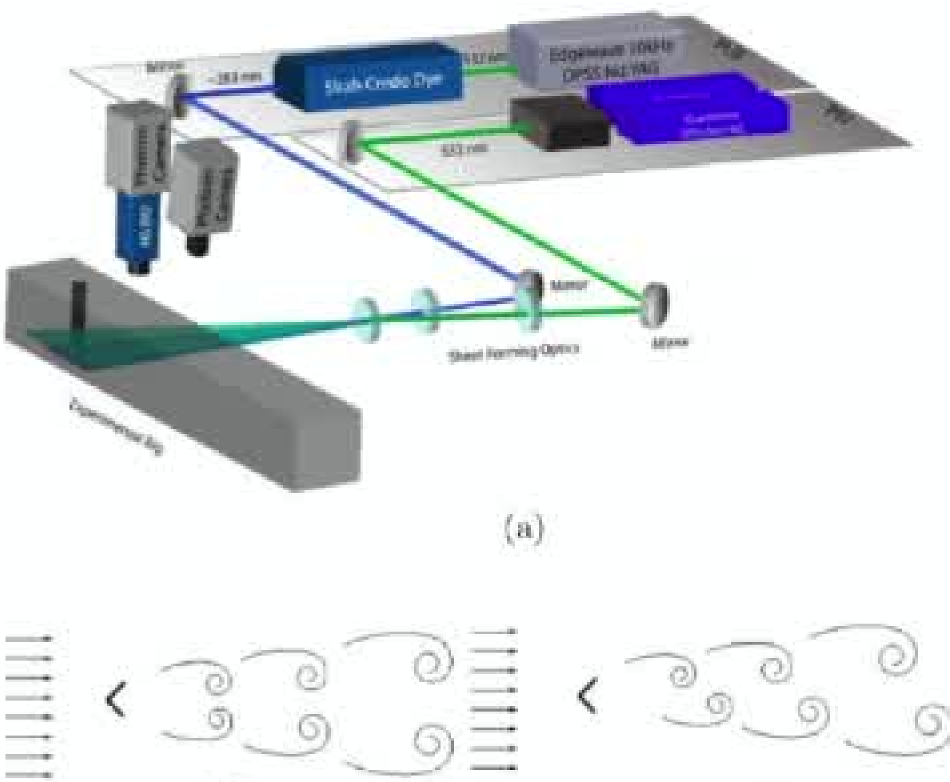


- The phase of a dynamic mode

$$\theta_k = \frac{1}{n} tan^{-1} \left[\frac{Im\ a_k(t)}{Re\ a_k(t)} \right]$$

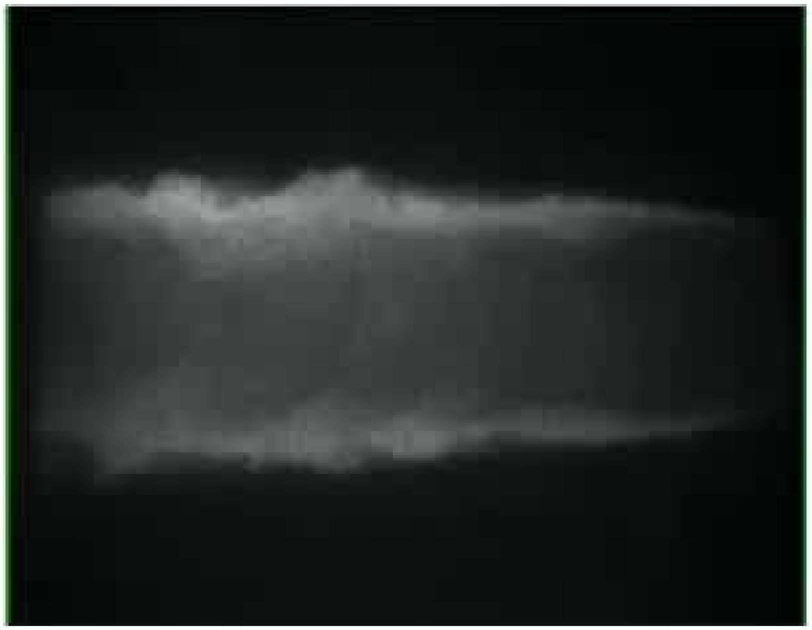
- Lissajous figure for a pair of modes in a single constituent (nearly) lies on on a curve
- For modes in different constituents, the Lissajous figure is space filling

Example: Reacting Flows behind a Buff-Body



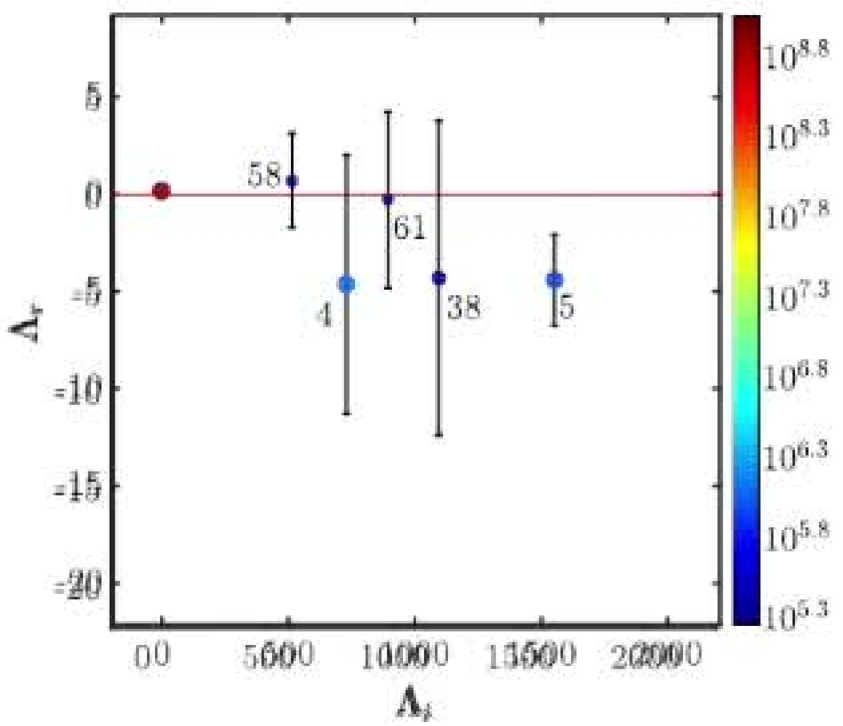
(a)

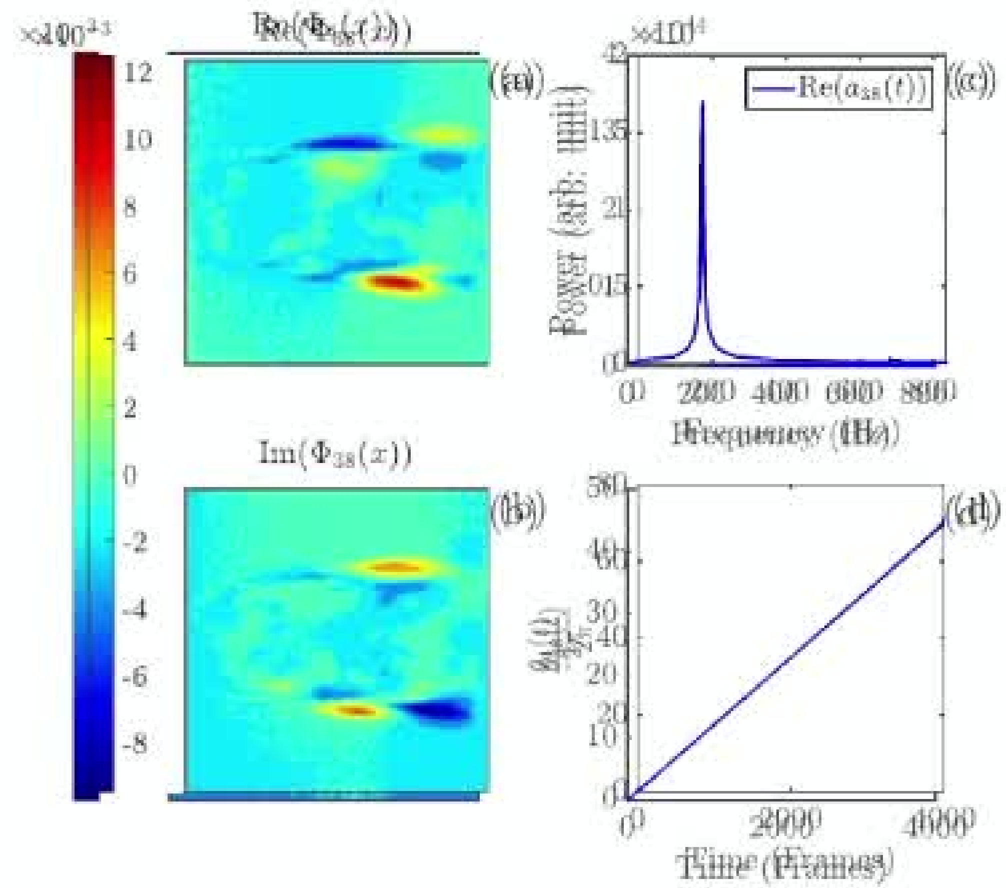
(b) (c)

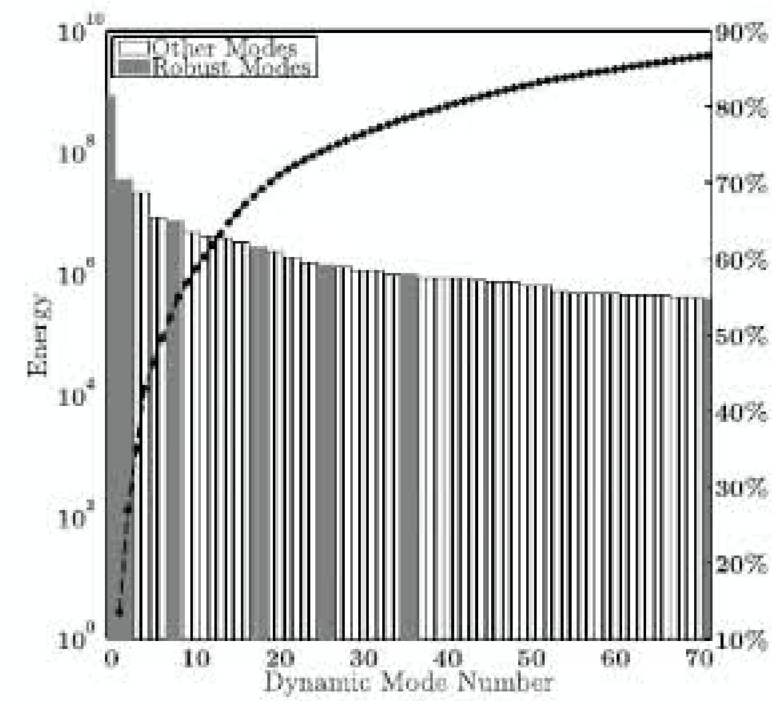


- Flow appears to contain
  - Symmetric vortex shedding
  - von Karman vortex shedding
- Combustion extinguishes soon after the appearance of von Karman shedding; wish to prevent its onset.

