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Coupling and Decoupling of Flow in Conduit and Flow in Porous Media

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Figure: Department of Mathematics, Southern University of Science and Technology (SUSTech)

Hiring at all levels: postdoc, junior and senior faculty, with multiple positions at each level in all areas of mathematical sciences.

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Flows in karstic geometry

Figure: Conceptual interacting flows

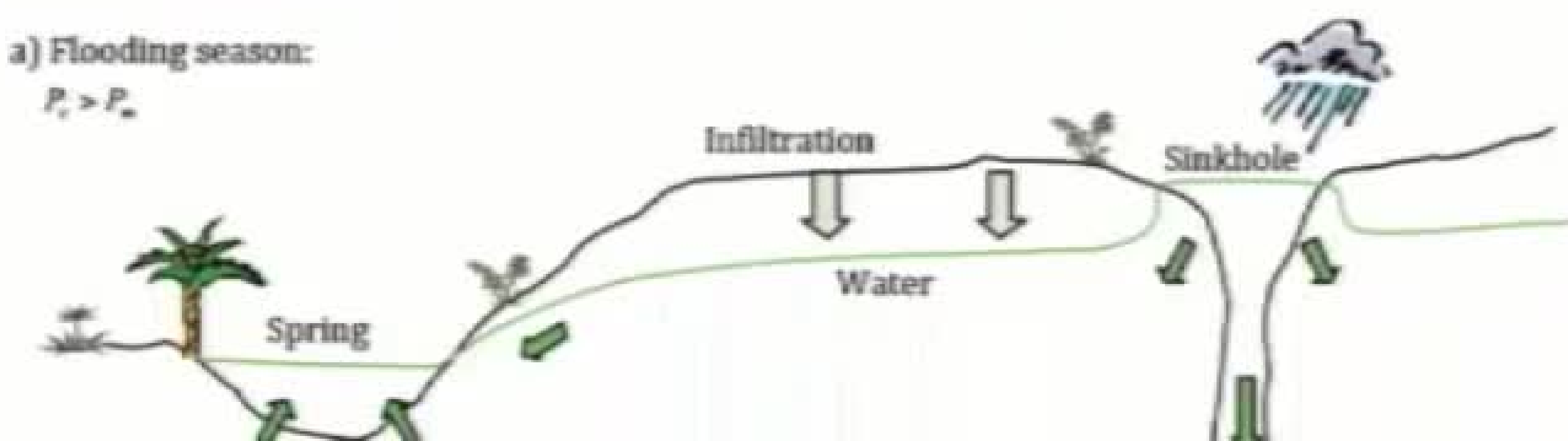

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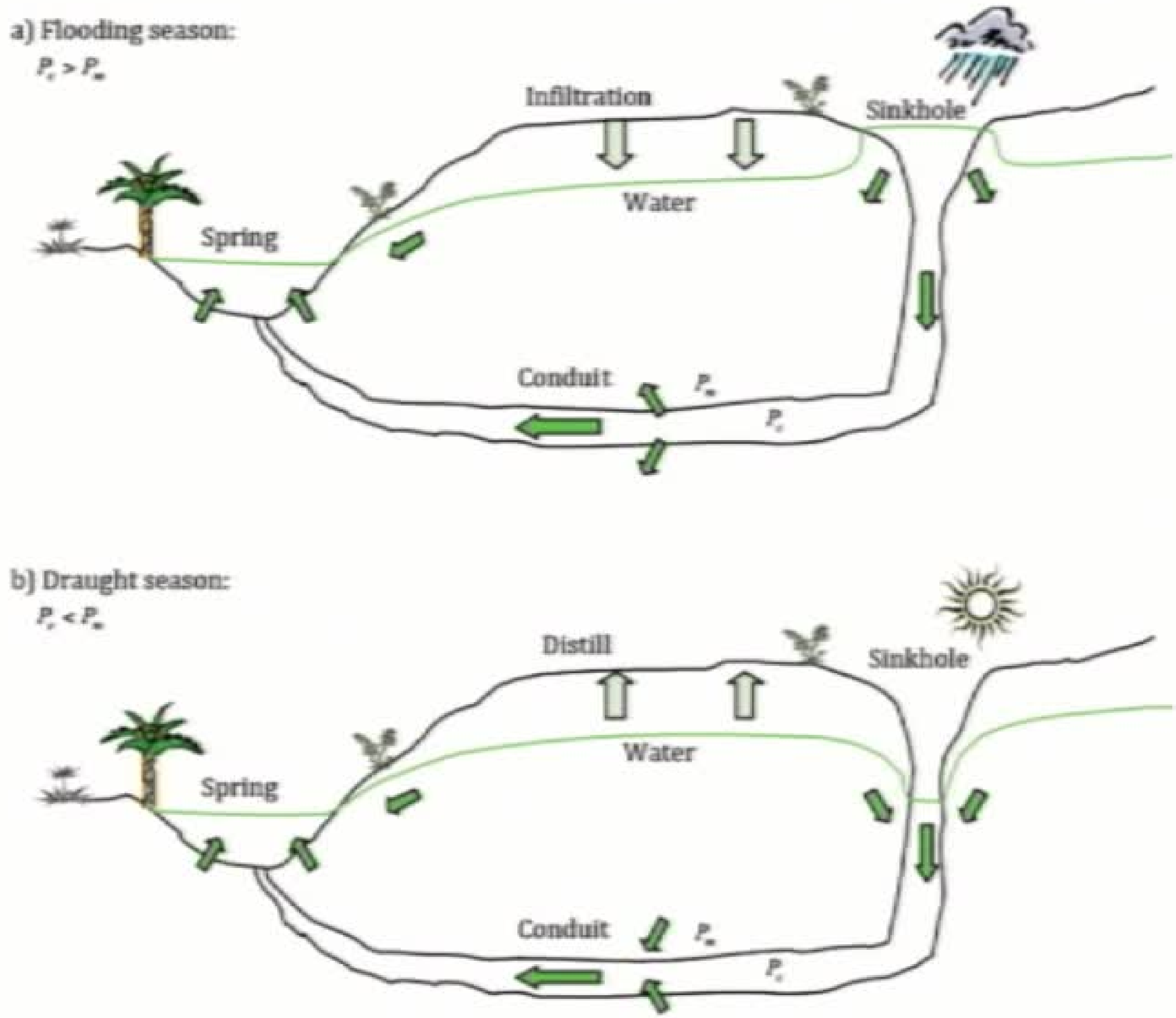
Figure: Conceptual interacting flows



