Dissipative Time Reversal Technique for Photoacoustic Tomography in a Cavity

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Outline

1. Free Space Photoacoustic Tomography
   - Time Reversal Technique

2. Photoacoustic Tomography in A Cavity
   - Mathematical Model and Difficulty
   - Current Time Reversal Techniques
   - Dissipative Time reversal

Photoacoustic Tomography

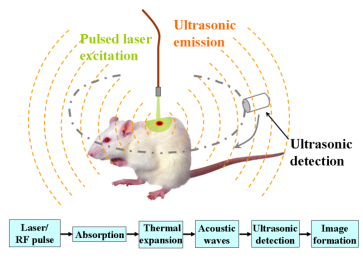


Figure: Photoacoustic tomography - Source: Wikipedia

- Ultrasonic pressure measured on observation surface S.
- One needs to find the initial pressure $f(x)$.

Mathematical Model

Wave propagation:

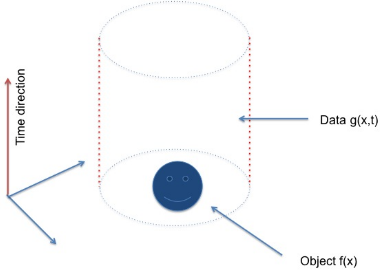
$$\begin{cases} u_{tt}(x,t) - c^2(x)\Delta u(x,t) = 0, \ x \in \mathbb{R}^n, \ t \geq 0, \\ u(x,0) = f(x), \ u_t(x,0) = 0, \quad x \in \mathbb{R}^n. \end{cases}$$

Data: $g = u|_{S\times(0,T)}$.

Problem: Find f from g.

Assumptions:

- S is a closed surface.
- T can be either a fixed finite number or infinity.



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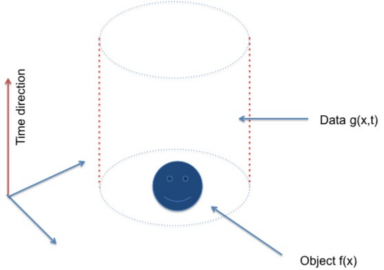

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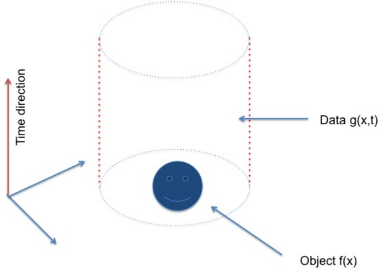

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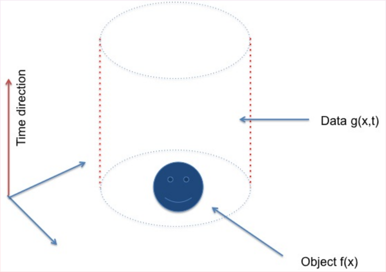

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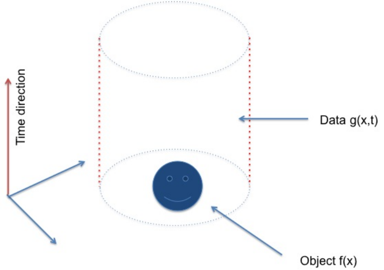

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Techniques

1. Exact Inversion formulas: Norton & Linzer ('79-'81), Xu & Wang ('02, '04), Finch, Patch & Rakesh ('04), Finch, Haltmeier & Rakesh ('07), Kunyansky ('07, '11,'15), N. ('09), Palamodov ('11 & 14), Natterer ('12), Salman ('12), Haltmeier ('11 & '13) . . .
2. Series solutions: Agranovsky & Kuchment ('07), Kunyansky ('07) . . .
3. Time-reversal: Finch, Patch & Rakesh ('04), Hristova, Kuchment & N. ('08), Hristova ('09), Stefanov & Uhlmann ('09 & '11), J. Qian, Stefanov, G. Uhlmann & H. Zhao ('11) . . .
4. Iterative method: Anastasio, Su, & Oraevsky ('12), Belhachmi, Glatz, & Scherzer ('15) . . .

. . .