Robust Mode Analysis



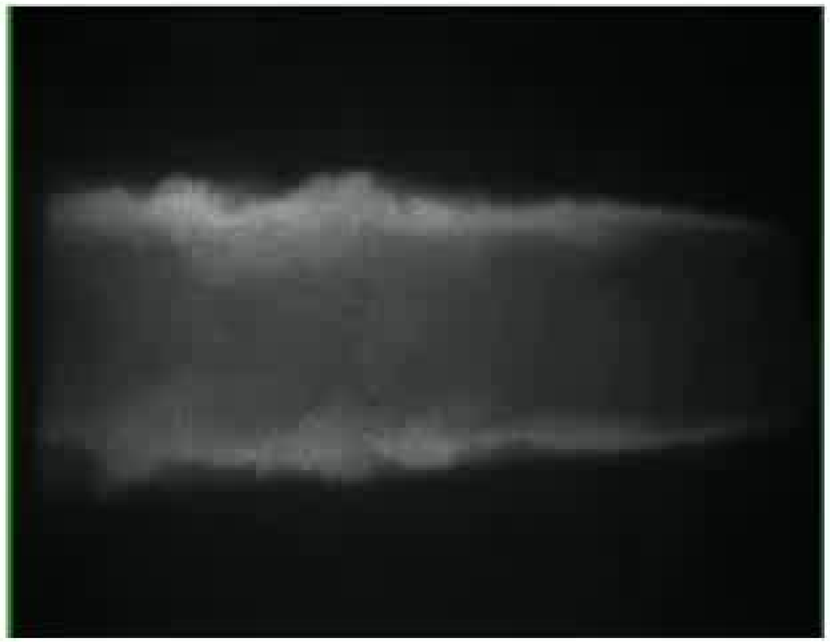


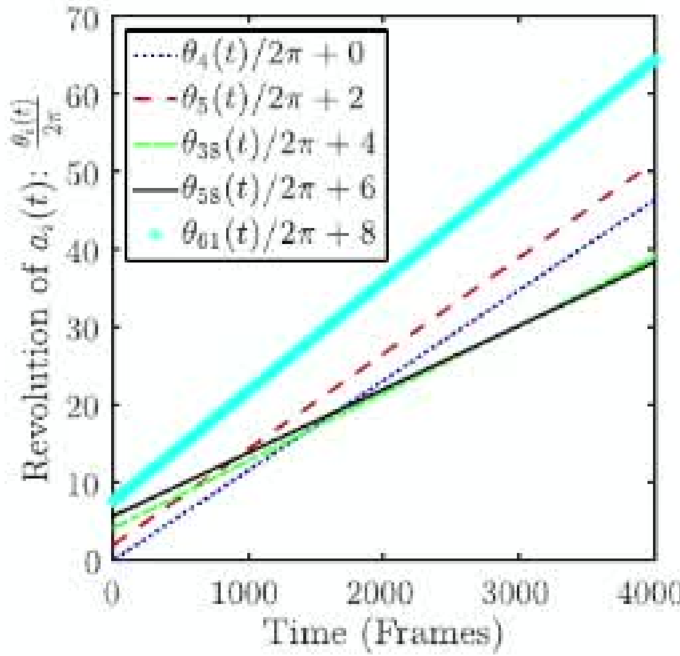


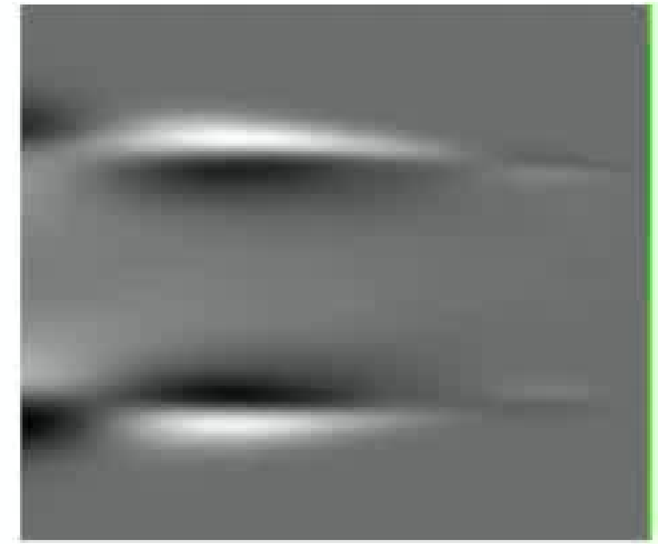
- Robust modes
  - subgroups of a single run
- (nearly) Persistent

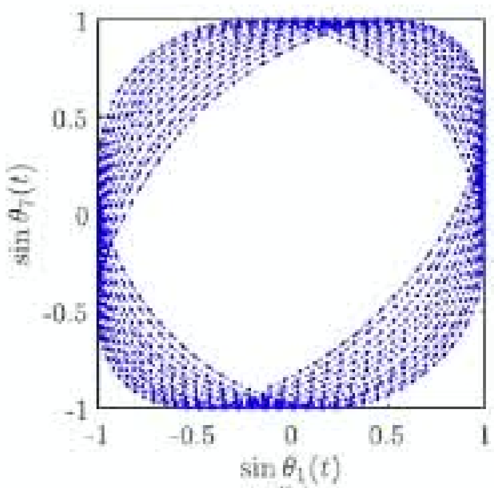
- Robust modes may not have the highest energy

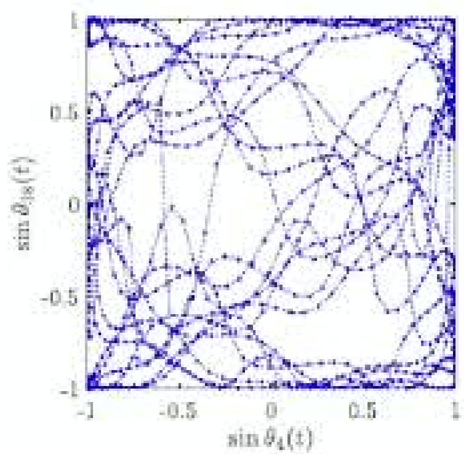
Flow Constituents





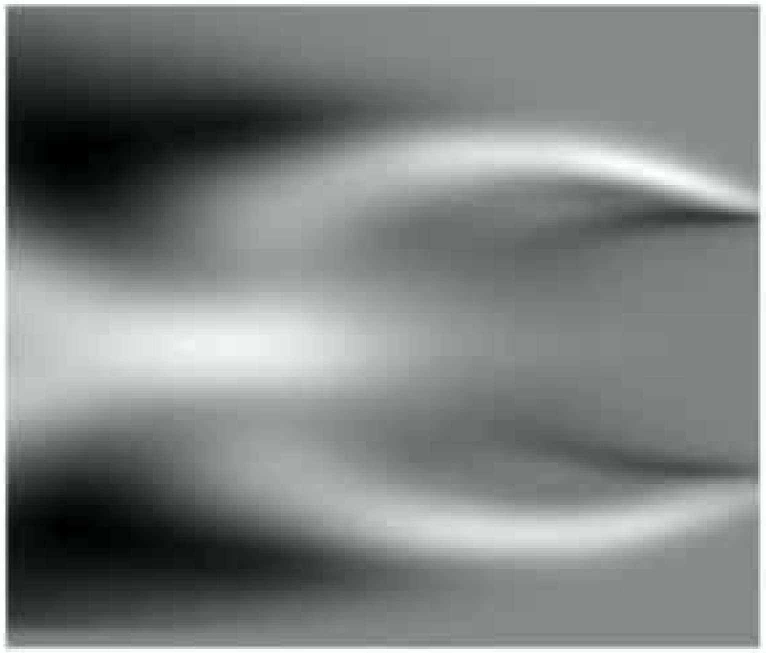






- DMD + reproducible modes ➔ robust flow features
- Phase dynamics + Lissajous ➔ flow constituents
- Reduced order models

Non-Robust Features of the Flow



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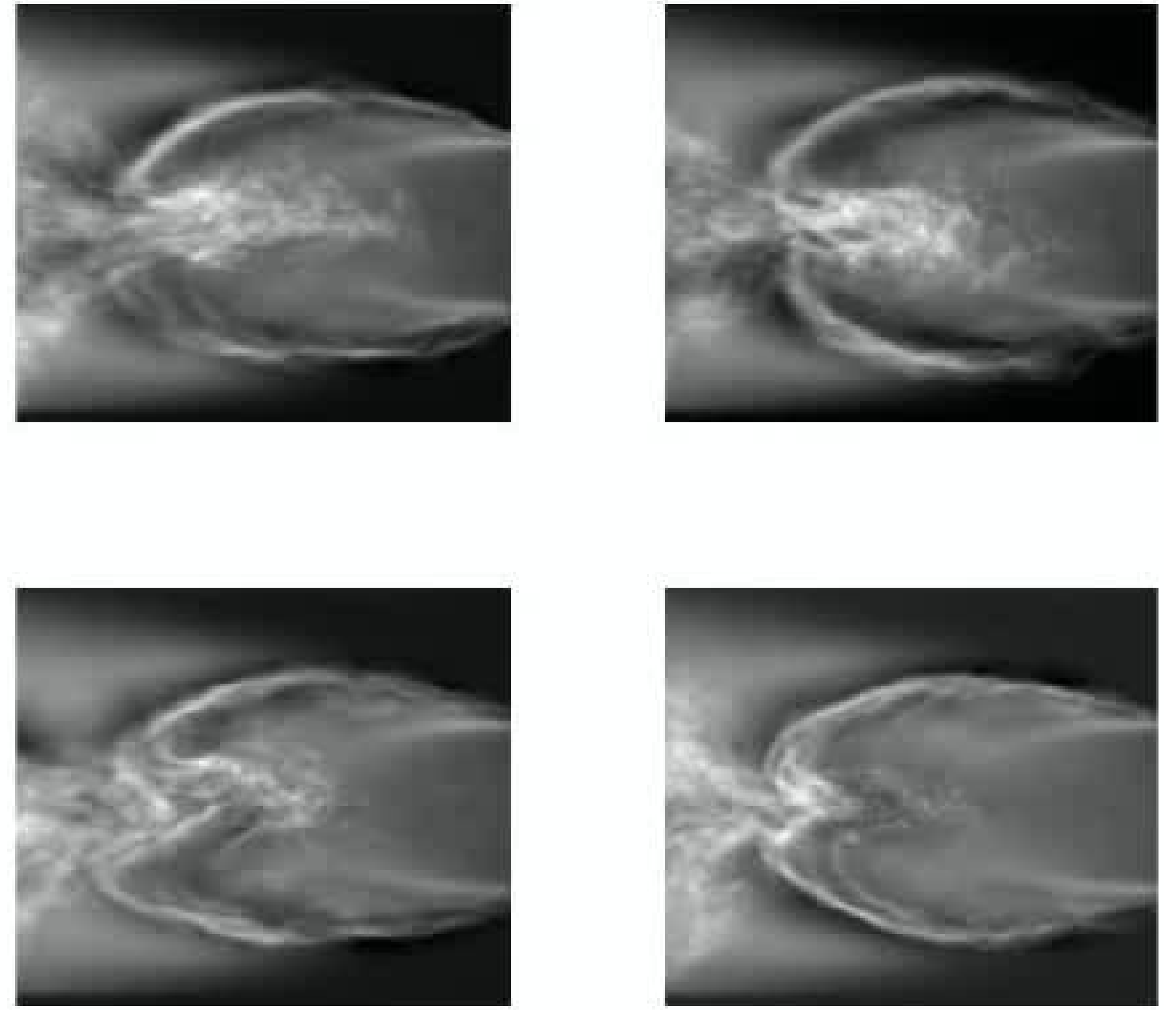
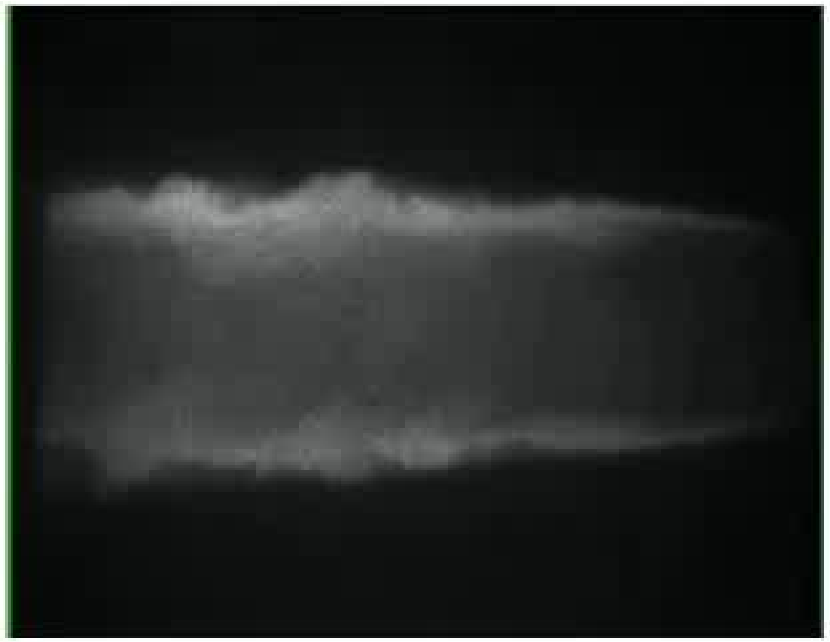
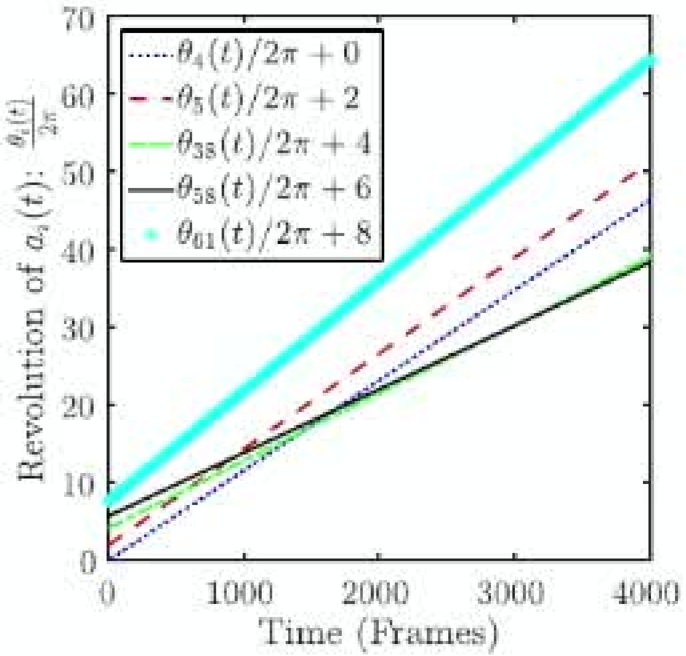
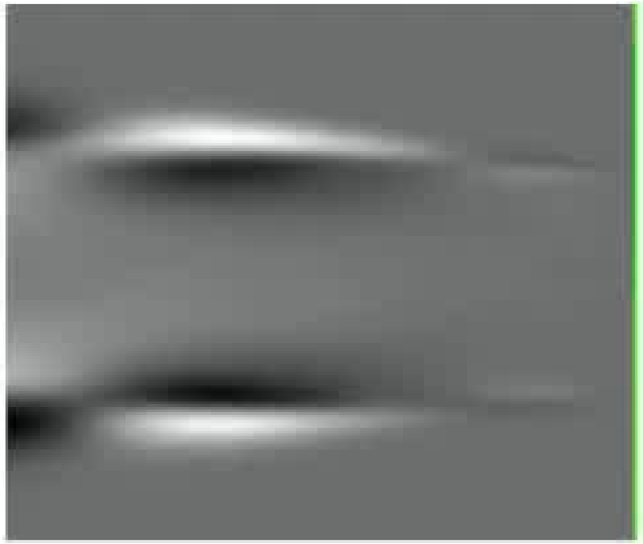


