Rigidity of nonlinear prestrained plates

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Results in collaboration with L. Codenotti, P. Goldstein, P. Hajłasz and M. Lewicka

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The prestrained Kirchhoff models

- **Arise as Γ-limit** (Friesecke, James and Müller (02, 06); Lewicka, P. (10), Lewicka, Ochoa, P. (14))
- $\Omega \subset \mathbb{R}^2$ and $\mathfrak{g} : \Omega \to Sym_+^{2\times 2}$ Riemannian metric.

  Minimize $I_{\mathfrak{g}}(y) = \int_\Omega |H_y|^2$ in the class of isometric immersions

  $\mathfrak{I}^{2,2}_{\mathfrak{g}} := \{y : (\Omega, \mathfrak{g}) \to \mathbb{R}^3; \quad (Dy)^T Dy = \mathfrak{g} \quad a.e., \quad \int_\Omega |D^2 y|^2 < \infty\}.$
- $\Omega \subset \mathbb{R}^2$, $f : \Omega \to \mathbb{R}$.

  Minimize $I_f(u) = \int_\Omega |D^2 u|^2$ in the class of Monge-Ampère solutions:

  $MA_f := \{u : \Omega \to \mathbb{R}; \quad \det D^2 u = f \quad a.e., \quad \int_\Omega |D^2 u|^2 < \infty\}$
- **Main problems**:

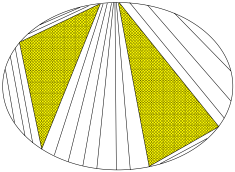
  Existence of admissible mappings/solutions,

  (with or without boundary conditions).

  Uniqueness and regularity of minimizers/ critical points.

  Rigidity or flexibility of isometric mappings/MA solutions.

A tale of two rigidities



- Rigidity for C^∞:
- Monge-Ampère equation:

  $f \equiv 0 \implies u$ is developable.

  $f > 0 \implies u$ is locally convex (modulo sign) in Ω.
- $\mathfrak{g} = \mathrm{Id}$ (Darboux).

  C^∞ isometric immersions of flat 2d domains into $\mathbb{R}^3$ are developable.

  Elliptic $\mathfrak{g}$ (Hilbert).

  Images of C^∞ isometries of elliptic surfaces into $\mathbb{R}^3$ are convex.

Rigidity and flexibility in Sobolev and Hölder spaces

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- $W^{2,2}$ solutions:
  Developability for $f \equiv 0$.
  Convexity for $f > 0$.
- $C^{1,\alpha}$ solutions, $\alpha > 2/3$:
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  Developability for $f \equiv 0$.
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  Total flexibility
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$W^{2,2}$ rigidity for elliptic $\mathfrak{g}$

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If $\mathfrak{g}$ is smooth with $K(\mathfrak{g}) > 0$ and $y \in \mathfrak{I}_{\mathfrak{g}}^{2,2}$, then y is smooth.

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- If $\forall x \in \Omega$, $y(\Omega)$ admits a tangent plane at $y(x)$, over which it is locally the graph of a scalar function $\longrightarrow$ Monge-Ampère equation.
- The normal $\mathbf{n} \in W^{1,2}(\Omega, \mathbb{S}^2)$ satisfies $\mathbf{n} \cdot (\mathbf{n}_{,1} \times \mathbf{n}_{,2}) > 0$ a.e. in $\Omega \subset \mathbb{R}^2$. Nontrivial result: $\mathbf{n} \in C^0$.

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  Theorem: (Vodopyanov and Goldstein-1975) If $v \in W^{1,2}(\Omega, \mathbb{R}^2)$ satisfies $\det(Dv) > 0$ a.e. in $\Omega \subset \mathbb{R}^2$, then $v \in C^0$.

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Theorem (Goldstein, Hajłasz and P. (15))

M, N orientable manifolds of dimension n, N closed and $\mathfrak{v} \in W^{1,n}(M, N)$. If the tangent map does not change orientation a.e. in M, then $\mathfrak{v} \in C^0(M, N)$.

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- Conjecture: If $\mathfrak{g}$ is C^2 and $y \in \mathfrak{I}_{\mathfrak{g}}^{2,2}$, then y is C^1.

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  Proof: Write $v := Du \in W^{1,2}(\Omega, \mathbb{R}^2)$, and
  $v_\delta(x_1, x_2) := v(x_1, x_2) + \delta(-x_2, x_1)$. Then

$$det\, Dv_\delta = det\, Dv + \delta^2 > \delta^2 - c_0 > 0$$

  for δ large enough.

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- Conjecture: $\Omega = \mathbb{R}^2$, $\mathfrak{g} \in C^2$ and $K(\mathfrak{g}) \equiv -1$. Then $\mathfrak{I}_{\mathfrak{g},loc}^{2,2} = \emptyset$.

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- Conjecture: $\mathfrak{g} \in C^2$. $\exists \rho > 0$, $\forall y \in \mathfrak{I}^{2,2}_{\mathfrak{g}}$, $\mathrm{diam}(y(\Omega)) > \rho$.

Flexibility for the Monge-Ampère equation

- The very weak Hessian determinant for $u \in W^{1,2}_{loc}$:

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Theorem (Lewicka and P. (15))

$\Omega \subset \mathbb{R}^2$ open, bounded, simply connected and of sufficient regularity. $f = -\Delta g$ for $g \in C^1(\bar{\Omega})$. Then for all $u_0 \in C^0(\bar{\Omega})$, and all $\alpha < 1/7$, there exists a sequence of weak solutions $u_k \in C^{1,\alpha}$:

$$\mathcal{D}et D^2 u_k = f$$

converging uniformly to u_0. If $u_0 \in C^1(\bar{\Omega})$, then u_k are uniformly bounded in C^1. Finally $\mathcal{D}et D^2$ is discontinuous at all $u \in W^{1,2}(\Omega)$.

Rigidity for *alpha* $> 2/3$

- Rigidity of weakest solutions for $\alpha > 2/3$:

Theorem (Lewicka and P. (15))

Let $u \in C^{1,\alpha}$, $\alpha > 2/3$. If $\mathcal{D}et D^2 u = 0$, then u is developable. If $\mathcal{D}et D^2 u \geq c > 0$ is Dini continuous, then u is locally convex and an Alexandrov solution.

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- **Key ingredient:** Let $\mathcal{D}et D^2 u = f = -\Delta g$, $g \in C^2(\bar{\Omega})$, $u \in C^{1,2/3+}$. If $U \subset \Omega$ open, $F \in C_c^\infty(\mathbb{R}^2 \setminus Du(\partial U))$ then:

$$\int_U (F \circ Du) f = \int_{\mathbb{R}^2} F(y) \deg(y, U, Du) \, \mathrm{d}y.$$

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&\quad \left. - \mathrm{curl}(g_\varepsilon \mathrm{Id}) \Big] \right| + \|g_\varepsilon - g\|_1 + \left| \int_U (F \circ Du_\varepsilon - F \circ Du) f \right| \leq C \varepsilon^{3\alpha - 2} + o(1).
\end{aligned}$$

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  $$\int_U (F \circ v_\delta)\delta^2 = \int_{\mathbb{R}^2} F(y) \deg(y, U, v_\delta)\, \mathrm{d}y.$$
- For all $y \in v_\delta(U) \setminus v_\delta(\partial U)$, $\deg(y, U, v_\delta) \geq 1$.
  v_δ converges uniformly to v.
  For all $y \in v(U) \setminus v(\partial U)$, $\deg(y, U, v) = 0$.

Thank you for your attention.