Näive Time Reversal

Consider the time reversed problem

$$\begin{cases} v_{tt}(x,t) - c^2(x)\,\Delta v(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ v(x,T) = 0, \quad v_t(x,T) = 0, & x \in \Omega, \\ v(x,t) = g(x,t), & (x,t) \in S \times (0,T). \end{cases}$$

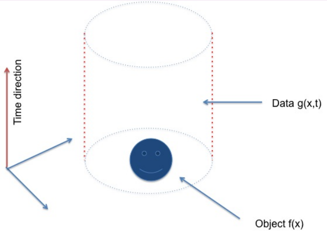

Figure: Forward model

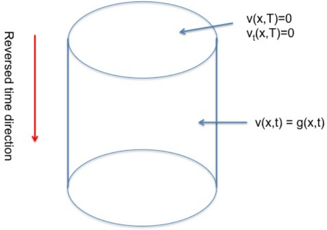

Figure: Time reversed

Näive Time Reversal

Denote: $Ag = v(.,0)$. Then,

$$Ag \to f \text{ in } H^1(\Omega), \quad T \to \infty.$$

Idea:

- $w = u - v$ solves wave equation with zero Dirichlet BC.
- Conservation of energy:

$$E_\Omega(w,0) = E_\Omega(w,T) = E_\Omega(u,T).$$

- Then,

$$\|Ag - f\|_{H^1(\Omega)} = \|w(.,0)\|_{H^1(\Omega)} \leq C\, E_\Omega(w,0) \to 0.$$

A technical issue: The initial condition and boundary condition may not be compatible.

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Fixing The Problem

- Fix 1: Hristova, Kuchment, & N. ('08), Hristova ('09).

$$\begin{cases} v_{tt}(x,t) - c^2(x)\,\Delta v(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ v(x,T) = 0, \quad v_t(x,T) = 0, & x \in \Omega, \\ v(x,t) = \chi(t) g(x,t), & (x,t) \in S \times (0,T). \end{cases}$$

Here, $\chi \equiv 1$ for $t \in [0, T-\epsilon]$ and $\chi(T) = 0$.

- Fix 2: Stefanov& Uhlmann ('09)

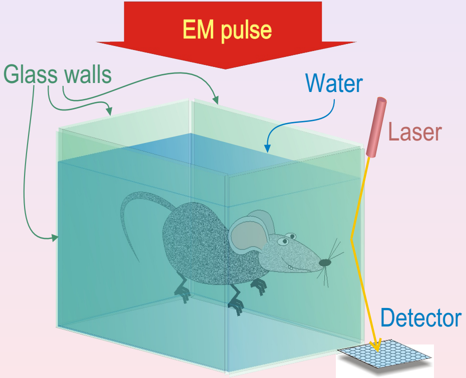
$$\begin{cases} v_{tt}(x,t) - c^2(x)\,\Delta v(x,t) = 0, & (x,t) \in \Omega \times (0,T], \\ v(x,T) = \phi(x), \quad v_t(x,T) = 0, & x \in \Omega, \\ v(x,t) = g(x,t), & (x,t) \in S \times (0,T). \end{cases}$$

Here, ϕ harmonic extension of $g(., T)$ on Ω.

Stefanov & Uhlmann obtained a Neumann series solution.

Photoacoustic Tomography in A Cavity

Photoacoustic setup at University College London:



Mathematical Model

Mathematical model:

$$\begin{cases} u_{tt}(x,t) - c^2(x)\Delta u(x,t) = 0, \ (x,t) \in \Omega \times (0,T), \\ u(x,0) = f(x), \ u_t(x,0) = 0, \quad x \in \Omega, \\ \partial_\nu u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

Data: $g = u|_{\Gamma \times (0,T)}$, where $\Gamma \subset \partial\Omega$.

Problem: Find f given g. Equivalently, invert $\Lambda : f \to g$.

Historical Remarks

Contributions: Kunyansky, Holman, & Cox ('13): series expansion; Holman & Kunyansky ('15): gradual time reversal; Stefanov & Yang ('15): averaged time reversal; Acosta & Montalto ('15): conjugate gradient method & time reversal for partially reflecting boundary.

Recently: Stefanov & Yang ('16): Landweber's iteration, Chernova & Okasen ('16): Time reversal.

Difficulty: The energy $E_{\Omega}(u, t)$ is conserved for all time t.