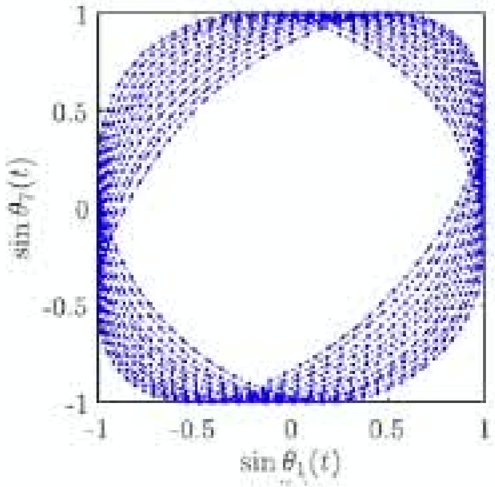
- Not “random”
- Need to characterize the non-robust flow as well
  - statistical properties & dynamical invariants

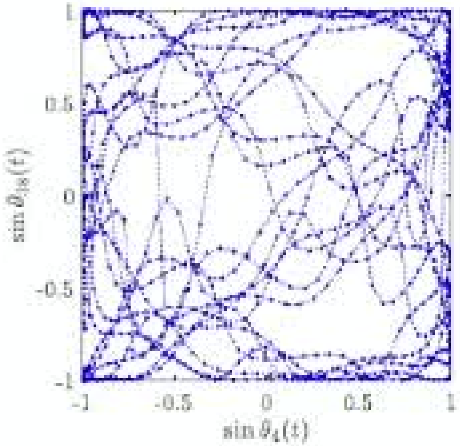
Flow Constituents











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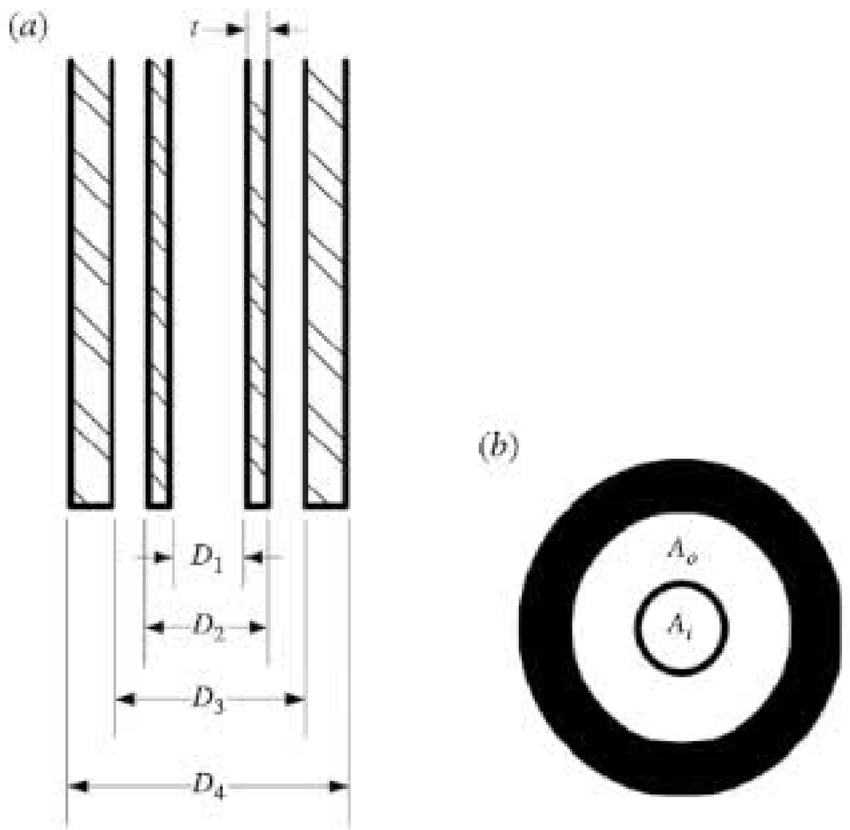
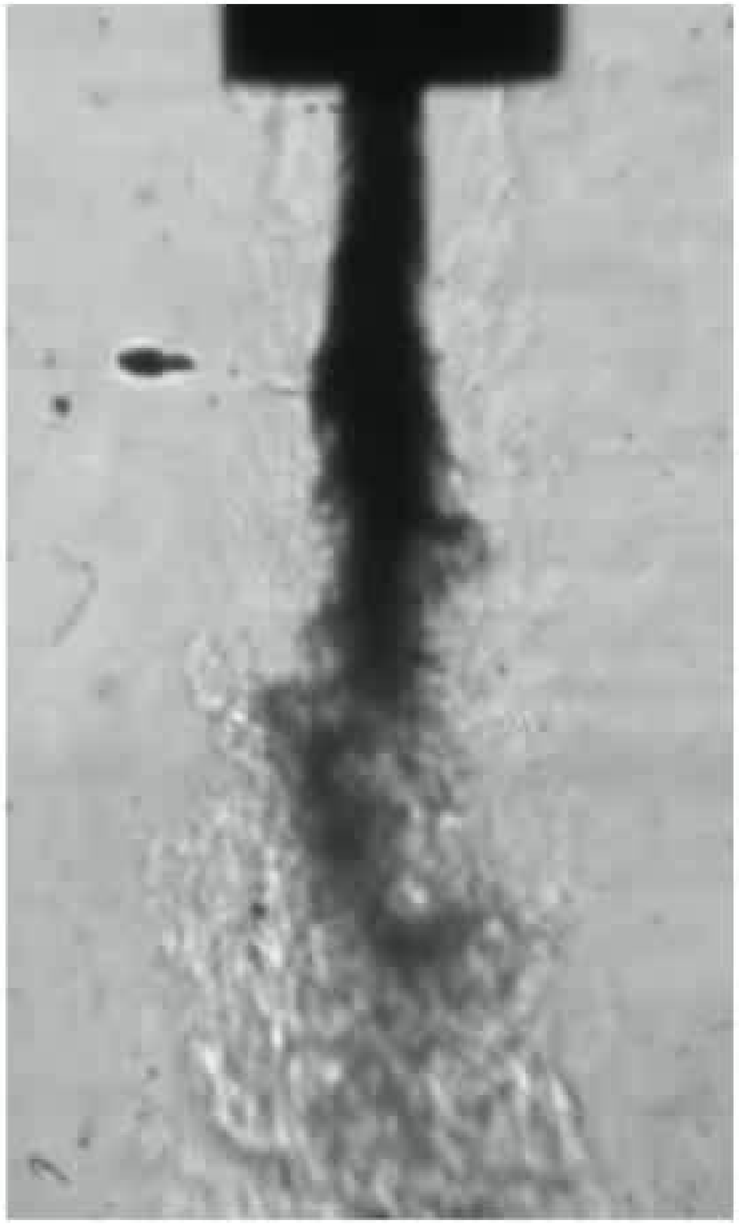
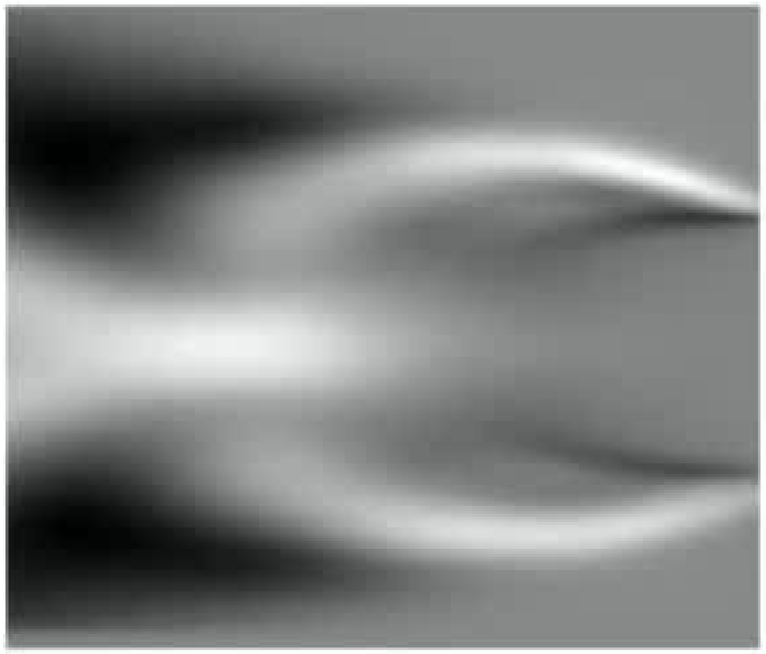
Shear Coaxial Jet Flows





Non-Robust Features of the Flow



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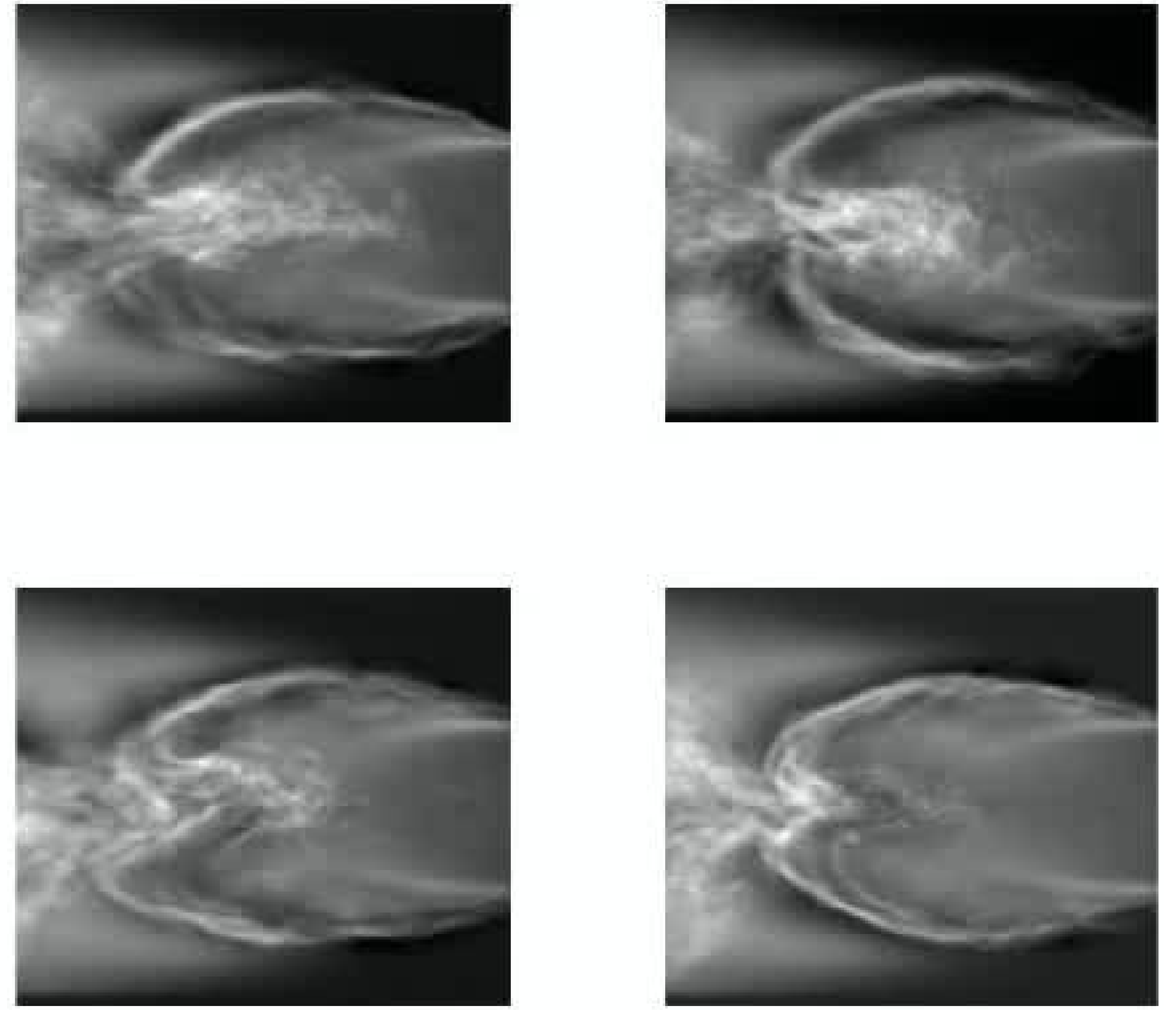

