Chimeras, Cluster States, and Symmetries: Experiments on the Smallest Chimera

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SIAM Dynamical Systems

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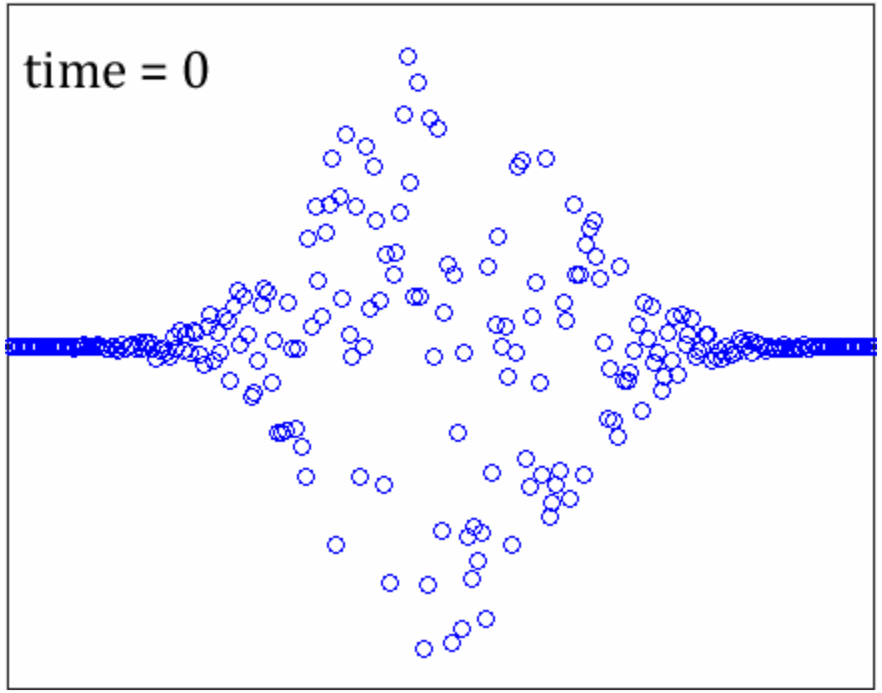
Chimeras

- Domains of coherence and incoherence
- Traditionally:
    - large networks
    - non-local coupling
    - only for special initial conditions

Non-local coupling

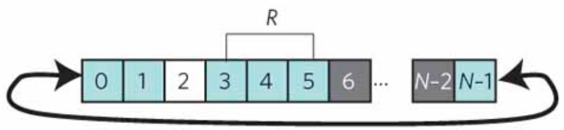
N=256



Abrams and Strogatz
Phys. Rev. Lett. **93**, 174102
22 October 2004

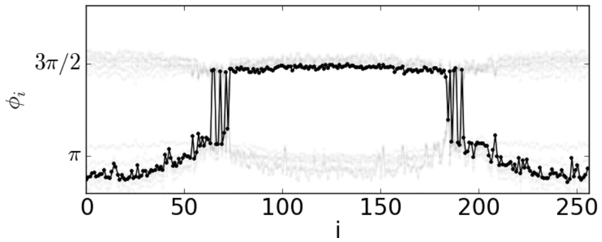
Kuramoto, Y., and D. Battogtokh. "Coexistence of Coherence and Incoherence in Nonlocally Coupled Phase Oscillators." *NONLINEAR PHENOMENA IN COMPLEX SYSTEMS* 5.4 (2002): 380-385.

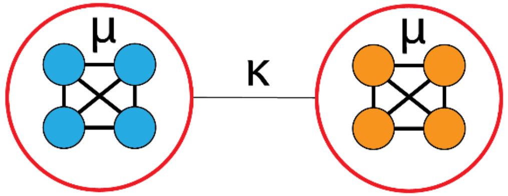
Experimental realizations of chimeras



Spatial light modulator feedback

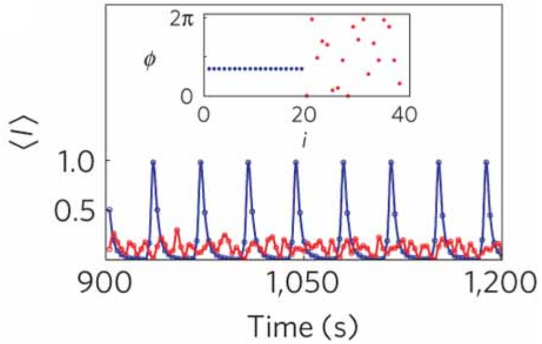
A. M. Hagerstrom, T. E. Murphy, R. Roy, P. Hövel, I. Omelchenko, and E. Schöll. Experimental observation of chimeras in coupled-map lattices. *Nature Physics*, **8**: 658 (2012)





Chemical oscillators

M. R. Tinsley, S. Nkomo, and K. Showalter. Chimera and phase-cluster states in populations of coupled chemical oscillators. *Nature Physics* **8**: 662 (2012)



Metronomes

E. A. Martens, S. Thutupalli, A. Fourrière, O. Hallatschek, PNAS **110**(26) 10563-10567 (2013)

Many others, most relatively large networks

Chimeras in small networks

Simulations by Böhm et al., Phys. Rev. E, **91** 040901 (2015): amplitude-phase coupling induces **chimeras**

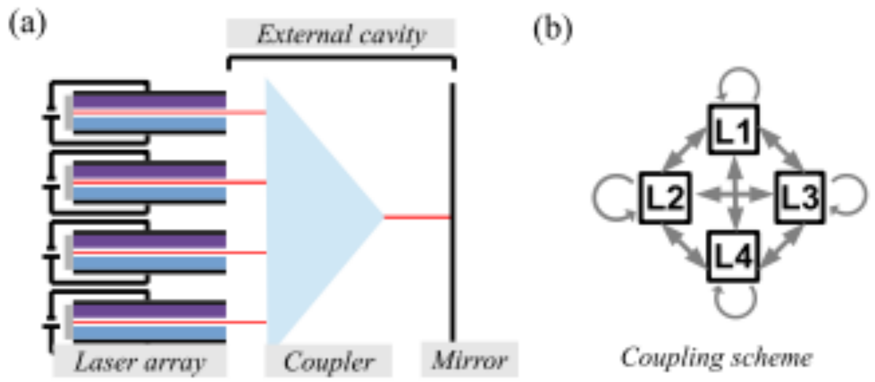
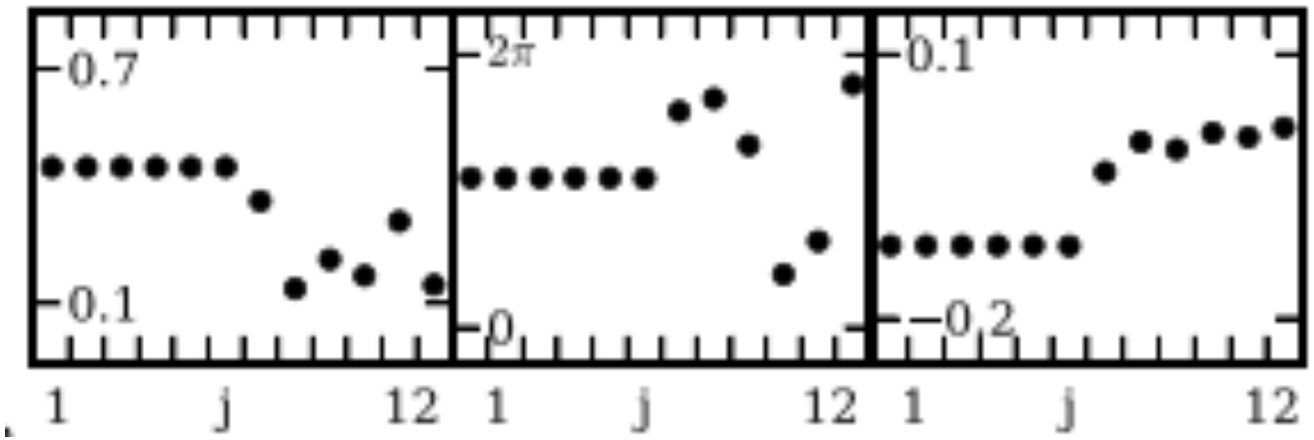
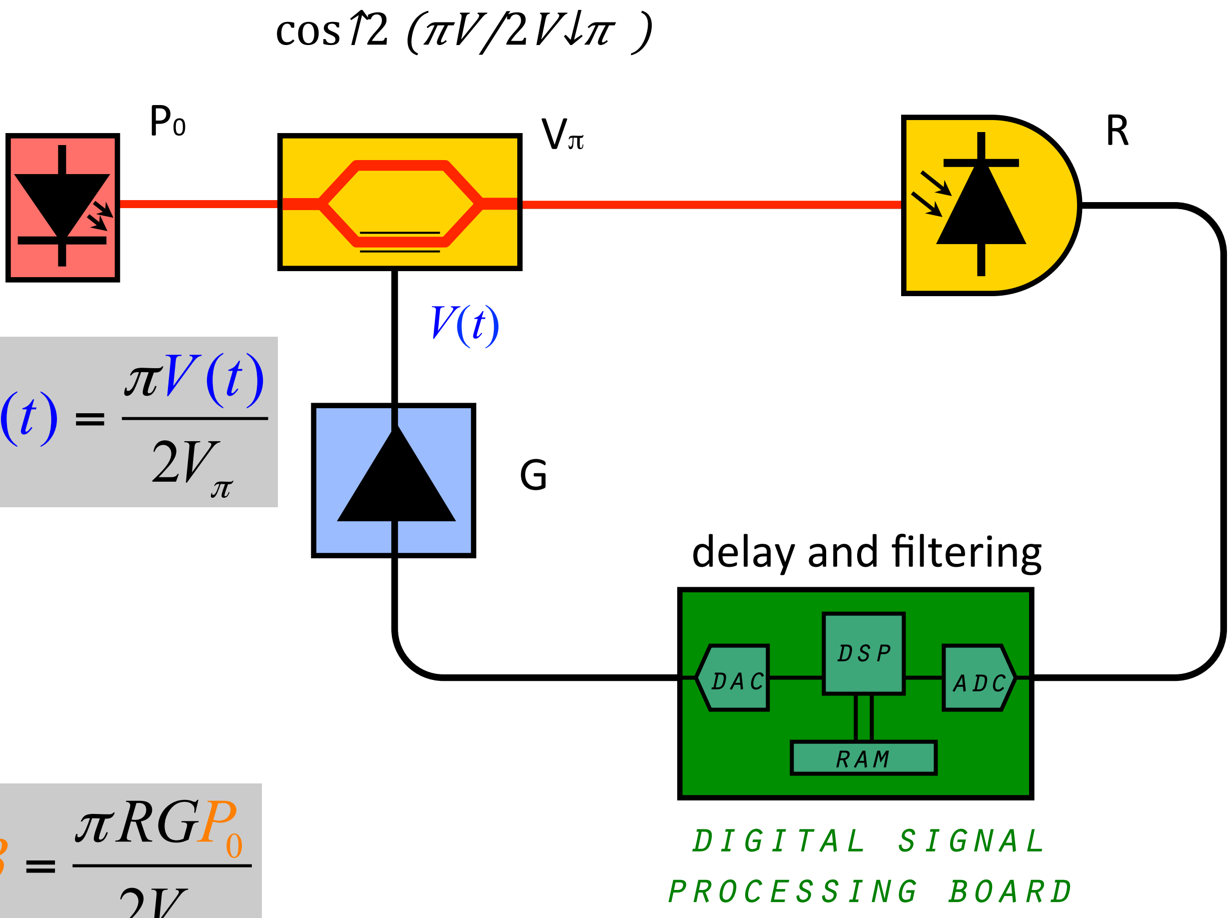