Gradual Time Reversal

Gradual time reversal - Holman & Kunyansky ('15):

$$\begin{cases} v_{tt}(x,t) - c^2(x)\,\Delta v(x,t) = 0, \; (x,t) \in \Omega \times (0,T), \\ v(x,T) = 0, \quad v_t(x,T) = 0, \quad x \in \Omega, \\ v(x,t) = \chi(\epsilon t)\, g(x,t), \quad (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

Here, $\epsilon = \frac{1}{T}$ and $\chi \in C_0^\infty[0,1)$. Define:

$$Ag = v(.,0).$$

Gradual Time Reversal

Then,

$$Ag = v(.,0) \to f \text{ weakly } H^1(\Omega) \text{ as } \epsilon \to 0,$$

under assumption on the eigenvalues of Laplacian with Dirichlet and Neumann boundary condition.

Features:

1. Adaptable to partial data problem (i.e., $\Gamma \neq \partial\Omega$).
2. Assumption on the eigenvalues not simple to verify.
3. The rate of convergence is unclear.

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Averaged Time Reversal

Rough idea - Stefanov & Yang ('15):

- Time reversal at all time $\tau \in (0, T]$ with Dirichlet boundary condition. Then average the result.
- Neumann series solution

$$f = \sum_{m=0}^{\infty} K^m A g = f_0 + f_1 + \dots,$$

where $K = I - A\Lambda$.

Averaged Time Reversal

Features:

1. Converges exponentially as the $N \to \infty$ as long as $T > \frac{1}{2} T(\Omega)$.
2. Not clear that $f_0 \to f$ as $T \to \infty$.
3. Theory not available for partial data problem (but some convincing numerical examples).

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New Boundary Condition

Consider the time reversed problem - N. & Kunyansky('16):

$$\begin{cases} v_{tt}(x,t) - c^2(x)\,\Delta v(x,t) = 0, \ (x,t) \in \Omega \times (0,T), \\ v(x,T) = 0, \quad v_t(x,T) = 0, \quad x \in \Omega, \\ \partial_n v(x,t) - \lambda\, v_t(x,t) = -\lambda\, g_t(x,t), (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$

Define: $Ag = v(.,0)$.

Results

Assume that $T(\Omega) < \infty$. Then,

1. If $T \to \infty$, then
$$Ag \to f \text{ exponentially.}$$
2. If $T > T(\Omega)$, one has the Neumann series formula:
$$f = \sum_{m=0}^{\infty} K^m A g = f_0 + f_1 + \dots,$$
where $K = I - A\Lambda$.

Quick Proof

Let $w = u - v$. Then,

$$\begin{cases} w_{tt}(x,t) - c^2(x)\,\Delta w(x,t) = 0, \ (x,t) \in \Omega \times (0,T), \\ w(x,T) = 0, \quad w_t(x,T) = 0, \quad x \in \Omega, \\ \partial_n w(x,t) - \lambda\, w_t(x,t) = 0, (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

We have

$$E_\Omega(w,T) - E_\Omega(w,0) = \iint\limits_{\partial\Omega \times (0,T)} \lambda |w_t(x,t)|^2 \, dx\, dt.$$

New Idea: Quick Proof

Key point: the energy of w leaks out on the boundary (going backward in time). Therefore Bardos, Lebeau & Rauch('92):

- As $T \to \infty$,

$$E_\Omega(w, 0) \to 0 \quad \text{exponentially.}$$

  That is, $v(., 0) \to f$ exponentially.

- If $T > T(\Omega)$, then

$$E_\Omega(w, 0) \leq (1 - \epsilon)\, E_\Omega(w, T).$$

  That is,

$$\|f - A\Lambda f\| \leq (1 - \epsilon)\|f\|_{H^1(\Omega)}.$$

  $\Longrightarrow$ Neumann series formula.

Partial Data Problem

Consider $\Gamma \neq \partial\Omega$. Assumption (GCC):
All the generalized geodesics rays hit Γ in a positive time $T_\Gamma < T$. Then

- Neumann series formula:

$$f = \sum_{m=0}^{\infty} K^m A g = f_0 + f_1 + \ldots,$$

  where $K = I - A\Lambda$.

- Moreover,

$$f_0 = Ag \to f \text{ exponentially as } T \to \infty.$$

Numerical Experiments (0/4)

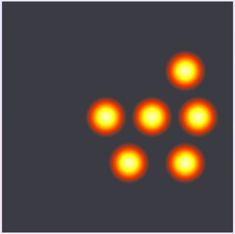


Figure: Phantom

Numerical setup:

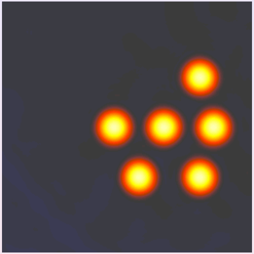
- Domain Ω is a a square of size 2×2.
- Sound speed $c = 1$, $T(\Omega) = 2\sqrt{2}$.
- Experiment for full data: $T > \sqrt{2}$.
- Experiment for partial data: left and lower edges, $T > T(\Omega) = 2\sqrt{2}$.