Interface bc: Navier-Stokes-Darcy II (BJSJ)

- Beavers-Joseph 1967 interface b.c. (empirical with flat interface)

$$-\boldsymbol{\tau}\cdot\mathbb{T}(\mathbf{v}_c, P_c)\mathbf{n}_{cm} = \frac{\alpha\eta\sqrt{3}}{\sqrt{\text{trace}(\boldsymbol{\kappa})}}\boldsymbol{\tau}\cdot(\mathbf{v}_c - \mathbf{v}_m)$$

equivalent formulation in the flat interface case

$$-\eta\left(\frac{\partial u_{c1}}{\partial y} + \frac{\partial u_{c2}}{\partial x}\right) = \frac{\alpha\eta\sqrt{3}}{\sqrt{\text{trace}(\boldsymbol{\kappa})}}(u_{c1} - u_{m1})$$

Heuristics: friction proportional to jump in velocity.

- Beavers-Joseph-Saffman-Jones 1971(Navier-slip) interface b.c.

$$\boxed{-\boldsymbol{\tau}\cdot\mathbb{T}(\mathbf{v}_c, P_c)\mathbf{n}_{cm} = \frac{\alpha\eta\sqrt{3}}{\sqrt{\text{trace}(\boldsymbol{\kappa})}}\boldsymbol{\tau}\cdot\mathbf{v}_c}$$

- $$\alpha = \alpha_{BJ}$$

Does α_{BJ} depend on the local geometry?
(Recent progress via Stokes-Brinkman a la Fei&Niu&W.)

$$< \partial_t \vec{u}_c, \vec{v}_c >_c + b(\vec{u}_c, \vec{u}_c, \vec{v}_c) + \frac{\cdot}{Re}(\mathbb{D}(\vec{u}_c), \mathbb{D}(\vec{v}_c))_c - Eu(h_c, \nabla \cdot \vec{v}_c)_c$$

$$+ \int_{\Gamma_i} \left(\frac{\beta(\vec{x})}{Re\sqrt{Da}} (\vec{u}_c \cdot \vec{\tau})(\vec{v}_c \cdot \vec{\tau}) + Eu\, h_m \vec{v}_c \cdot \vec{n} \right) + Eu(\nabla \cdot \vec{u}_c, q_c)_c$$

$$+ \frac{1}{2} \int_{\Gamma_i} \vec{u}_c \cdot \vec{n} \vec{u}_c \cdot \vec{v}_c \quad = \quad 0;$$

$$< \frac{1}{\chi} \partial_t \vec{u}_m, \vec{v}_m >_m + (\frac{1}{Da\, Re} \vec{u}_m, \vec{v}_m)_m + Eu(\nabla h_m, \vec{v}_m)_m$$

$$- Eu(\vec{u}_m, \nabla q_m)_m - Eu \int_{\Gamma_i} \vec{u}_c \cdot \vec{n}\, q_m \quad = \quad 0;$$

where

$$\vec{v}_c \in \mathbf{H}_{c,0}, q_c \in L^2(\Omega_c); \vec{v}_m \in$$

$$b(\vec{u}, \vec{v}, \vec{w}) = \frac{1}{2} \int_{\Omega_c} ((\vec{u} \cdot \nabla \vec{v})\vec{w} - (\vec{u} \cdot \nabla \vec{w})\vec{v}) + \frac{1}{2} \int_{\Gamma_i} (\vec{v} \cdot \vec{w} \vec{u} \cdot \vec{n} - \vec{u} \cdot \vec{v} \vec{w} \cdot \vec{n})$$