FIG. 1. (Color online) (a) Scheme of a laser array coupled by a common mirror via an external cavity. (b) All-to-all coupling scheme.



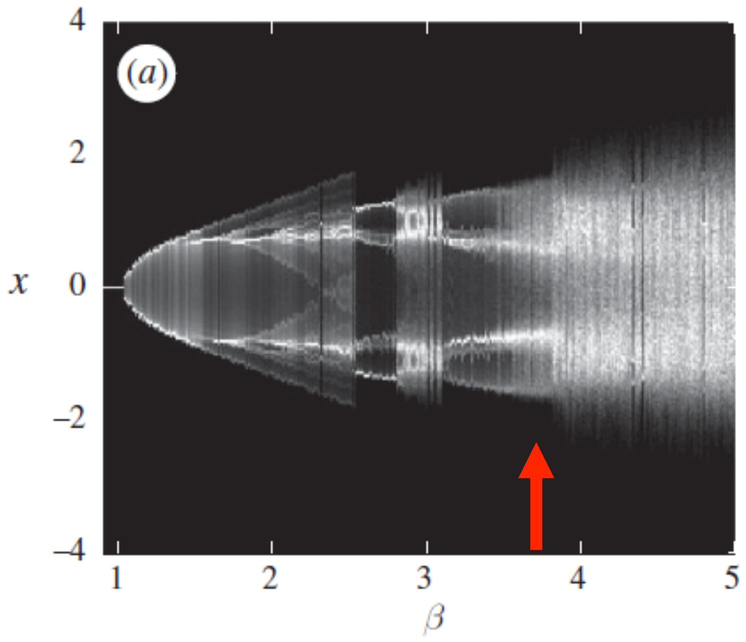
Amplitude Phase Inversion

Optoelectronic oscillators



$$x(t) = \frac{\pi V(t)}{2V_\pi}$$

$$\beta = \frac{\pi R G P_0}{2V_\pi}$$

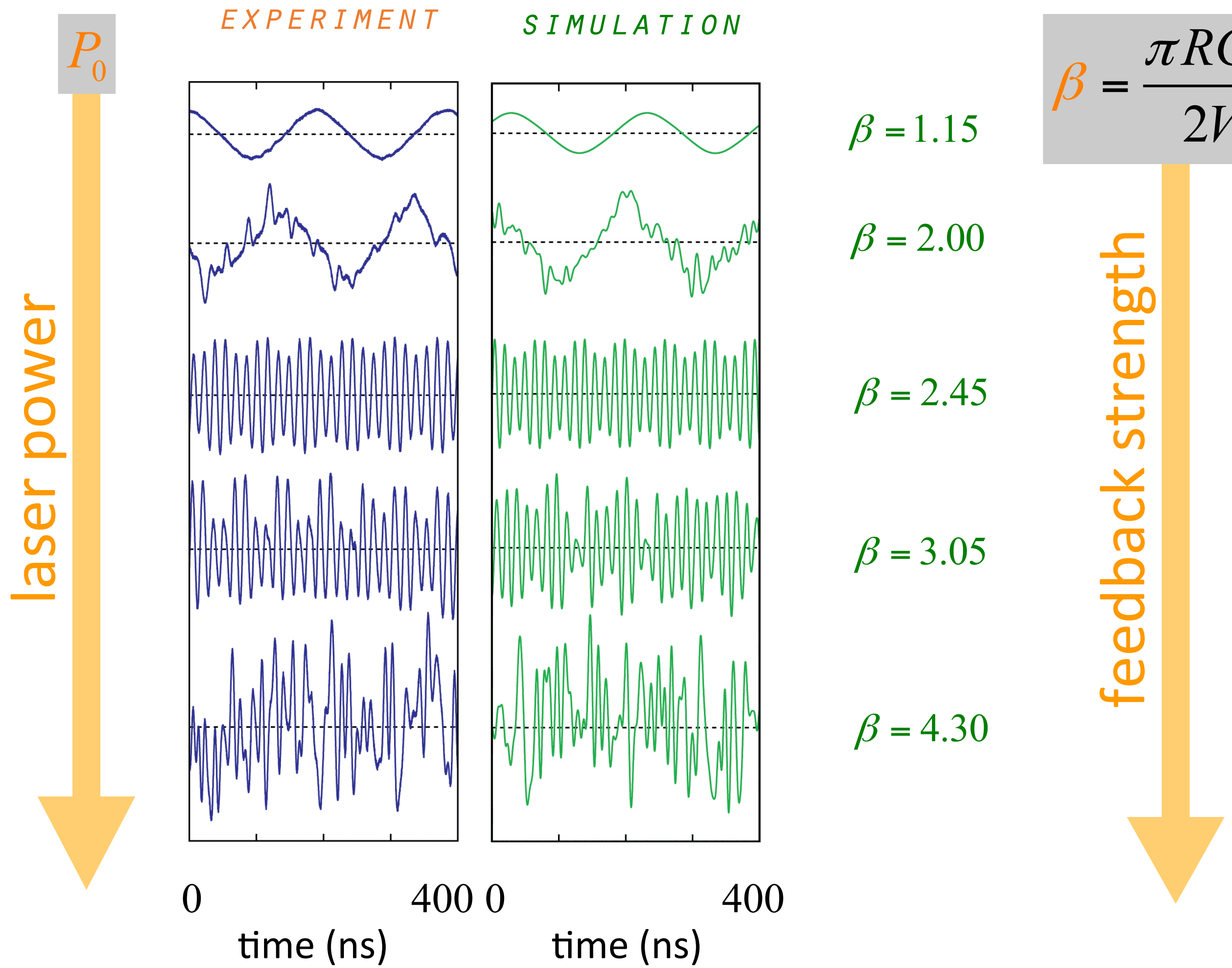


Y. Chembo Kouomou *et al.*, *Phys. Rev. Lett.* 95, 203903 (2005)

T. E. Murphy *et al.*, *Phil. Trans. R. Soc. A* 368, 343 (2010)

Optoelectronic chaos

$$x(t) = \frac{\pi V(t)}{2V_\pi}$$

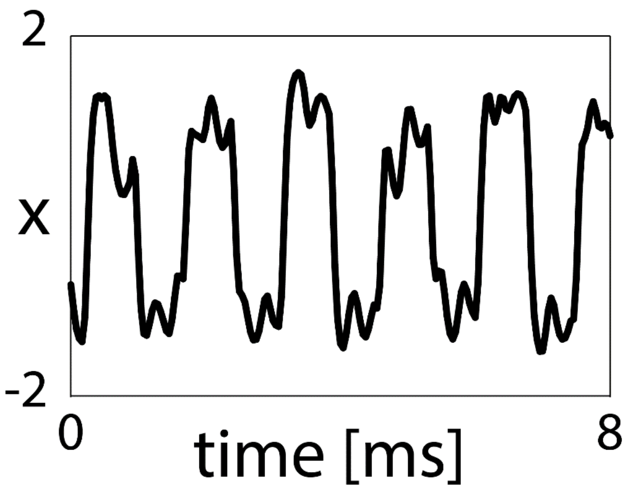


$$\beta = \frac{\pi R G P_0}{2V_\pi}$$

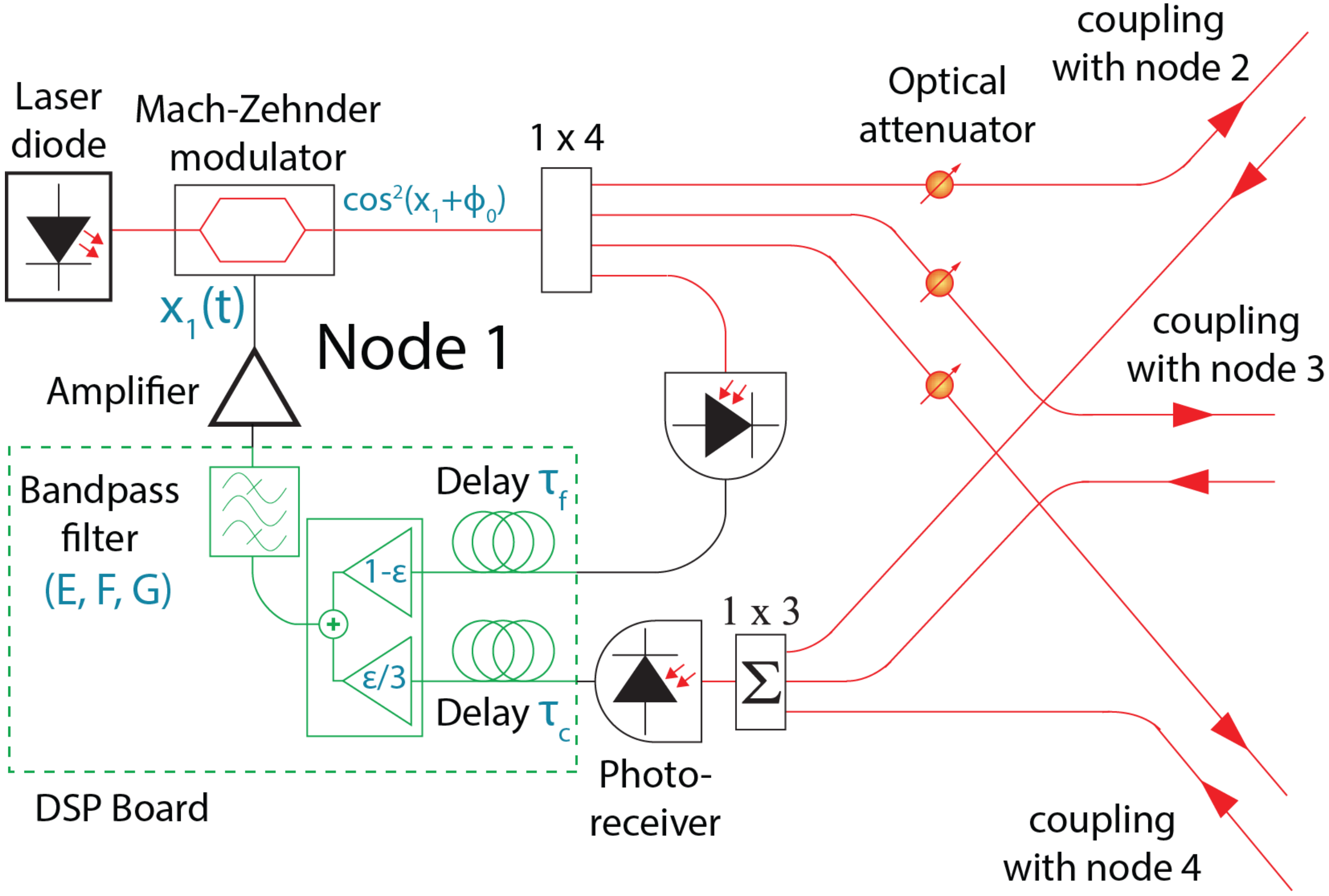
A. B. Cohen *et al.*, *Phys. Rev. Lett.* 101, 154102 (2008)

Networks of opto-electronic oscillators

Uncoupled oscillator



$$A_{ij} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



β=3.8 τ_f=1.4 ms

Model