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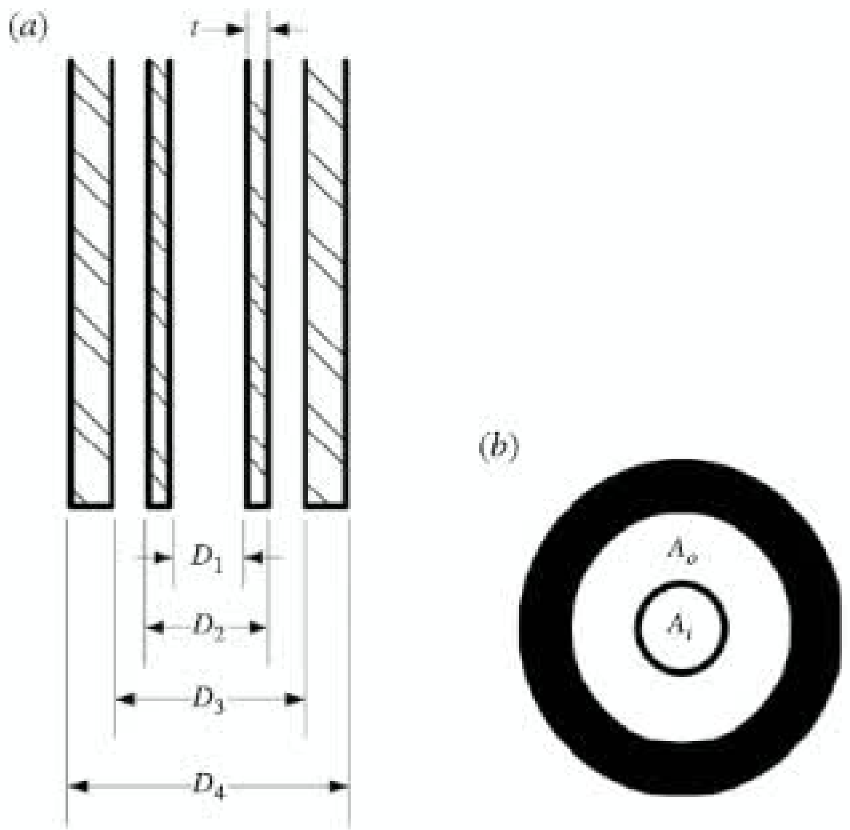
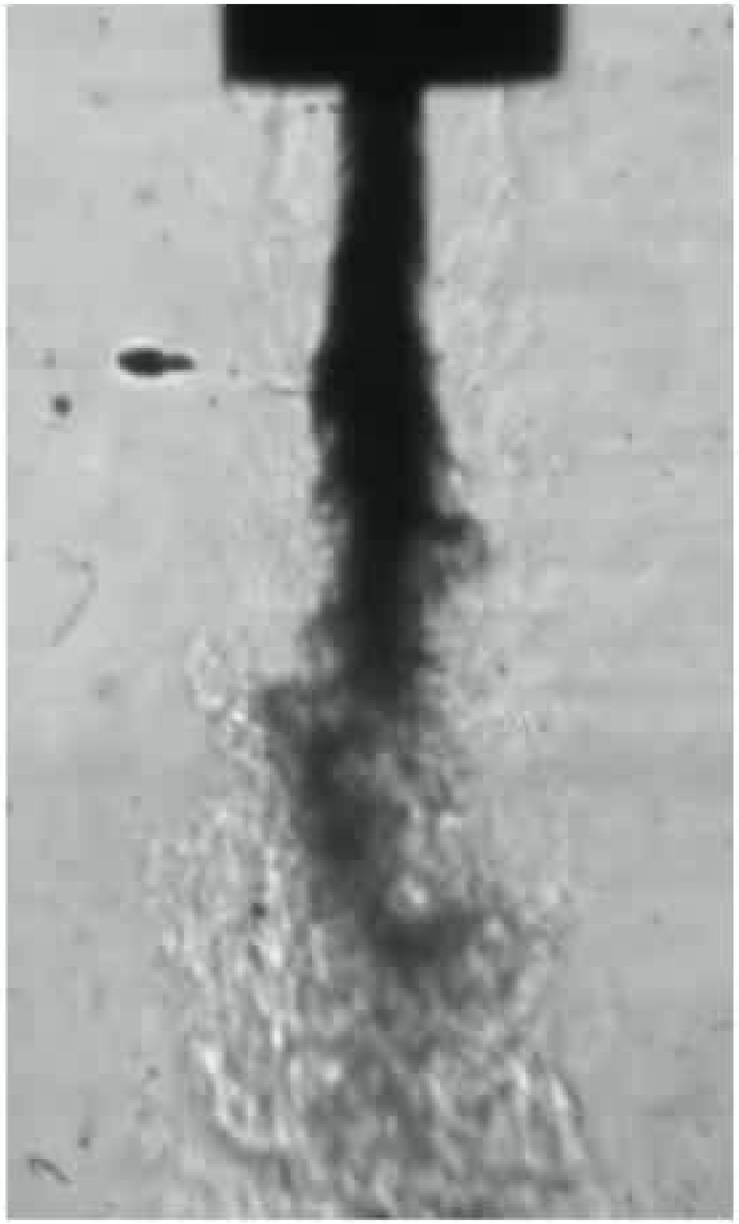
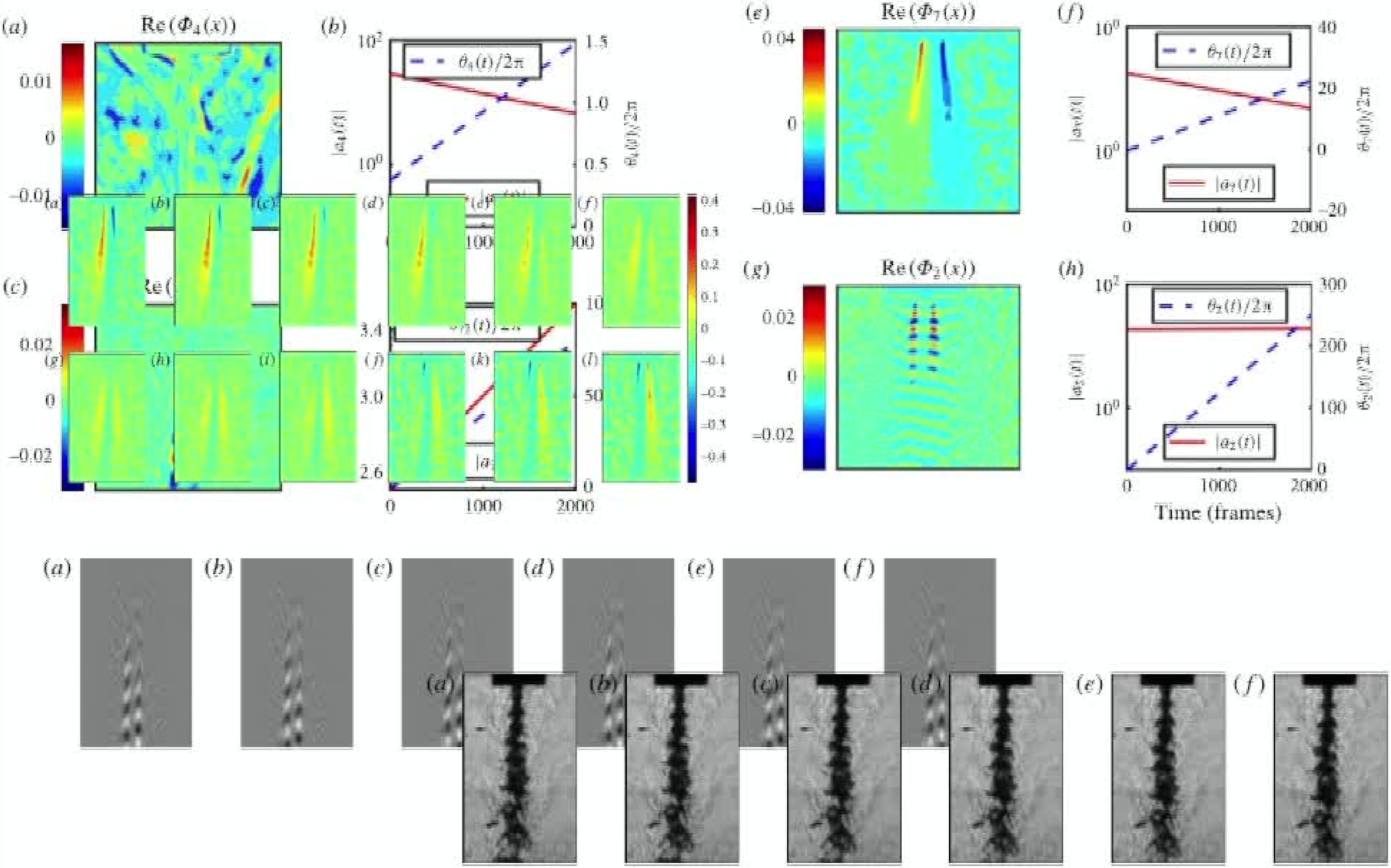




Robust Modes



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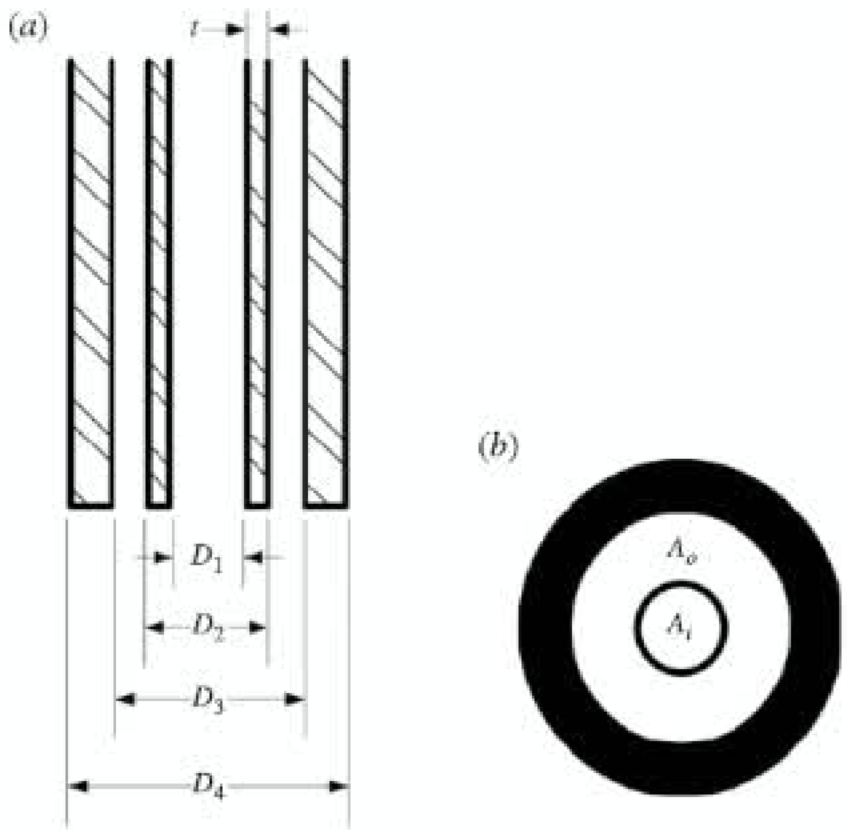