- The orange term disappears when we use the total pressure in the balance of the normal component of the normal stress.
- The nonlinear term treatment is a generalization of the Temam approach.
- Existing uniqueness result require smallness assumption (small Reynolds number and/or generalized Grashoff number ...): Quarteroni, Discacciati, Miglio, Girault, Riviere, Cesmilioglu, Chidyagwai, Layton, ...

Two-dimensional well-posedness (W.-Wu 2018)

- First order (independent of choice of interface b.c.)

$$\frac{\partial \vec{u}_c^1}{\partial t} + (\vec{u}_c^1 \cdot \nabla)\vec{u}_c^0 + (\vec{u}_c^0 \cdot \nabla)\vec{u}_c^1 - \frac{2}{Re}\nabla \cdot \mathbb{D}(\vec{u}_c^1) + Eu\nabla h_c^1 = 0, \quad \nabla \cdot \vec{u}_c^1 = 0, \text{ in } \Omega_c;$$
$$\vec{u}_m^1 = 0 \text{ in } \Omega_m;$$
$$\vec{u}_c^1 \cdot \vec{n} = 0, \quad 2\vec{\tau} \cdot \mathbb{D}(\vec{u}_c^0)\vec{n} = -\beta \vec{u}_c^1 \cdot \vec{\tau} \text{ on } \Gamma_i,$$

- Second order (influence of choice of interface b.c. on free-flow)

$$\frac{\partial \vec{u}_c^2}{\partial t} + (\vec{u}_c^2 \cdot \nabla)\vec{u}_c^0 + (\vec{u}_c^0 \cdot \nabla)\vec{u}_c^2 - \frac{2}{Re}\nabla \cdot \mathbb{D}(\vec{u}_c^2) + Eu\nabla h_c^2 = -(\vec{u}_c^1 \cdot \nabla)\vec{u}_c^1, \quad \nabla \cdot \vec{u}_c^2 = 0,$$
$$\vec{u}_m^2 = -Re\,Eu\nabla h_m^0, \quad \nabla \cdot \vec{u}_m^2 = 0 \text{ in } \Omega_m;$$
$$h_m^0 = h_c^0 \text{ on } \Gamma_i$$
$$\vec{u}_c^2 \cdot \vec{n} = \vec{u}_m^2 \cdot \vec{n}, \quad 2\vec{\tau} \cdot \mathbb{D}(\vec{u}_c^2)\vec{n} = -\beta \vec{u}_c^2 \cdot \vec{\tau}, \quad -\frac{2}{Re\,Eu}\vec{n} \cdot \mathbb{D}(\vec{u}_c^2)\vec{n} + h_c^2 + \frac{1}{2Eu}|\vec{u}_c^1|^2 = h_m^2,$$

- $\vec{u}_c^2, h_c^2, h_m^0, \vec{u}_m^2$ independent of the choice of interface b.c., i.e., the three different interface boundary conditions are all consistent up to the order of ε^3. The impact will be on $\vec{u}_m^4$ through its boundary condition on Γ_i.
- fluid exchange rate, $\vec{u}_c \cdot \vec{n}$, of the order of $\varepsilon^2 = Da$. Challenging numerics (pressure on the boundary).

A first order unconditionally energy stable and decoupled scheme in the nonlinear case

- A uniquely solvable, linear, energy stable, fully discrete scheme

$$
\begin{aligned}
&< \delta_t \vec{u}_c^{n+1}, \vec{v}_c >_c + \tilde{b}(\vec{u}_c^{n}, \vec{u}_c^{n+1}, \vec{v}_c) + \frac{2}{Re}(\mathbb{D}(\vec{u}_c^{n+1}), \mathbb{D}(\vec{v}_c))_c - Eu(h_c^{n+1}, \nabla\cdot\vec{v}_c)_c \\
&\quad + \int_{\Gamma_i}\left(\frac{\beta(\vec{x})}{Re\sqrt{Da}}(\vec{u}_c^{n+1}\cdot\vec{\tau})(\vec{v}_c\cdot\vec{\tau}) + Eu\, h_m^{n+1}\vec{v}_c\cdot\vec{n}\right) + Eu(\nabla\cdot\vec{u}_c^{n+1}, q_c)_c = 0; \\
&< \frac{1}{\chi}\delta_t \vec{u}_m^{n+1}, \vec{v}_m >_m + \left(\frac{1}{Da\, Re}\vec{u}_m^{n+1}, \vec{v}_m\right)_m + Eu(\nabla h_m^{n+1}, \vec{v}_m)_m \\
&\quad + C_r Eu^2 \delta t(\nabla h_m^{n+1}, \nabla q_m)_m - Eu(\vec{u}_m^{n+1}, \nabla q_m)_m - Eu\int_{\Gamma_i}\vec{u}_c^{n}\cdot\vec{n}\, q_m = 0;
\end{aligned}
$$