Breaks Laplacian coupling and induces multistability

$$\dot{\mathbf{u}}_i(t) = \mathbf{E}\mathbf{u}_i(t) - \mathbf{F}\beta\cos^2(x_i(t) + \phi_0),$$

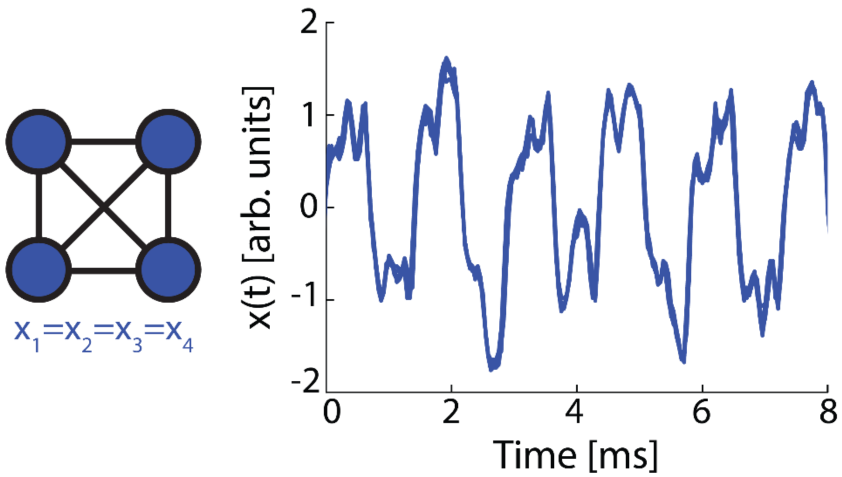
$$x_i(t) = \mathbf{G}\left(\mathbf{u}_i(t-\tau_f) + \frac{\varepsilon}{n_{in}}\sum_j A_{ij}\boxed{\left(\mathbf{u}_j(t-\tau_c) - \mathbf{u}_i(t-\tau_f)\right)}\right)$$

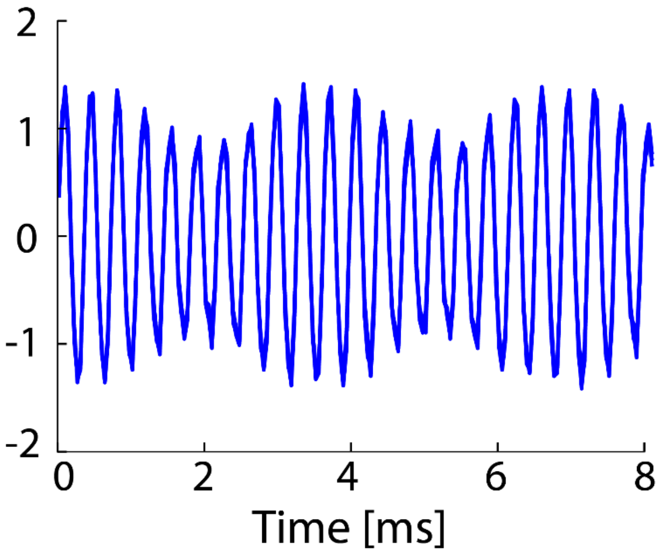
$$\mathbf{E} = \begin{bmatrix} -(\omega_L + \omega_H) & -\omega_L \\ \omega_H & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \omega_L \\ 0 \end{bmatrix}, \quad \text{and } \mathbf{G} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$\boldsymbol{u}(t) \equiv$ filter state vector
$A_{ij} \equiv$ adjacency matrix
ω_L and $\omega_H \equiv$ band-pass filter cutoffs

T. E. Murphy *et al.*, *Phil. Trans. R. Soc. A* 368, 343 (2010)
C.R.S. Williams *et al.*, *Chaos*, **23**(4) 043117 (2013)