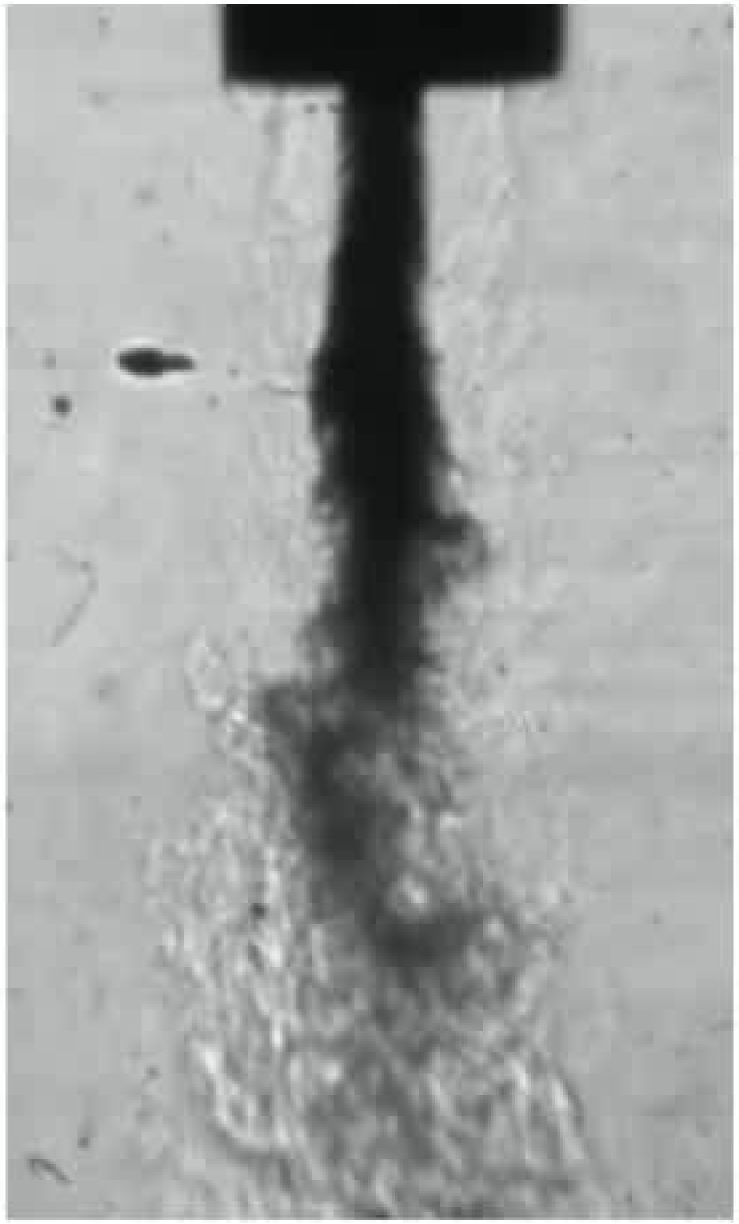

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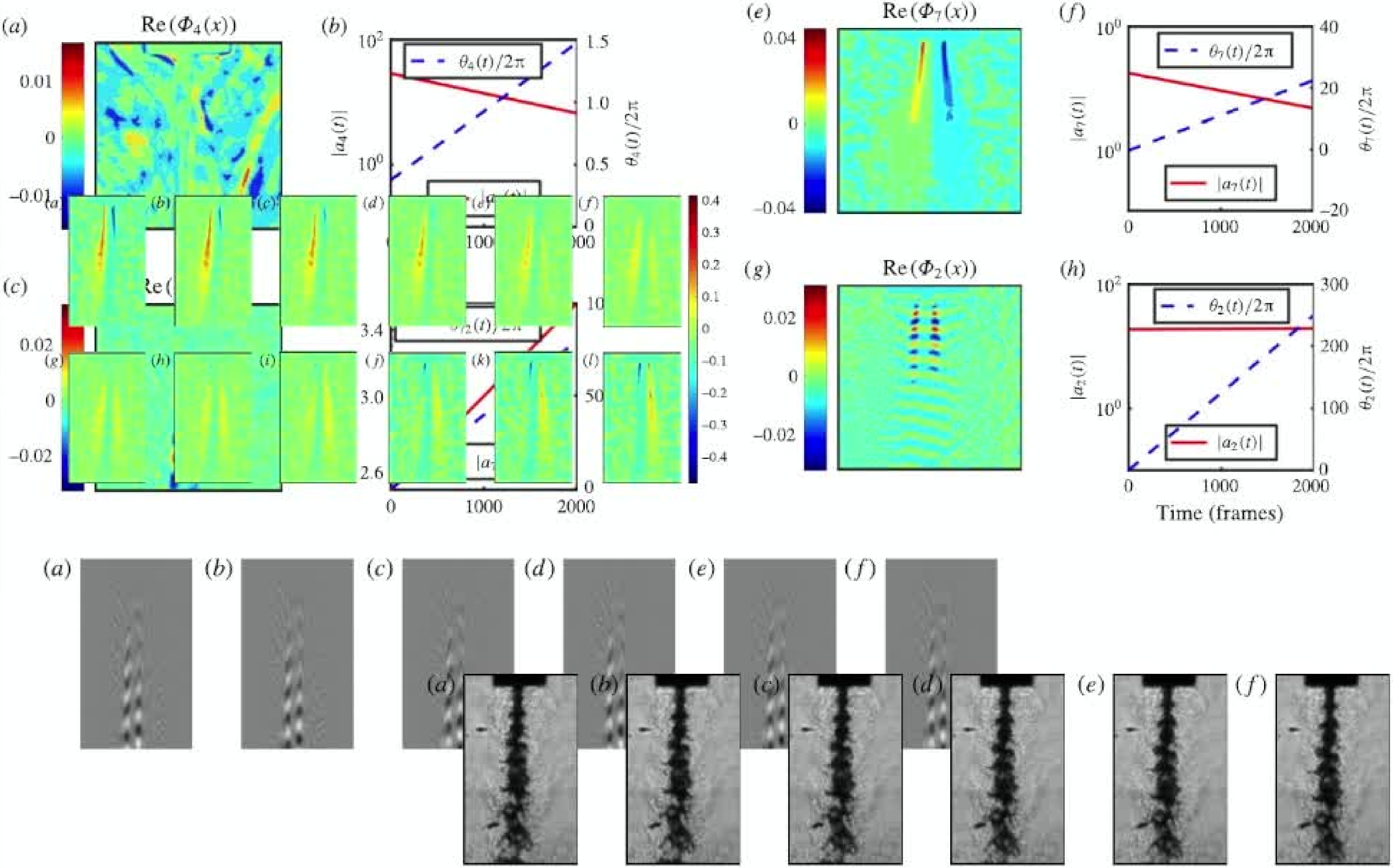
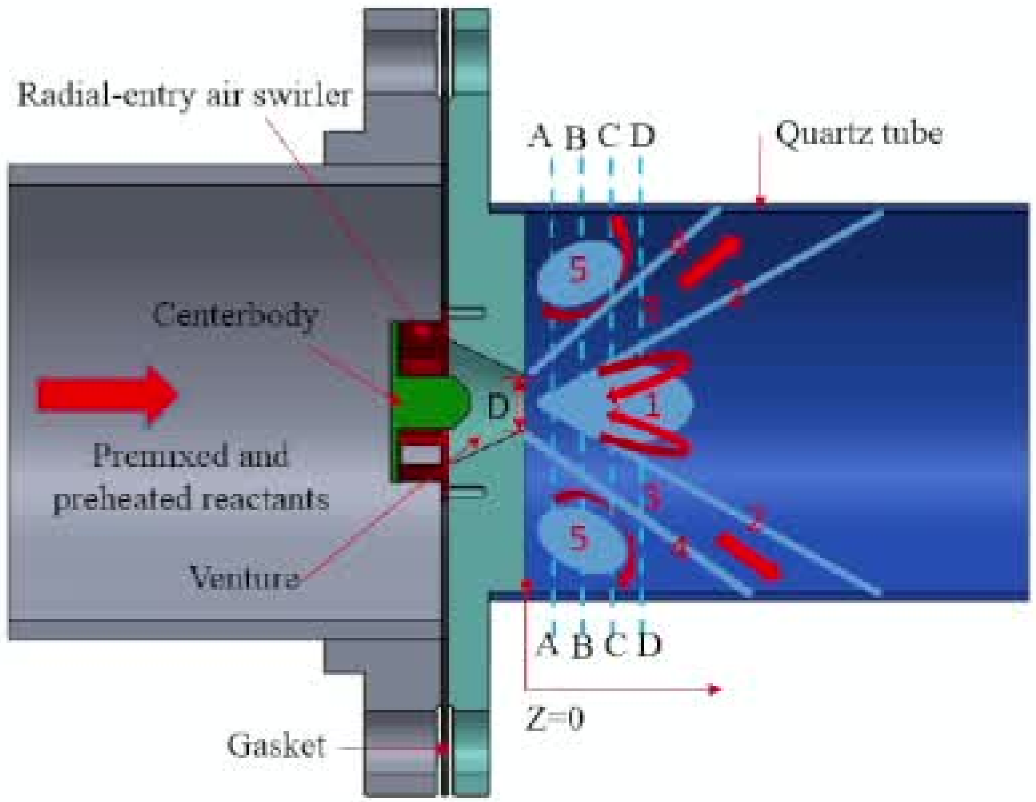
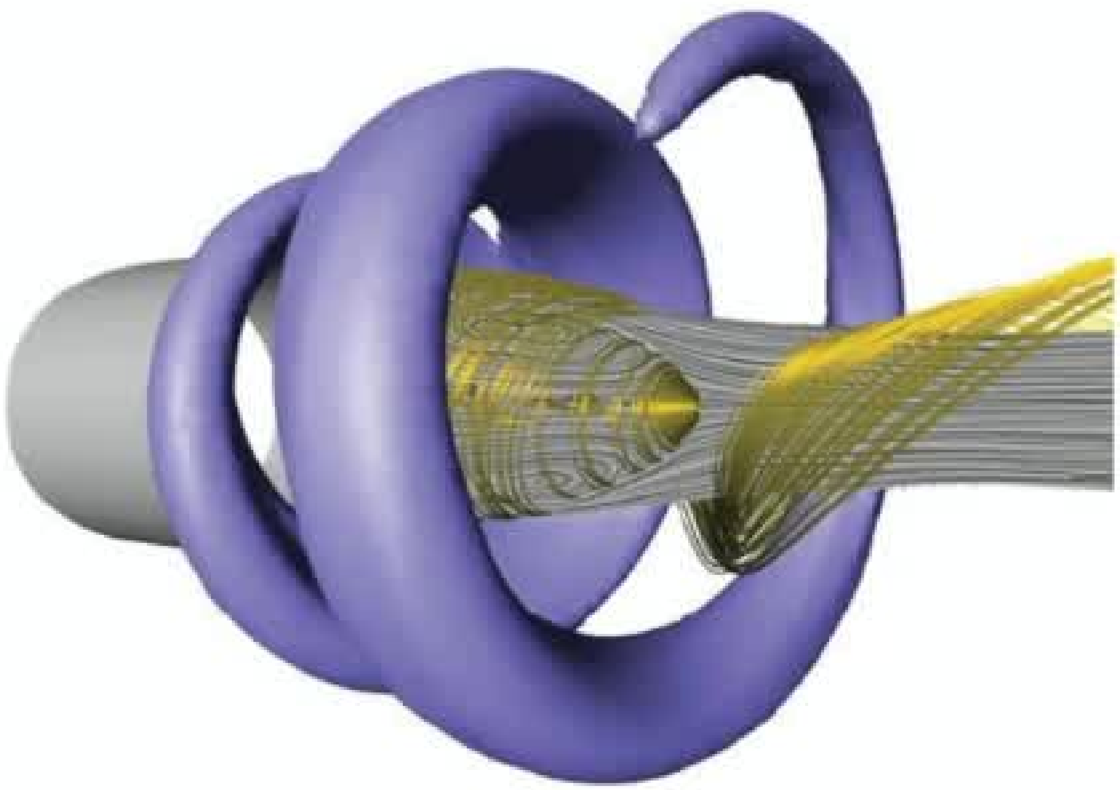
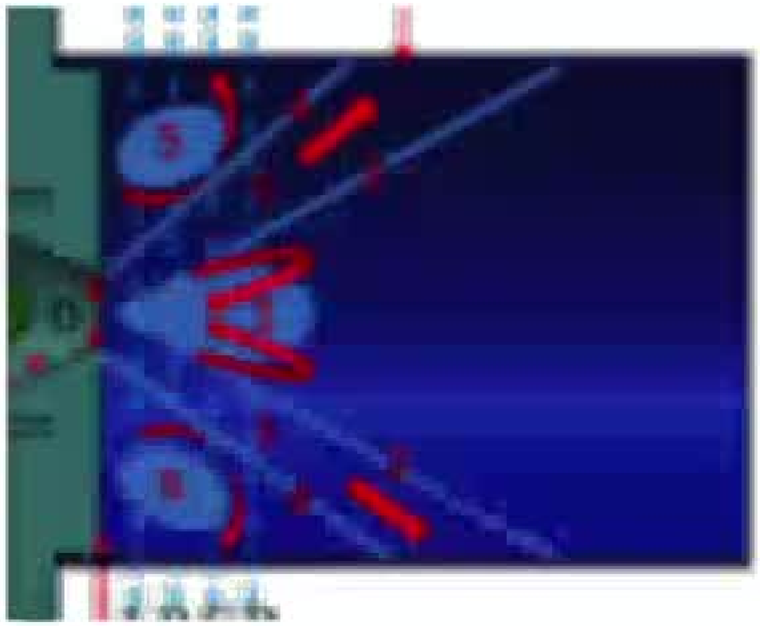