- modified trilinear term (generalization to Temnam's permutation method)

$$\tilde{b}(\vec{u}, \vec{v}, \vec{w}) = \frac{1}{2}\int_{\Omega_c}((\vec{u}\cdot\nabla\vec{v})\vec{w} - (\vec{u}\cdot\nabla\vec{w})\vec{v}) + \frac{1}{2}\int_{\Gamma_i}(\vec{u}\cdot\vec{w}\vec{v}\cdot\vec{n} - \vec{u}\cdot\vec{v}\vec{w}\cdot\vec{n})$$

  - Formal first order follows from the fact that $b(\vec{u}, \vec{v}, \vec{w}) - \tilde{b}(\vec{u}, \vec{v}, \vec{w}) \approx \vec{u} - \vec{v}$
  - $\tilde{b}(\vec{u}, \vec{v}, \vec{v}) = 0$

- First order regularizing term (red) explicit time derivative in Darcy equation is retained. (Pressure regularization is related to artificial compressible

定理 (Energy stability of the linear decoupled scheme (Chen-W.-Zhang 2018))

- *The first order, linear, decoupled scheme is energy stable for the Navier-Stokes-Darcy system with classical Darcy or Groundwater equation is the sense that*

$$\mathcal{E}^{k+1} + \frac{\tau}{2Re}\|\mathbb{D}(\vec{u}_{c,h}^{k+1})\|_{L^2(\Omega_c)}^2 + \frac{\tau}{Re\,Da}\|\vec{u}_{m,h}^{k+1}\|_{L^2(\Omega_m)}^2 + \frac{\tau\beta}{Re\sqrt{Da}}\|\vec{u}\cdot\vec{\tau}\|_{L^2(\Gamma_i)}^2 \leq \mathcal{E}^k$$

- *The scheme is energy stable in the presence of the explicit time-derivative term in the Darcy's equation provided that the spatial discretization satisfies the usual inf-sup condition and a standard inverse Poincaré type inequality:*

$$\sup_{\mathbf{v}_h\in\mathbf{X}_c^h}\frac{(\nabla\cdot\mathbf{v}_h,q_h)_c}{\|\mathbf{v}_h\|_{H^1}} \geq c\|q_h\|_{L^2}, \quad \forall q_h\in M_c^h$$
$$\sup_{\mathbf{v}_h\in\mathbf{X}_m^h}\frac{(\mathbf{v}_h,\nabla q_h)_m}{\|\mathbf{v}_h\|_{L^2}} = \|\nabla q_h\|_{L^2} \geq c\|q_h\|_{L^2}$$
$$h\|\mathbf{v}_h\|_{H^1(\Omega_c)} \leq C_p\|\mathbf{v}\|_{L^2(\Omega_c)}$$

  and the regularizing term is sufficiently strong in the sense that there exists a C_0, depending on the C_p and the geometry of the domain, so that the scheme is energy stability for $C_r \geq C_0$

- The pressure stabilizing method is related to Temam's artificial compressibility approach.

Summary and Questions

- Conduit flow and porous media flow can be coupled with appropriate coupling conditions resulting a well-posed Navier-Stokes-Darcy system.
- The different choices of the balance of the normal component of the normal stress (linear or nonlinear) lead to the same leading order behavior at the physically significant small Darcy number regime.
- Leading order dynamics are decoupled with the zero order expansion rigorously justified.
- A decoupled, linear, uniquely solvable, and energy stable first-order in time scheme has been developed.
- More physics can be added: thermal effect, Boussinesq effect, turbulent effect, two-phase, chemical reaction,...
- Transport can be dealt with utilizing enriched FEM for discontinuous permeability to avoid oscillation.
- Rigorous validation of the formal asymptotic expansion to higher orders for the nonlinear case? Exchange rate?
- Better analytical understanding of the long-time behavior of the coupled nonlinear system?
- More efficient and accurate schemes in the nonlinear case, say higher order in time?
- Comparison with experimental data at relatively high Reynolds number?
- Uncertainty in hydraulic conductivity (permeability) and geometry?
- Inverse problem using sparse measurement data to infer hydraulic conductivity and conduit geometry
- Morphology of riverbed or conduits (karst genesis)?
- Complex fluids?