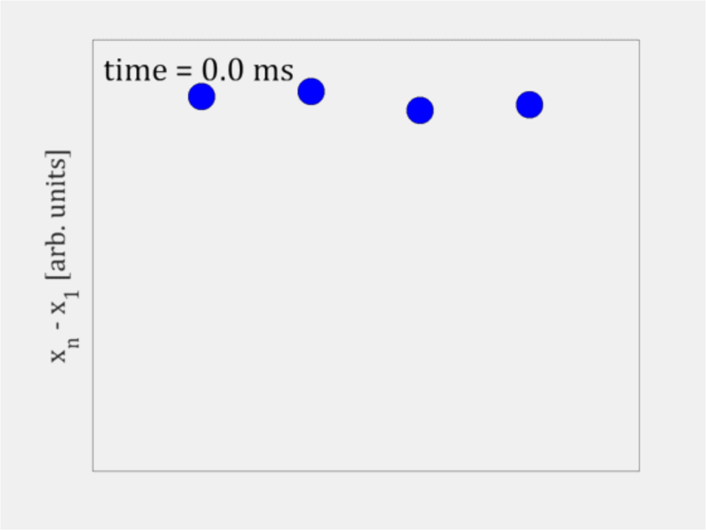
Global synchronization

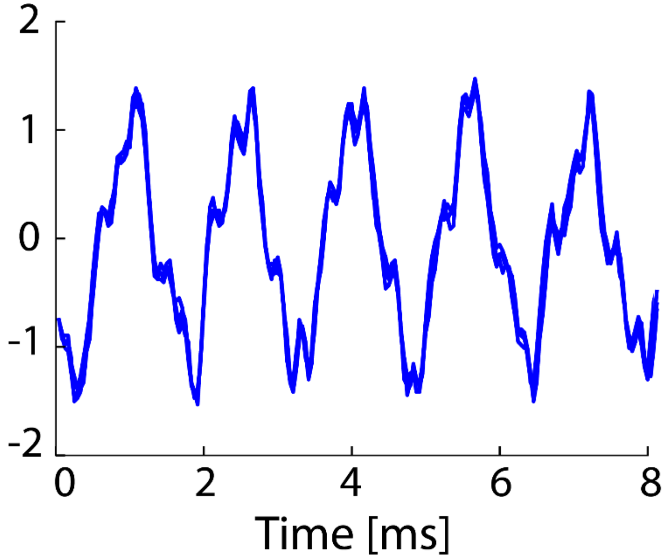




τ_c=1.8 ms, ε=0.50

τ_c=1.8 ms
ε=0.40

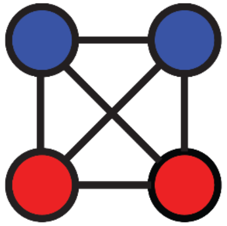


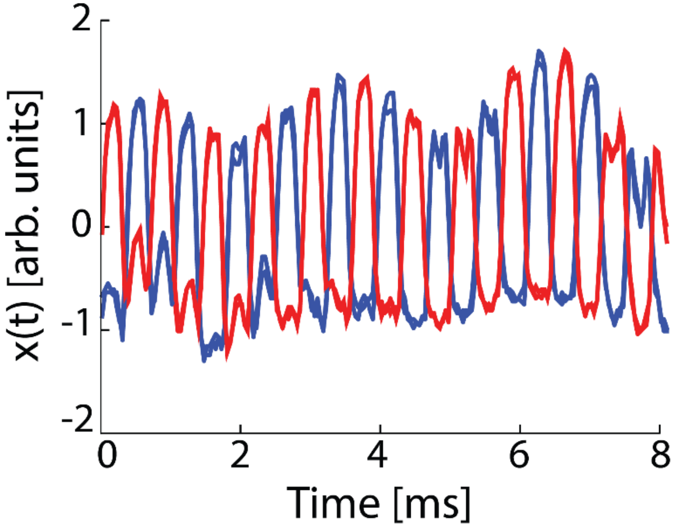


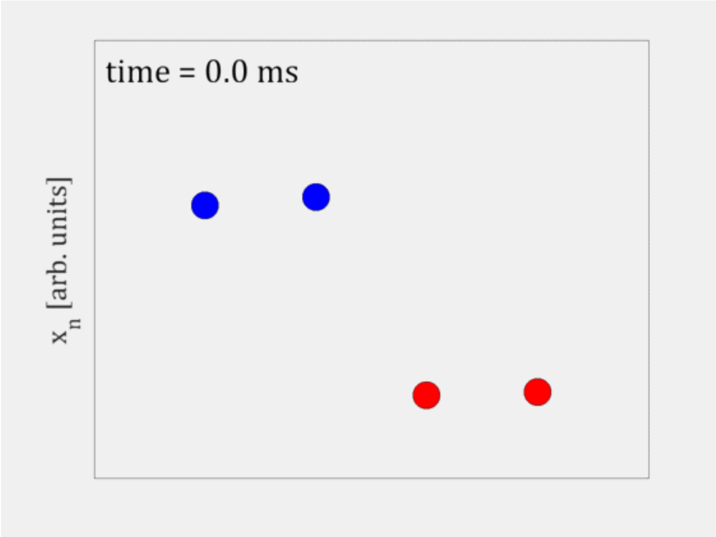
τ_c=1.8 ms, ε=0.50

Cluster synchronization

Doublet-doublet

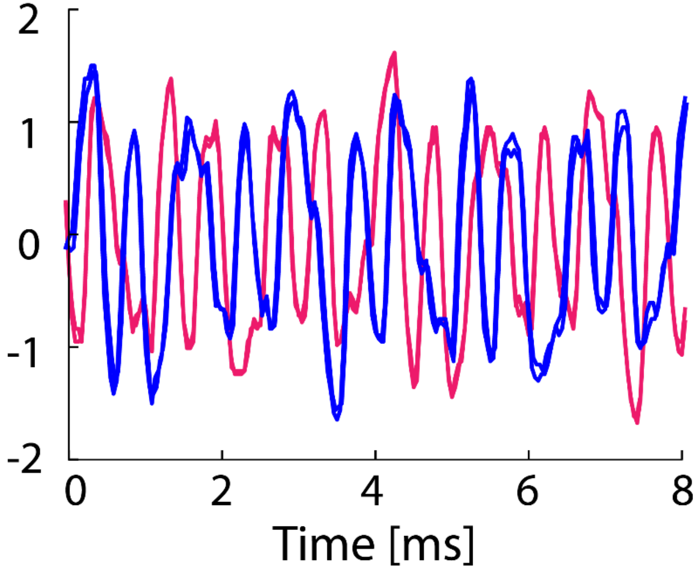






τ_c=1.8 ms, ε=0.45

τ_c=2.5 ms, ε=0.50



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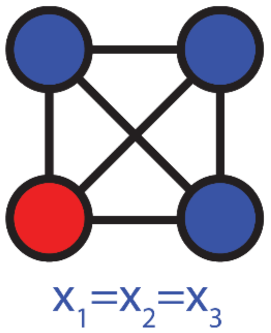
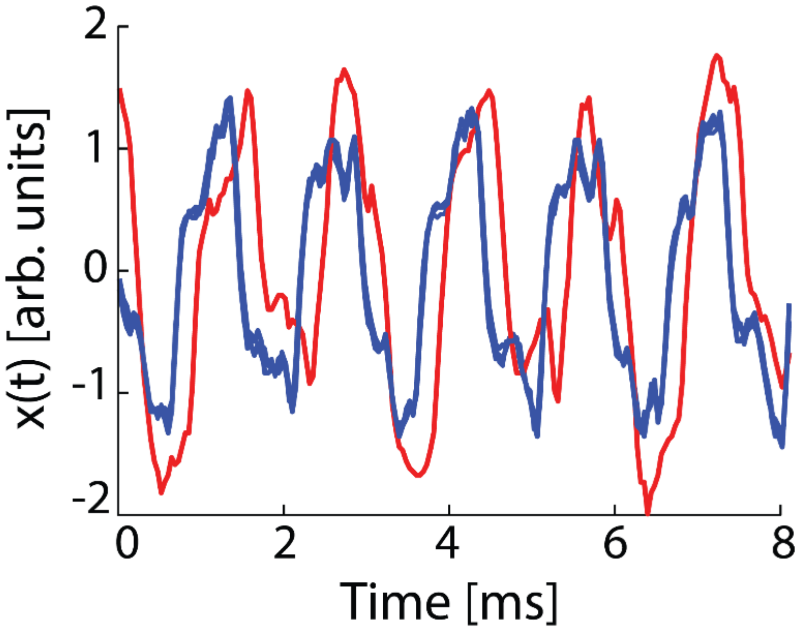
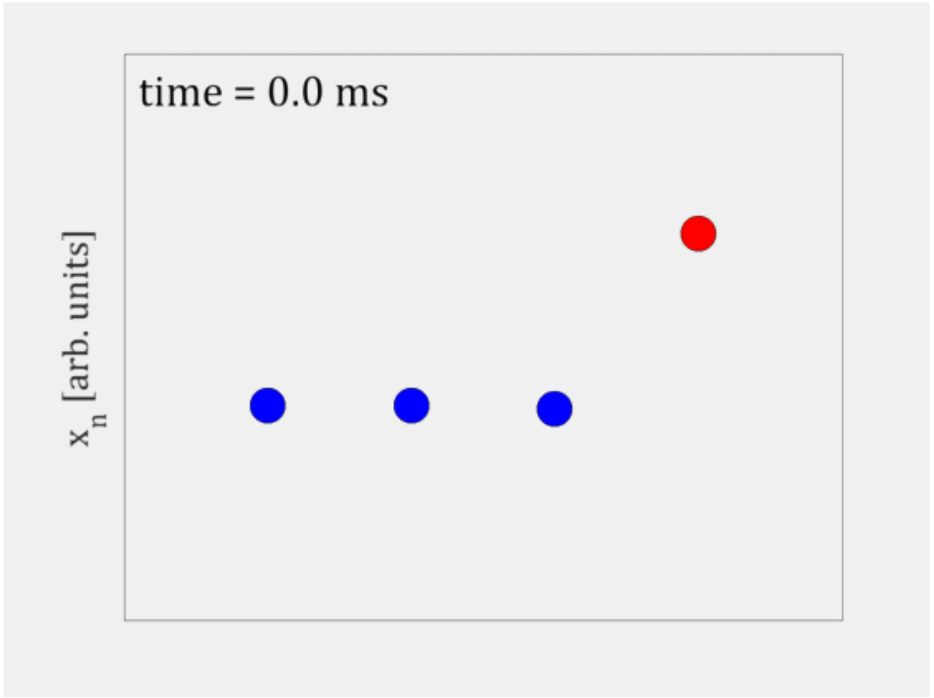
Triplet-singlet