Swirling Reacting Flows

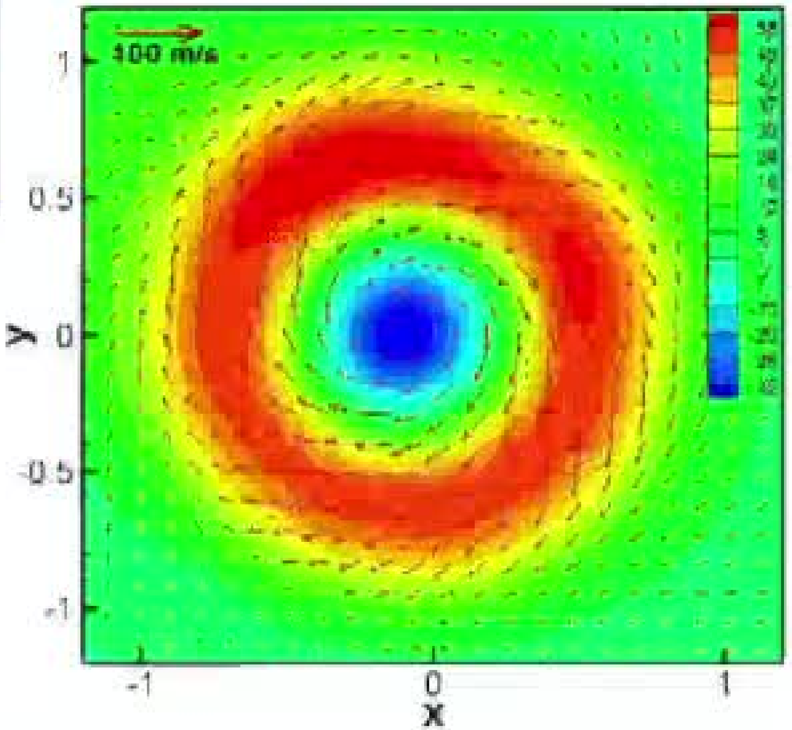


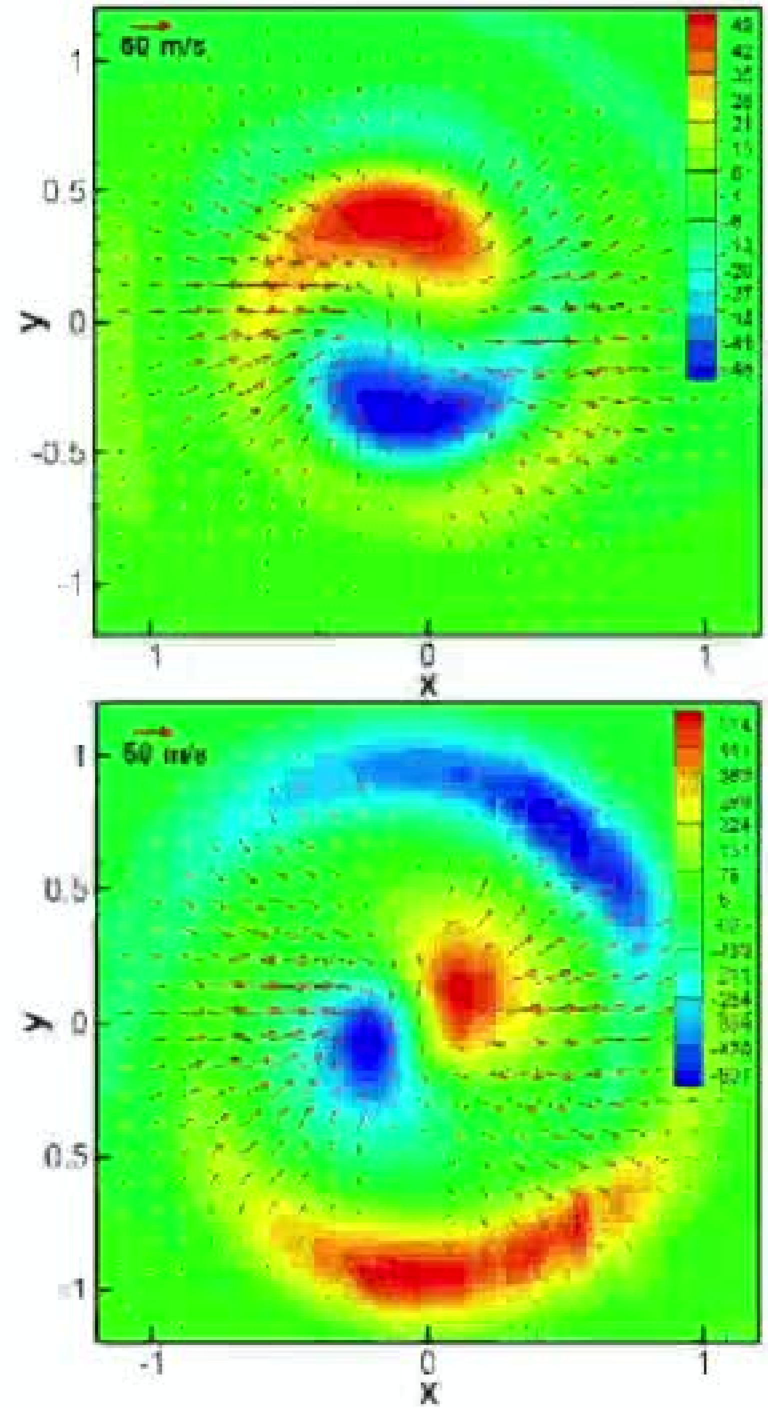


Robust Modes in Swirl Flow



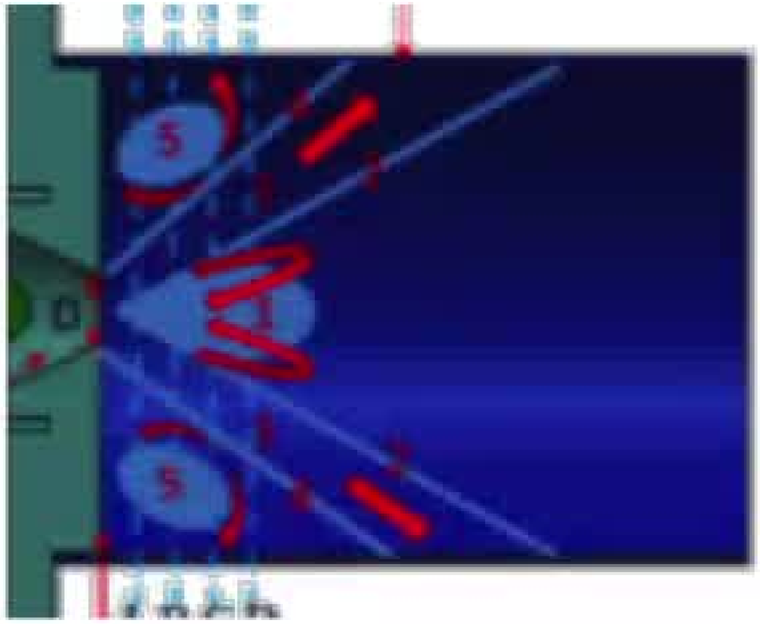
Mean Mode





Robust Mode

Robust Modes in Swirl Flow



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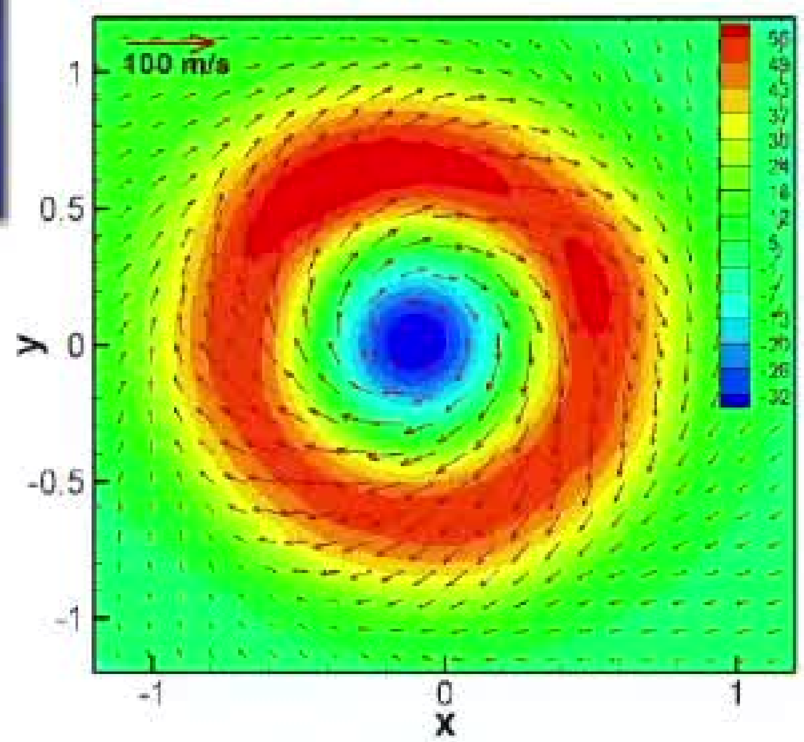
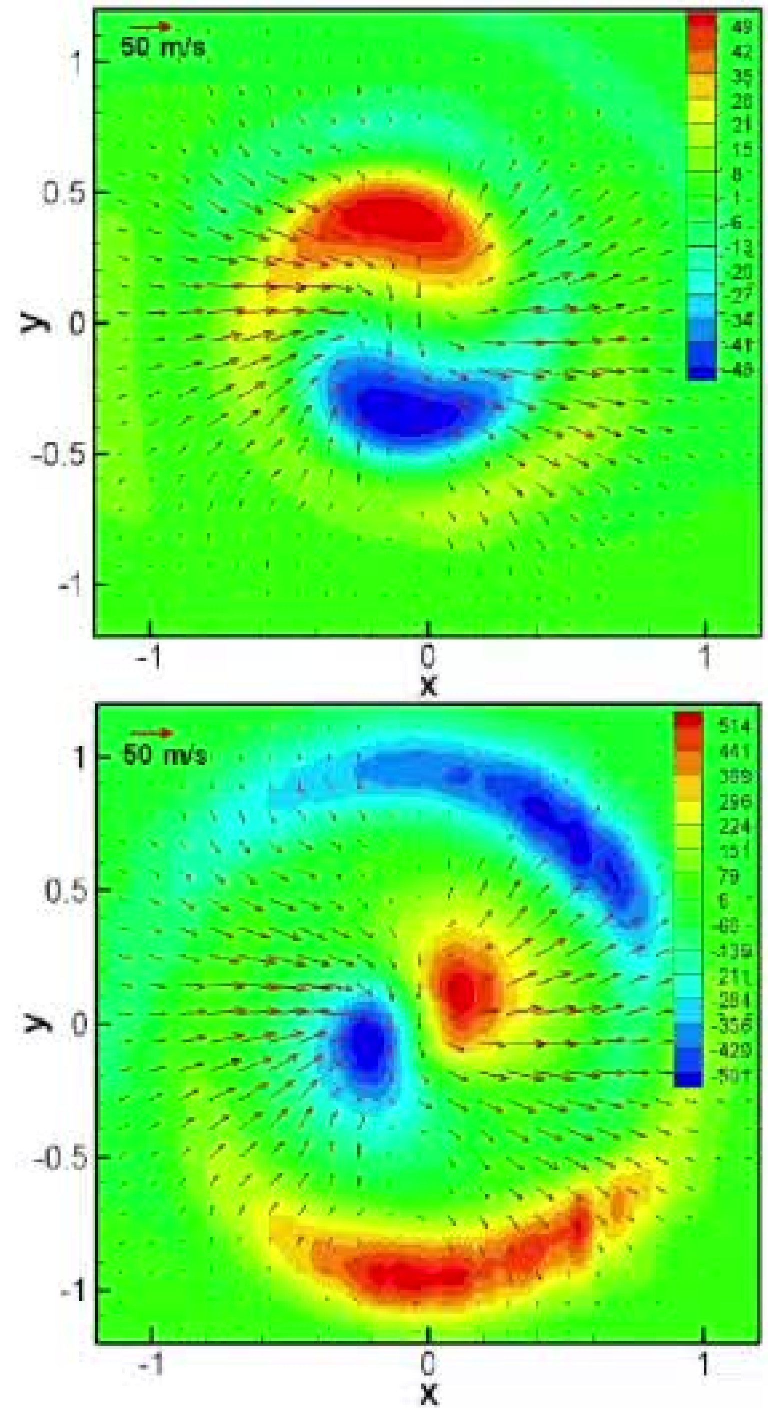
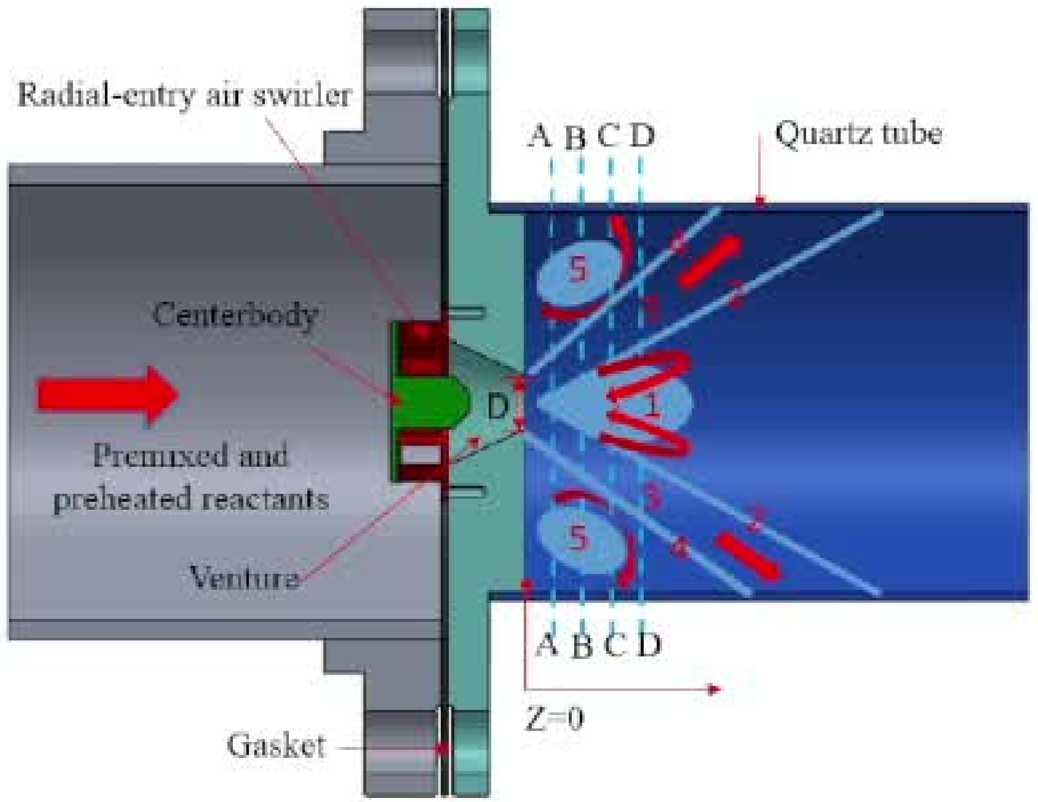
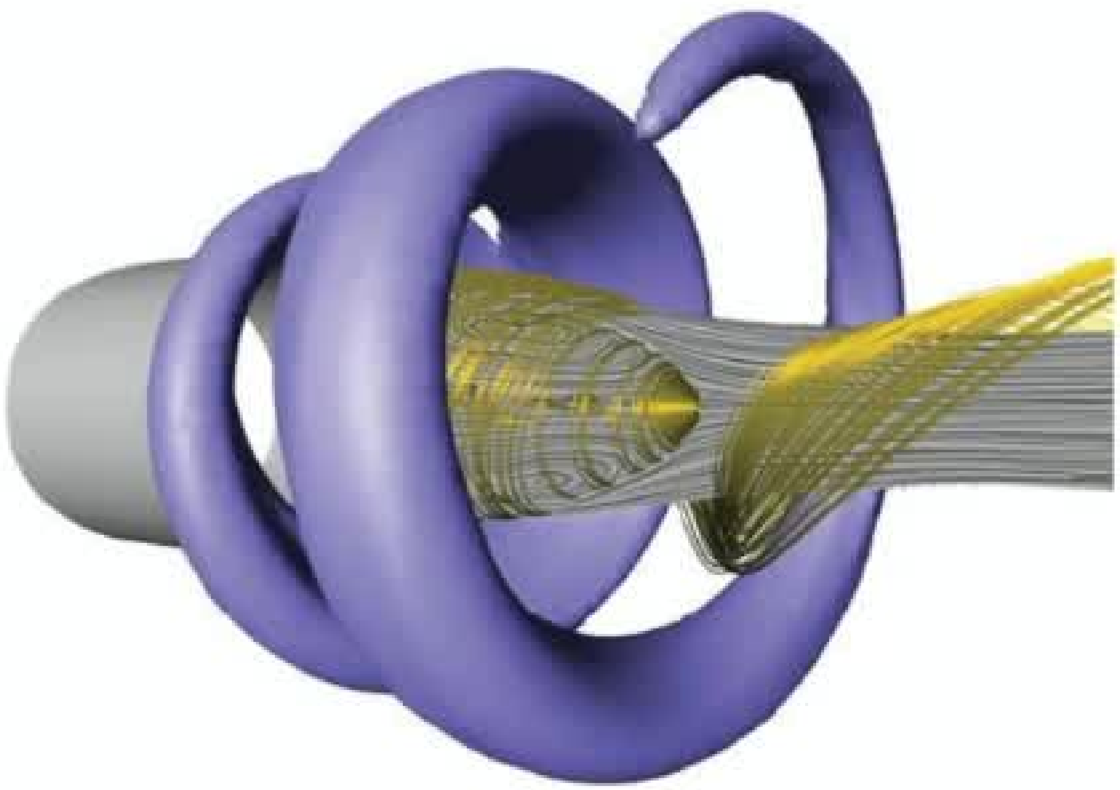