Numerical Experiments (1/4)

Reconstruction with large time: $T = 5$.

Phantom

From full data



From partial data

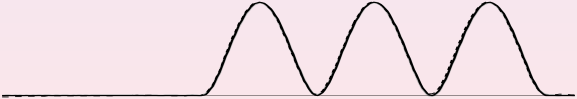
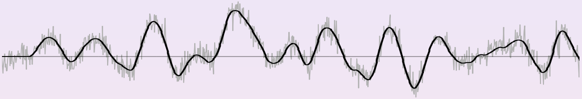


Figure: Gray: phantom, dashed: partial data, black: full data

Numerical Experiments (2/4)

Reconstruction with large time: $T = 5$. Noisy data: 50% noise (in L^2 norm).



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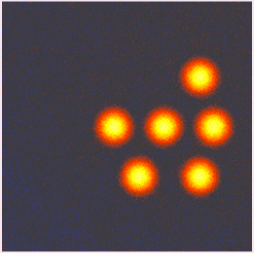

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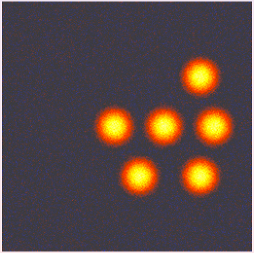


From full data

Numerical Experiments (2/4)

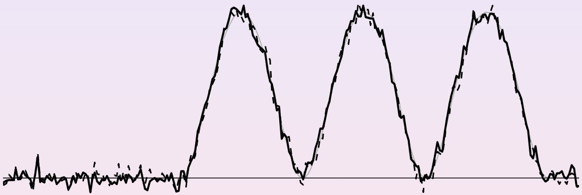
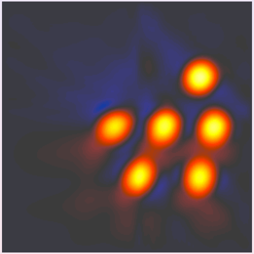


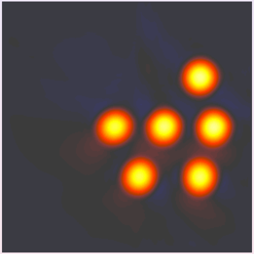
Figure: Central line profile

Numerical Experiments (3/4)

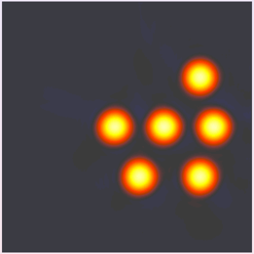
Reconstruction for full boundary data with short time $T = 1.6 > \sqrt{2}$.



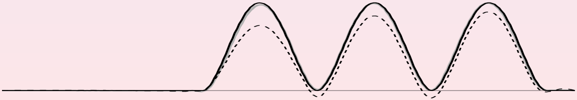
Iteration #1



Iteration #2

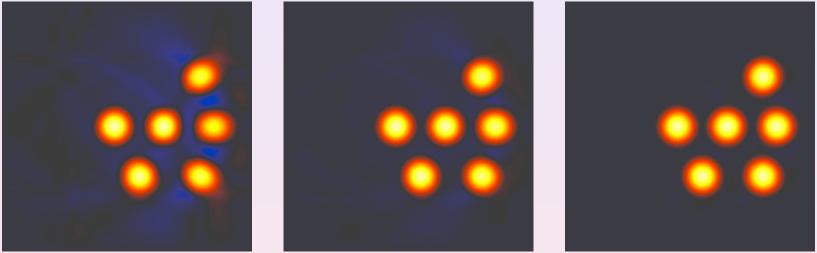


Iteration #5

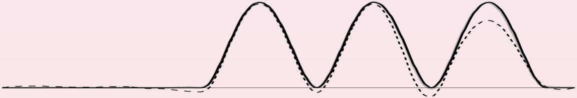


Numerical Experiments (4/4)

Reconstruction for partial data with short time $T = 3 > 2\sqrt{2}$.



Iteration #1 Iteration #2 Iteration #5



Questions

Thank you for your attention!