τ_c=1.8 ms, ε=0.45

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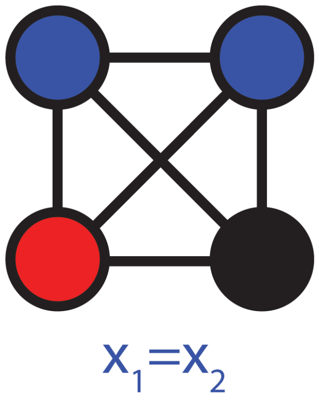
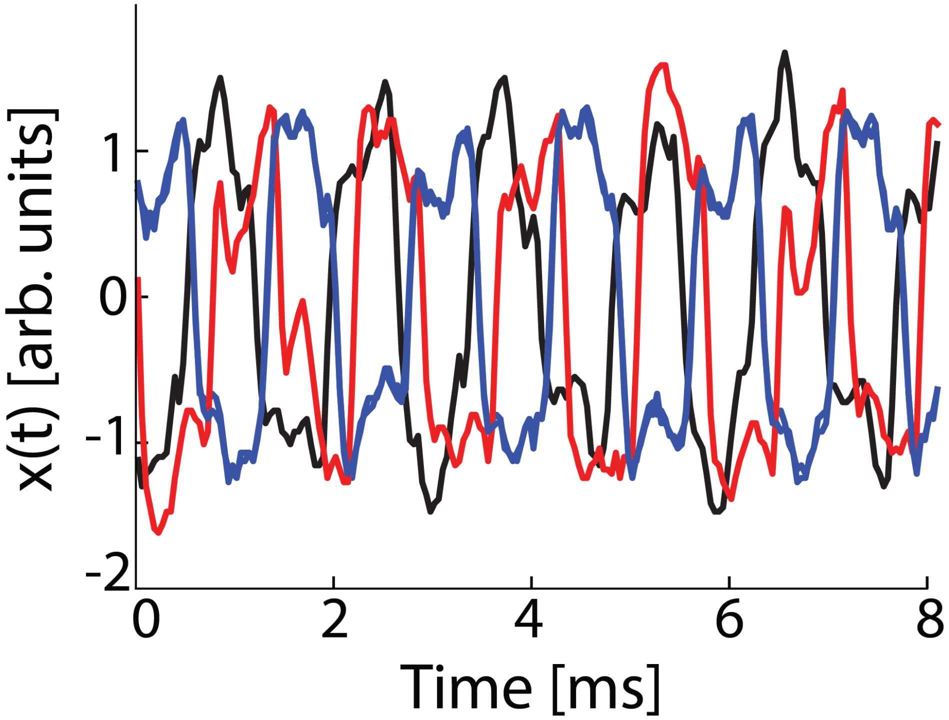
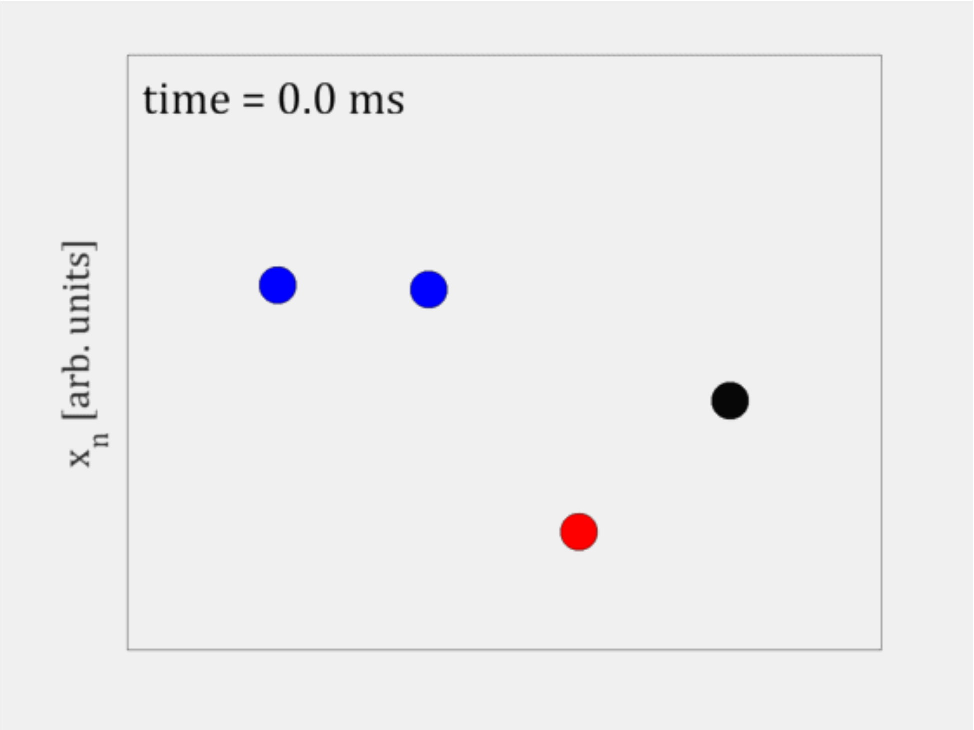
Chimera states







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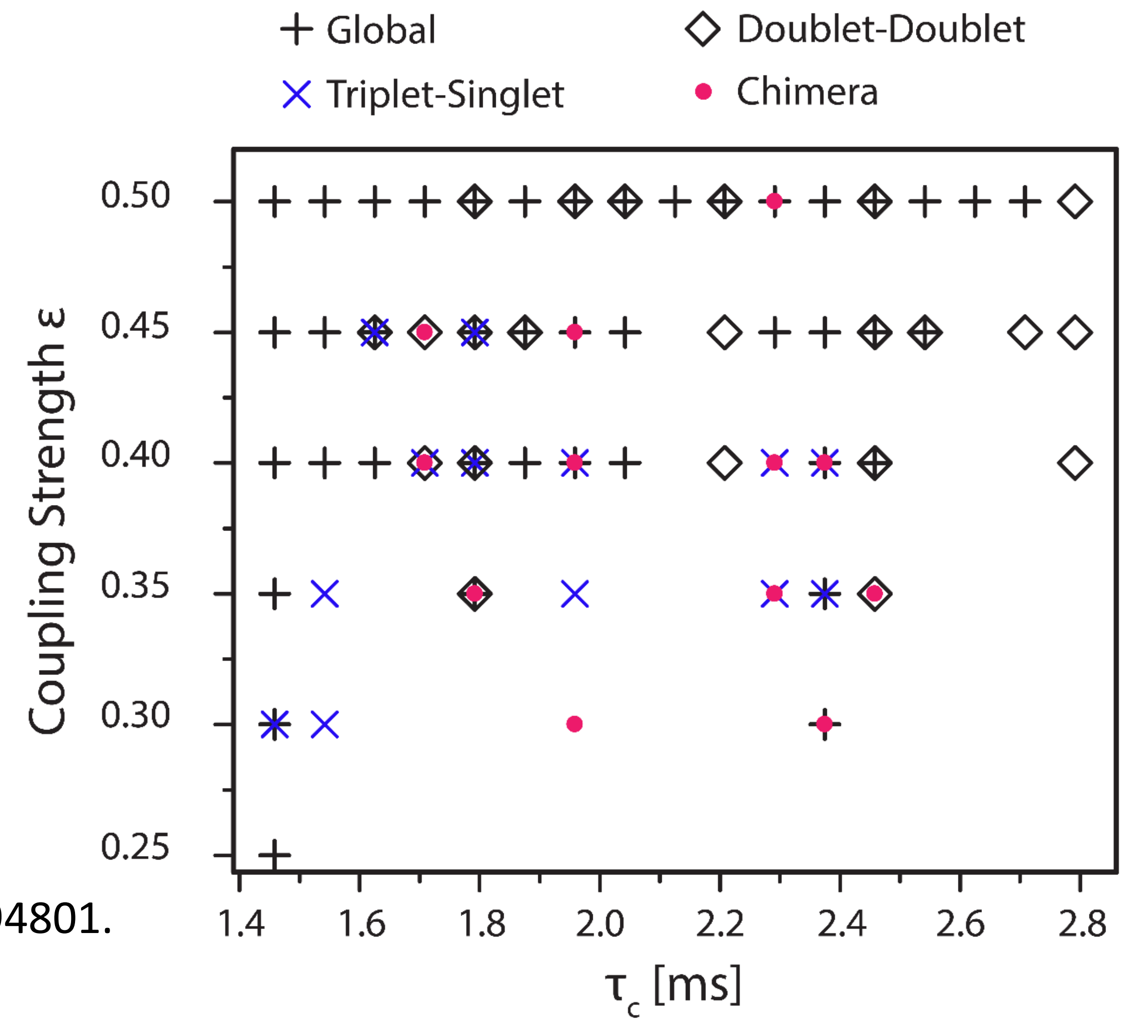
τ_c=2.3 ms, ε=0.40