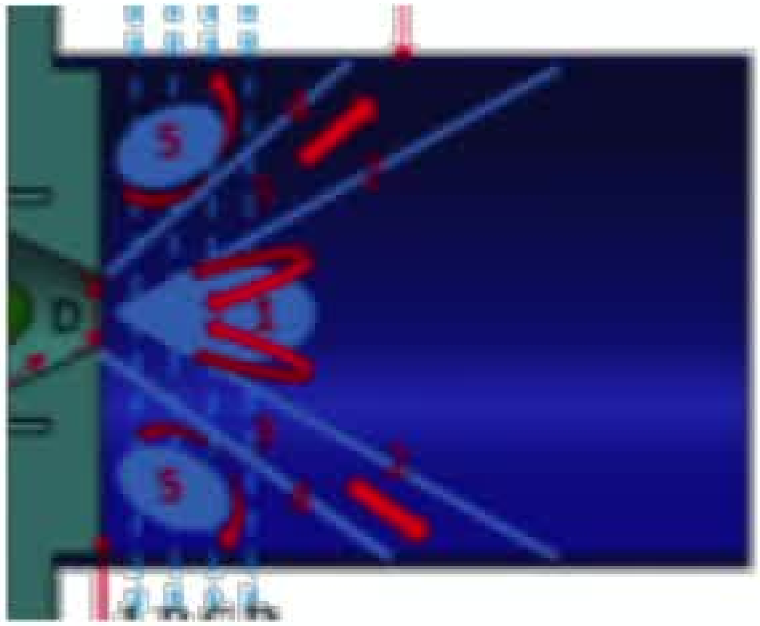
Robust Mode

Swirling Reacting Flows

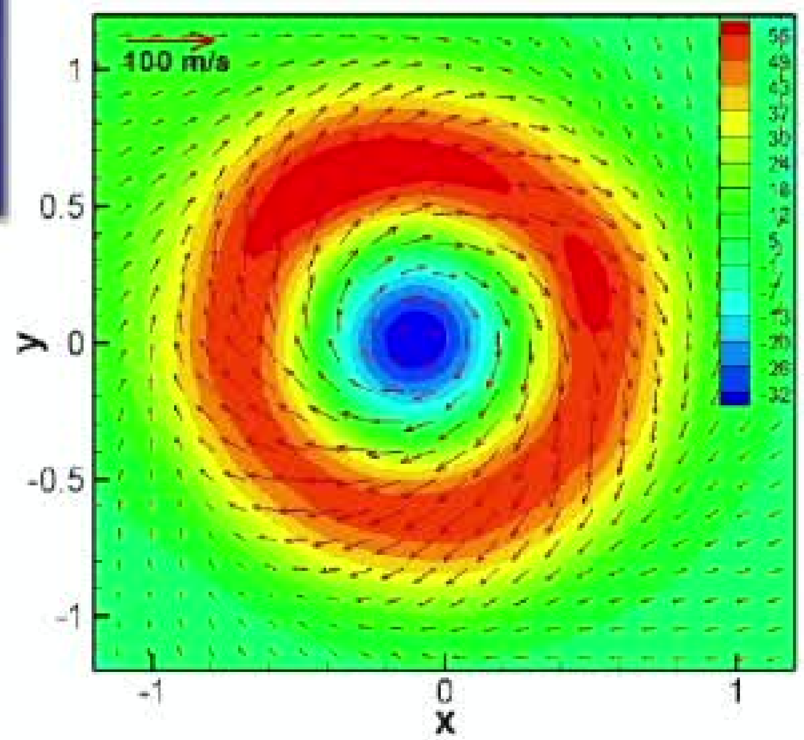


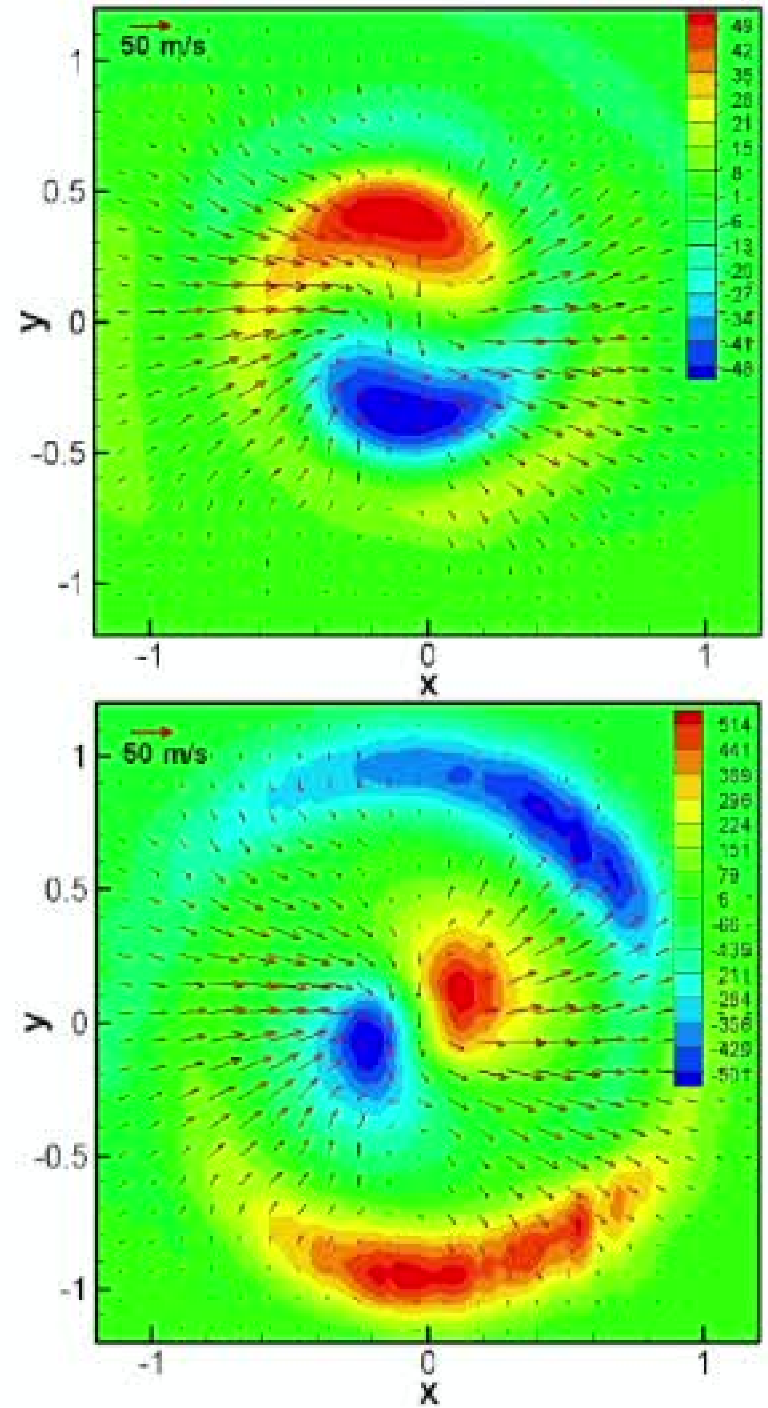


Robust Modes in Swirl Flow



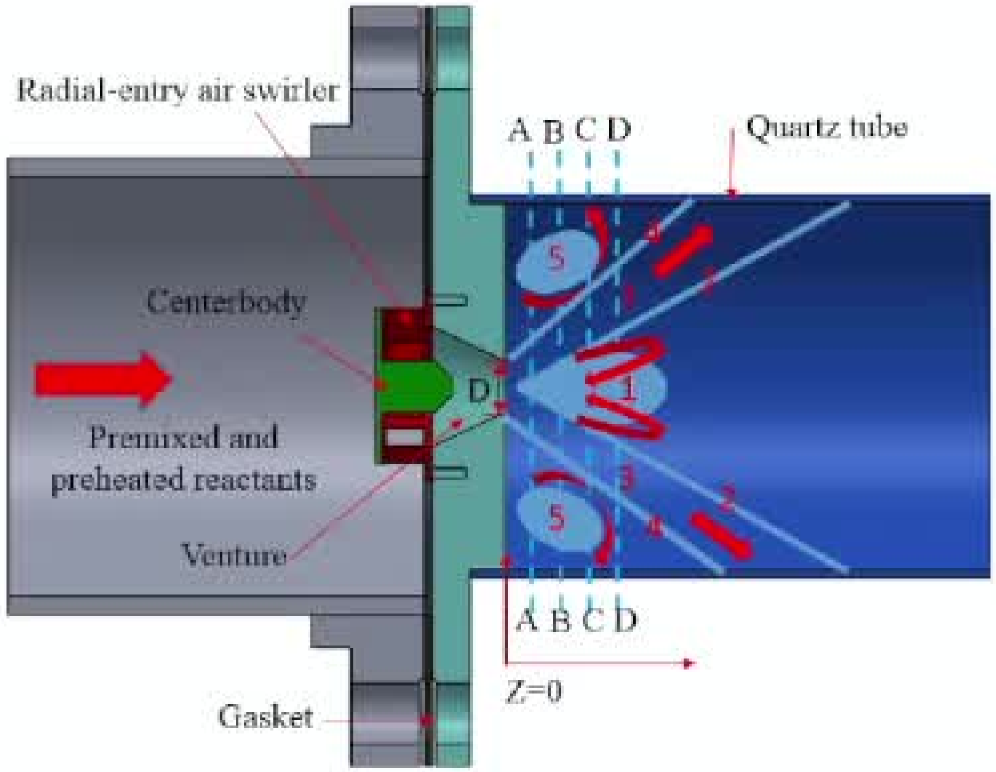
Mean Mode

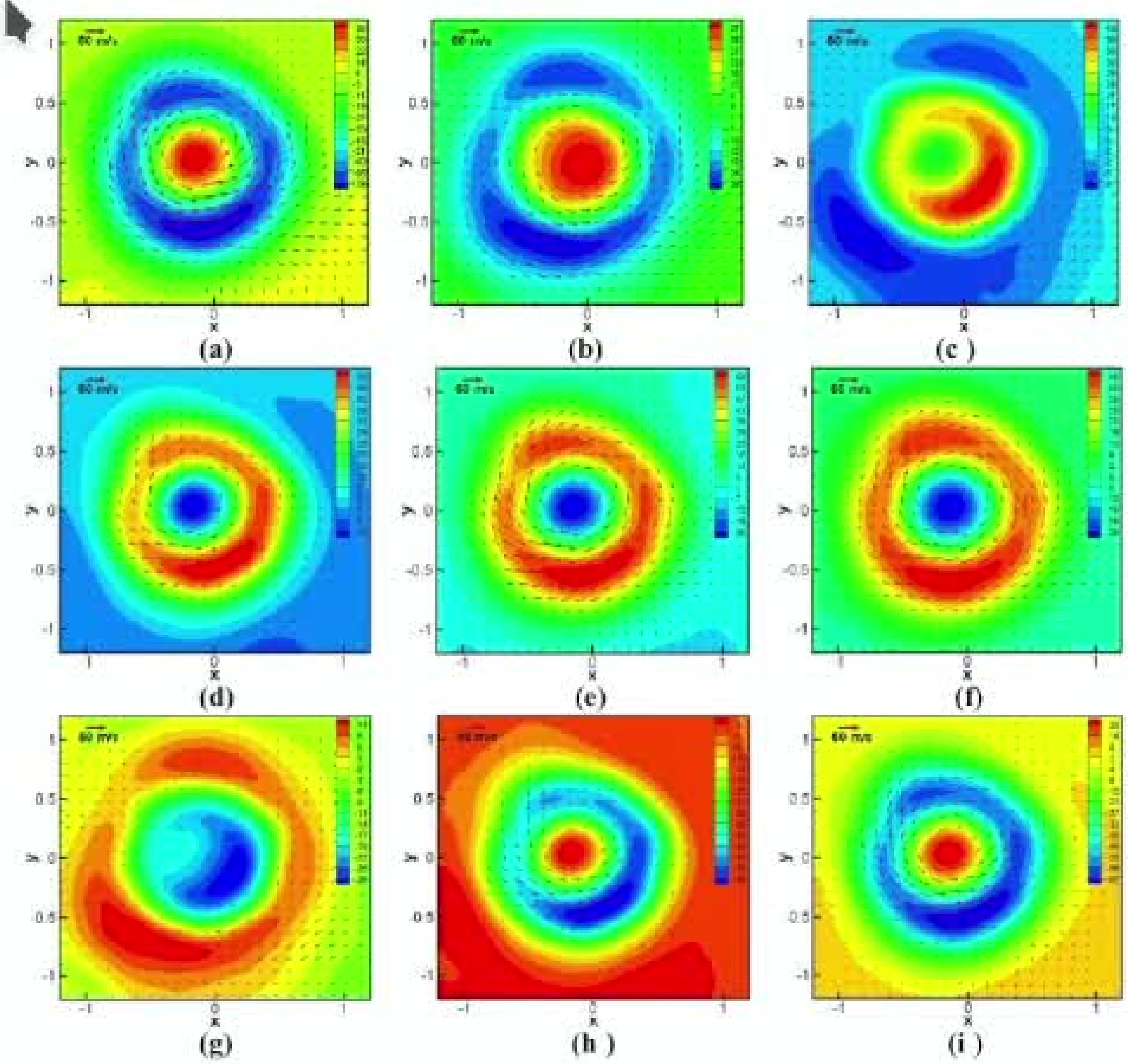




Robust Mode

Unstable Swirling Reacting Flows