Summary of experimental results



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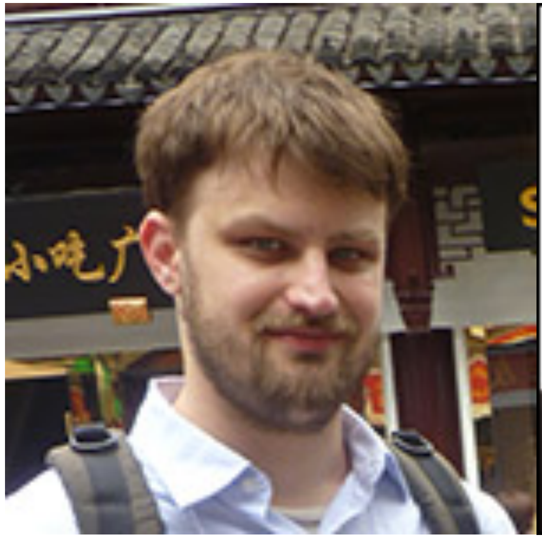
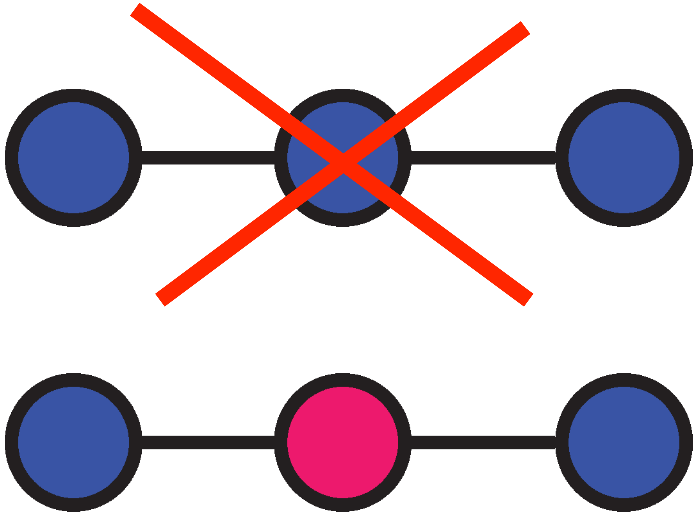
Cluster synchronization and isolated desynchronization in complex networks with symmetries

Louis M. Pecora[1], Francesco Sorrentino[2], Aaron M. Hagerstrom[3,4], Thomas E. Murphy[4,5] & Rajarshi Roy[4,6,7]






Symmetries and cluster synchronization

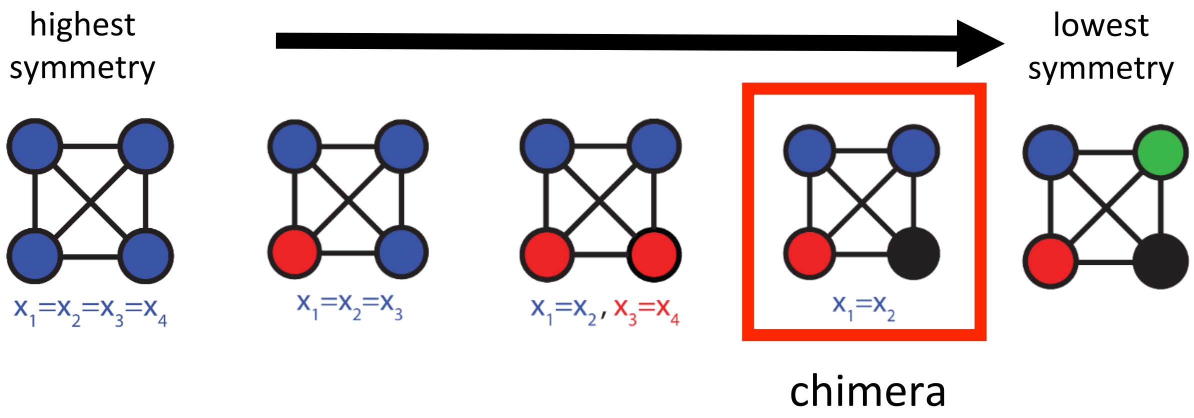


- Orbits of the symmetry group and subgroups of the adjacency matrix can help do this more quickly

Pecora et al. *Nat. Commun.* **5**, 5079 (2014).

Symmetries and cluster synchronization

- Orbits of the symmetry group and subgroups of the adjacency matrix determine which clusters can form



Global coupling: symmetries allow **ANY** combination of nodes to form a cluster.
BUT, stability determines *whether* a given pattern of sync can be observed.

Pecora et al. *Nat. Commun.* **5**, 5079 (2014).

Symmetries and Stability of Synchronization Patterns

- Choose the synchronization pattern.
- Linearize the equation of motion about the synchronization state.
- Transform node co-ordinate system to new co-ordinate system to decouple synchronization manifold and transverse directions.

$$\boldsymbol{B = TAT^{-1}}$$

- Finding out T is nontrivial, but can be done in software.

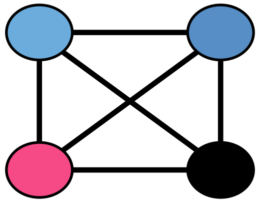
Stability Calculations: Chimera State

1. Choose the synchronization state:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$u_i(t) = E u_i(t) - F\beta\cos^2(x_i(t) + \phi_0)$$

$$x_i(t) = G\left(u_i(t-\tau_f) + \frac{\varepsilon}{3}\sum_j A_{ij}\left(u_j(t-\tau_c) - u_i(t-\tau_f)\right)\right)$$



i = d, s, s
j = d, s, s

2. Write variational equation:

$$\frac{d}{dt}\Delta u_i(t) = E\Delta u_i(t) + F\beta\sin(2x_i(t) + 2\phi_0)\,\Delta x_i(t)$$

$$\Delta x_i(t) = G\left((1-\varepsilon)\Delta u_i(t-\tau_f) + \frac{\varepsilon}{3}\sum_j A_{ij}\,\Delta u_j(t-\tau_c)\right)$$

i = d, s, s
j = d, s, s

3. Transformation of co-ordinate systems

- In new co-ordinate system $\Delta \boldsymbol{v}_{\boldsymbol{i}} \equiv T_{ij}\, \Delta \boldsymbol{u}_{\boldsymbol{i}}$

$$\mathbf{T} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \mathbf{TAT}^{-1} = \begin{bmatrix} 1 & \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 & 0 \\ \sqrt{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-1} \end{bmatrix}$$

Synchronization Manifold

Transverse Direction

- Only transverse component is required for stability calculation

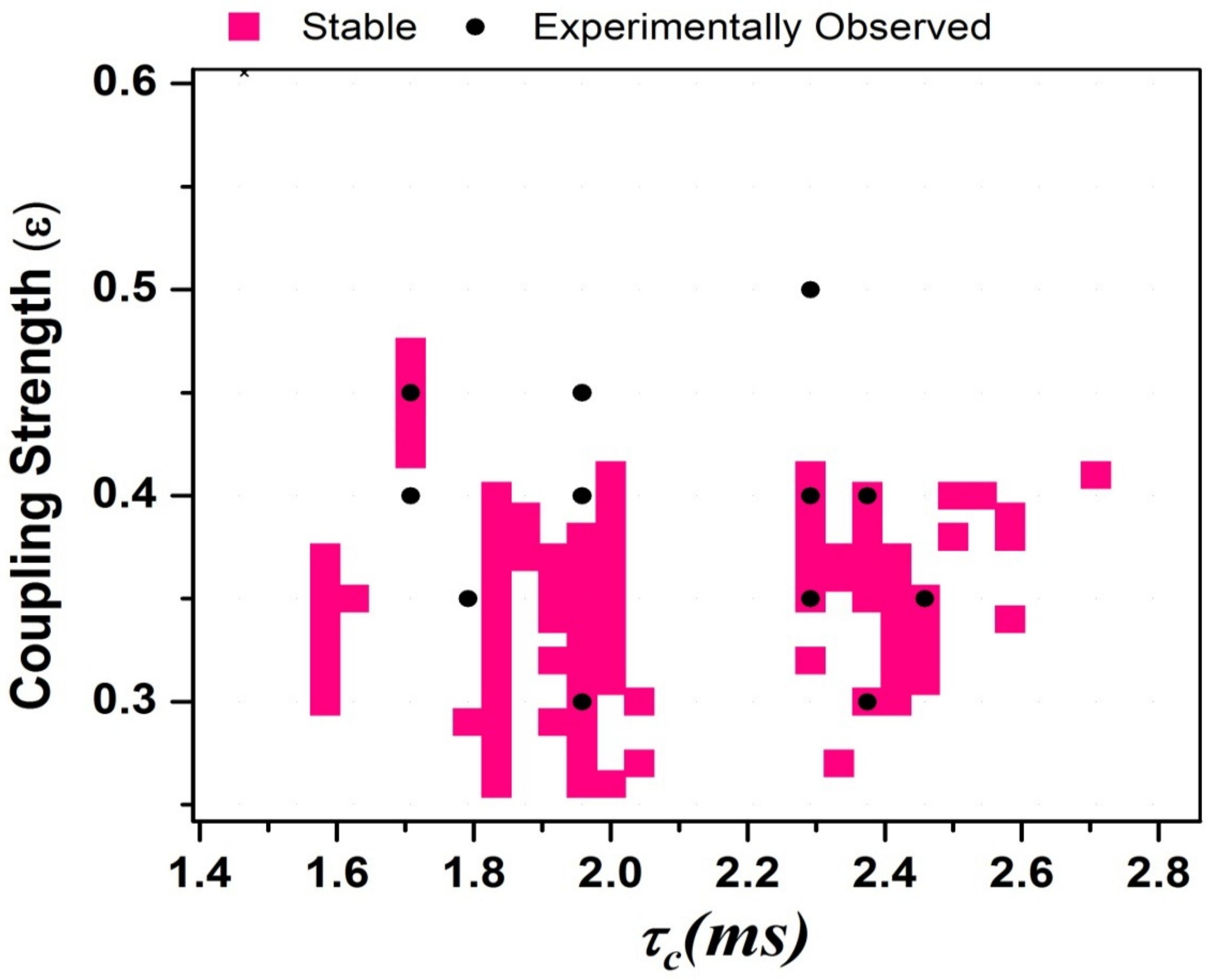
$$d/dt\, \Delta \boldsymbol{v}_T(t) = \boldsymbol{E}\boldsymbol{\Delta}\boldsymbol{v}_T(t) + \boldsymbol{F}\beta \sin(2x_s(t) + 2\phi_0)\, \Delta x_T(t)$$

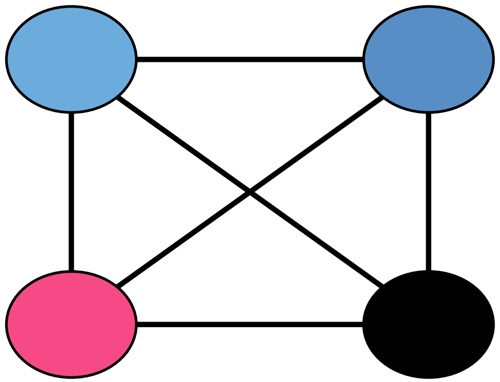
$$\Delta x_T(t) = \boldsymbol{G}[(\mathbf{1}-\boldsymbol{\varepsilon})\boldsymbol{\Delta}\boldsymbol{v}_T(t-\tau_f) + \boldsymbol{\varepsilon}/\mathbf{3} \sum_{j} \boldsymbol{B}_{\boldsymbol{Tj}}\, \boldsymbol{\Delta}\boldsymbol{v}_j(t-\tau_c)]$$

- Performing the sum:

$$\boxed{\Delta x_T(t) = \boldsymbol{G}((\mathbf{1}-\boldsymbol{\varepsilon})\boldsymbol{\Delta}\boldsymbol{v}_T(t-\tau_f) - \boldsymbol{\varepsilon}/\mathbf{3}\, \boldsymbol{\Delta}\boldsymbol{v}_T(t-\tau_c))}$$

Stable Chimera States





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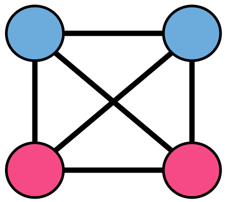
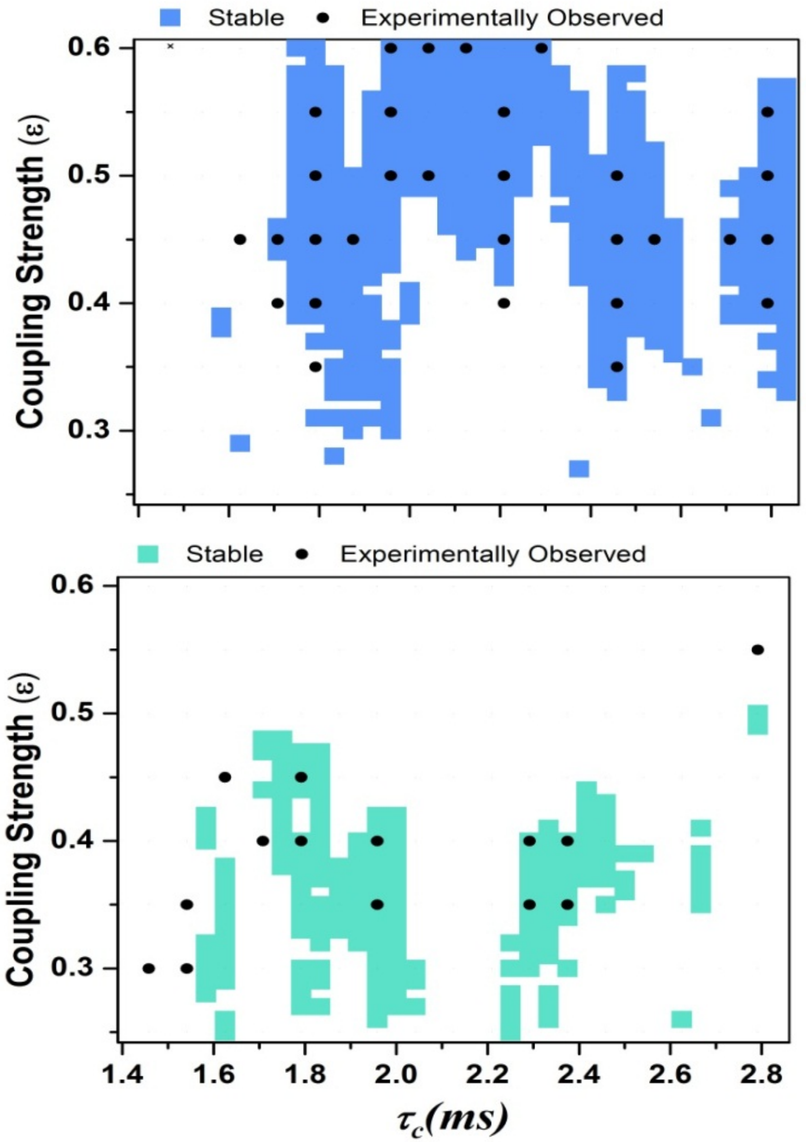
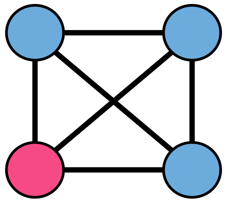
Stability of cluster states







Doublet – Doublet State

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

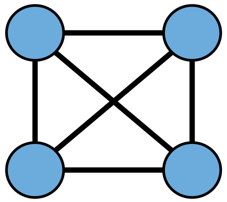
$$\mathbf{B} = \mathbf{TAT}^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Triplet – Singlet State