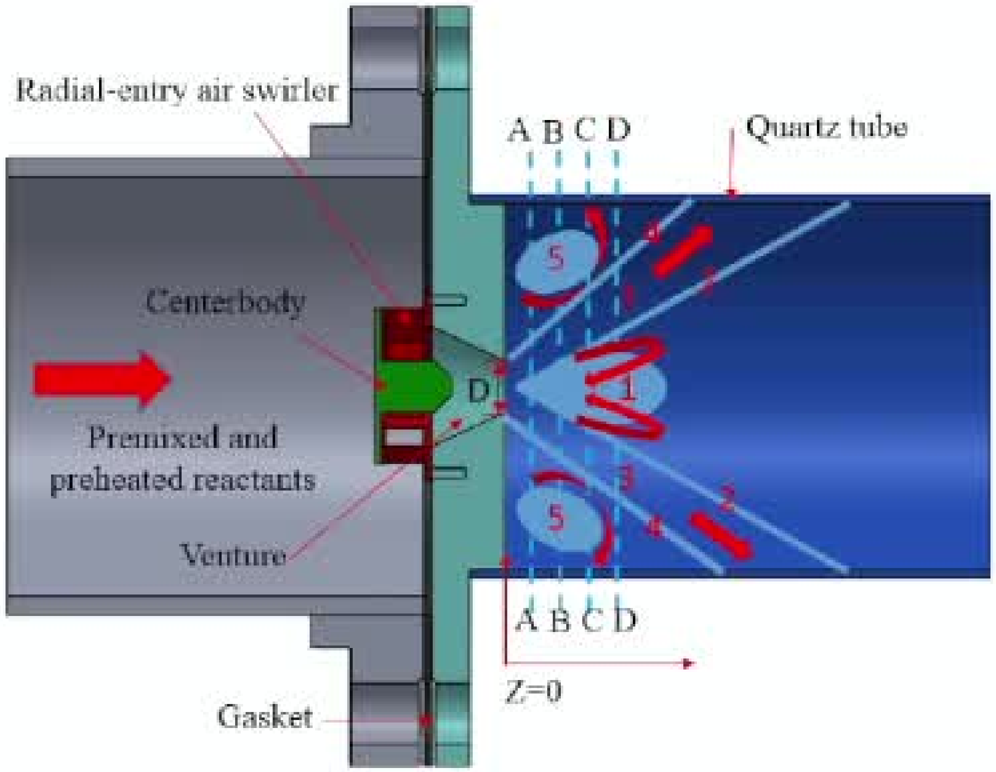


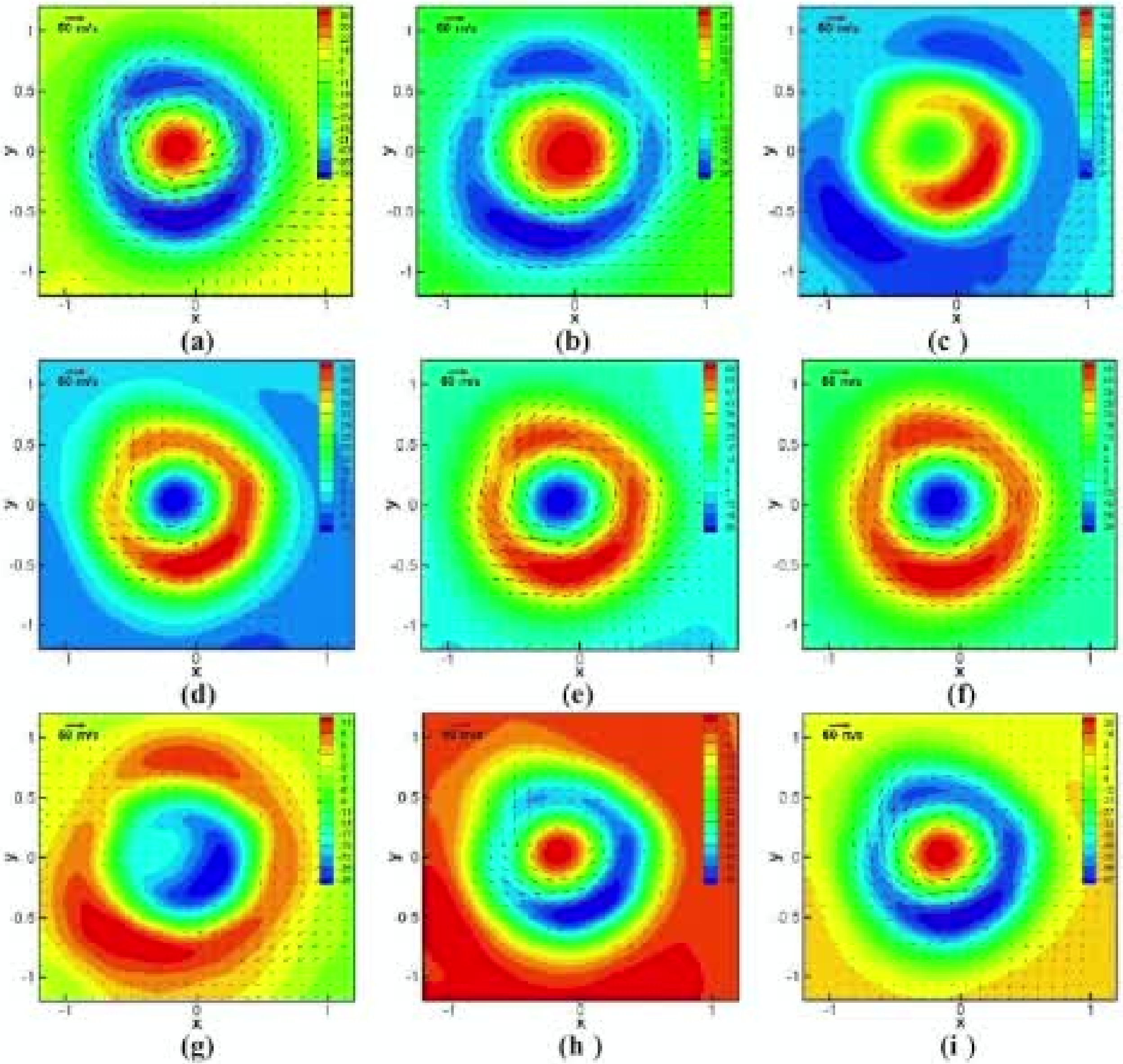


Financial Markets!

- Use the valuations of (567) stocks, each of whose market capitalization exceeded 7.5 million dollars between November 2002 and October 2014
- Search for “cycles”

Unstable Swirling Reacting Flows





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