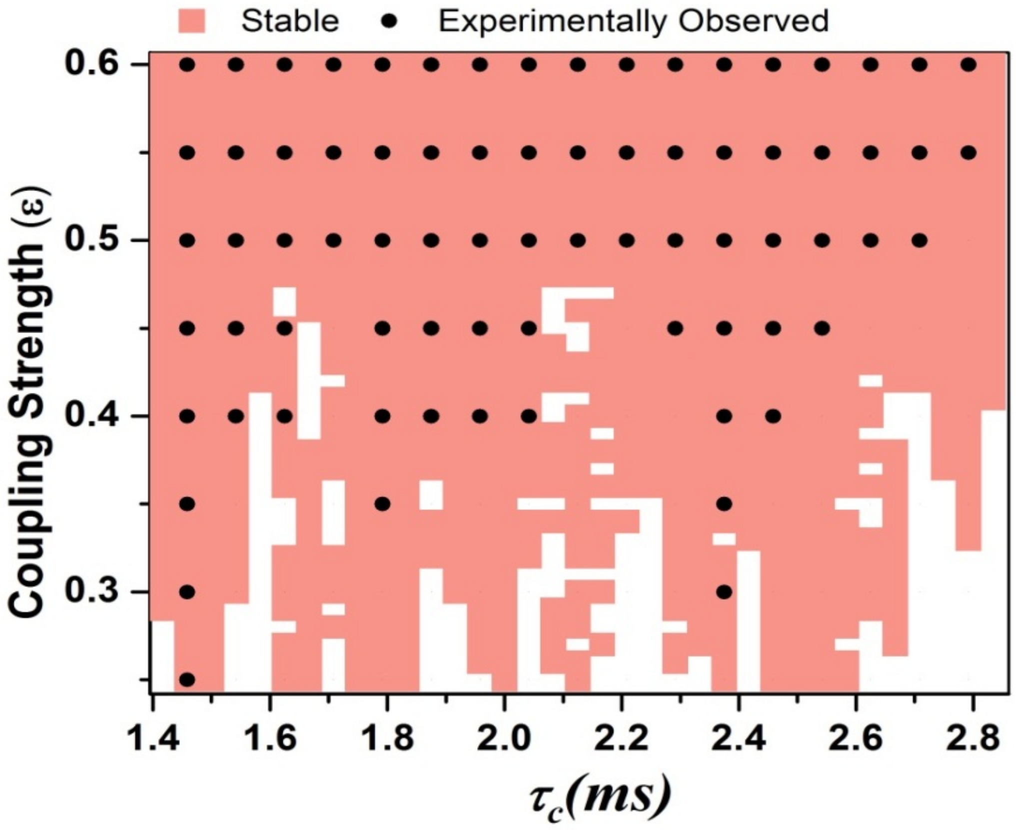
$$\mathbf{T} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

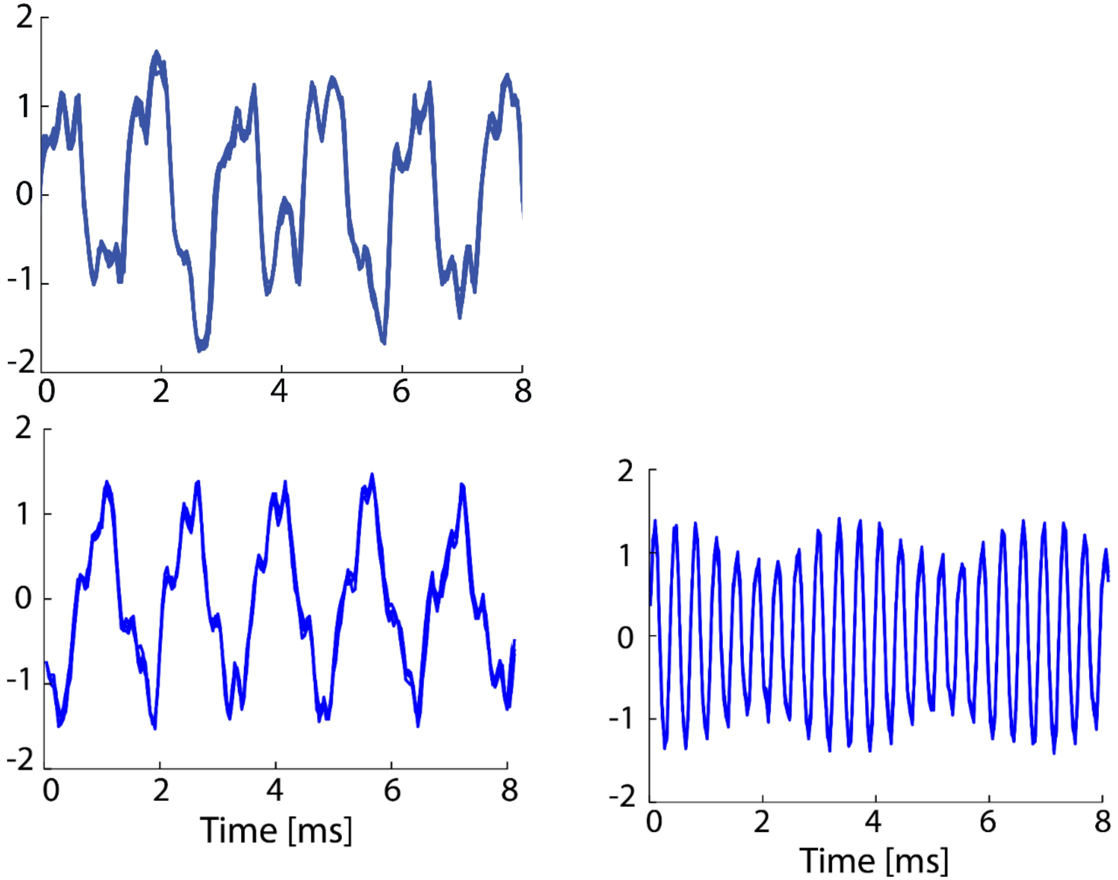
$$\mathbf{B} = \mathbf{TAT}^{-1} = \begin{bmatrix} 2 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Stability of Global Synchrony

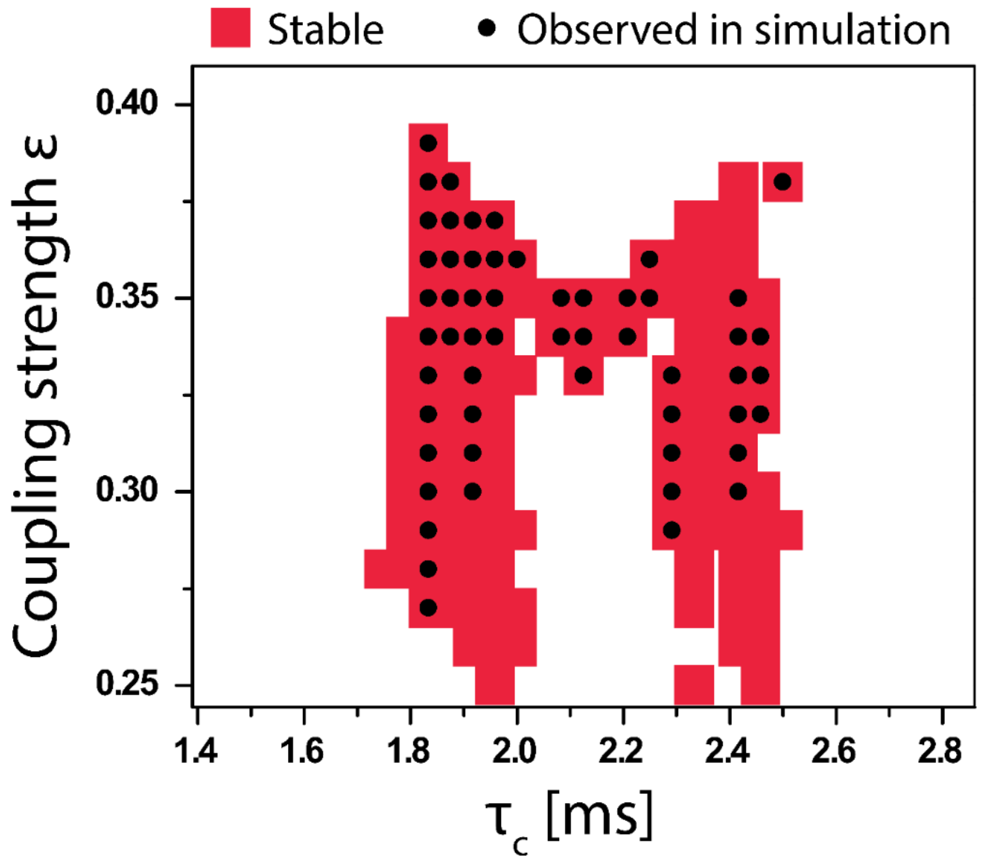


- Stability for all the global synchrony patterns observed.





10 node globally coupled network



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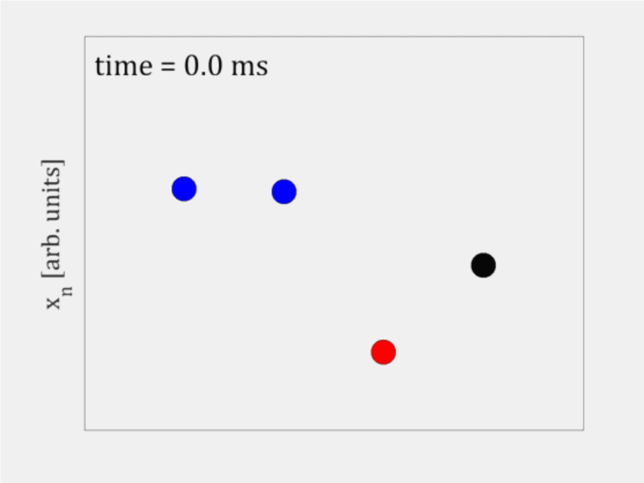
Conclusions

- Observed stable chimeras in the minimal globally coupled network
- These chimeras can be understood using the same methods recently developed for cluster synchronization
- Cluster stability analysis should work for networks of different sizes and topologies.



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