Slow Modulation & Large-Time Dynamics Near Periodic Waves

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Jointly with Mathew Johnson (Kansas), Pascal Noble (INSA Toulouse), Kevin Zumbrun (Indiana). Formerly Lyon/Indiana.

Part of a larger project also including Blake Barker (Brown).

Outline.

Outline.

Ordinary differential equations.

$\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ smooth.
$\mathbf{U} : \mathbf{R}_+ \to \mathbf{R}^n$ solving

$$\mathbf{U}_t \;=\; \mathbf{F}(\mathbf{U}) \quad \text{and} \quad \mathbf{U}(0) = \mathbf{U}_0\,.$$

Steady solution, $0 = \mathbf{F}(\underline{\mathbf{U}})$.

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Steady solution, $0 = \mathbf{F}(\underline{\mathbf{U}})$.

Stability

$\forall \delta > 0$, $\exists \varepsilon > 0$,

$$d(\mathbf{U}_0, \underline{\mathbf{U}}) < \varepsilon \quad \Rightarrow \quad (\exists!\, \mathbf{U} \quad \text{and} \quad (\forall t \geq 0,\ d(\mathbf{U}(t), \underline{\mathbf{U}}) < \delta))\ .$$

Asymptotic stability

Stability + $\exists \varepsilon_0 > 0$,

$$d(\mathbf{U}_0, \underline{\mathbf{U}}) < \varepsilon_0 \quad \Rightarrow \quad \left(\exists!\, \mathbf{U} \quad \text{and} \quad d(\mathbf{U}(t), \underline{\mathbf{U}}) \overset{t\to\infty}{\to} 0\right)\ .$$

From the spectrum to the nonlinear dynamics.

Spectral stability

$$\sigma(\mathrm{d}\mathbf{F}(\underline{\mathbf{U}})) \subset \{\lambda \mid \mathrm{Re}\lambda < 0\}\,.$$

Theorem

Spectral stability implies asymptotic stability.
Decay is exponentially fast.

From the spectrum to the nonlinear dynamics.

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Theorem

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Decay is exponentially fast.

Goal : PDE version for periodic plane waves.
$\mathbf{U} : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}^d$ solving

$$\mathbf{U}_t \;=\; \mathbf{F}(\mathbf{U},\mathbf{U}_x,\cdots) \quad \text{and} \quad \mathbf{U}(0,\cdot) = \mathbf{U}_0(\cdot)\,.$$

- Adapted notion of nonlinear stability for periodic waves.
- Adapted notion of spectral stability, depending on the PDE type.

Outline.

Roll-waves in thin films.

In a channel.

Courtesy of Neil Balmforth (British Columbia).

The Saint-Venant system (SV)

$$
\begin{aligned}
h_t &+ (hu)_x = 0,\\
(hu)_t &+ \left(hu^2 + \frac{h^2}{2F^2}\right)_x\\
&= h - |u|u + (h\,u_x)_x.
\end{aligned}
$$

$F > 2$, primary instability.

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$F > 2$, primary instability.

The Korteweg-de Vries/Kuramoto-Sivashinsky equation (KdV-KS)

$$U_T + \left(\tfrac{1}{2}U^2\right)_X + U_{XXX} + \delta(U_{XX} + U_{XXXX}) = 0.$$

Near threshold, $\delta \sim \sqrt{F-2}$.

Outline.

Outline.

About a wave.

$t > 0$ time, $x \in \mathbf{R}$ space, $\mathbf{U}(x,t) \in \mathbf{R}^d$ unknown. $\quad$ $\mathbf{f}$ smooth.

$$(P) \qquad \mathbf{U}_t \; + \; (\mathbf{f}(\mathbf{U}))_x \; = \; \mathbf{U}_{xx}\,.$$

Traveling wave. $\mathbf{U}(t,x) = \underline{\mathbf{U}}(x - ct)$ moving with speed c.

Profile $\underline{\mathbf{U}}$ solving $-\,c\,\underline{\mathbf{U}}_x \; + \; (\mathbf{f}(\underline{\mathbf{U}}))_x \; = \; \underline{\mathbf{U}}_{xx}\,.$

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Generator

$$L := \partial_x^2 + \underline{c}\partial_x - \partial_x \mathrm{d}\mathbf{f}(\underline{\mathbf{U}})$$

in the frame of the wave.

Integral transform.

Bloch-wave decomposition

$$g(x) \;=\; \int_{-\pi}^{\pi} e^{\mathrm{i}\xi x}\;\check{g}(\xi,x)\;\mathrm{d}\xi,$$

where ξ is a Floquet exponent

$$\begin{aligned}\check{g}(\xi,x+1) &= \check{g}(\xi,x),\\ e^{\mathrm{i}\xi(x+1)}\,\check{g}(\xi,x+1) &= e^{\mathrm{i}\xi}\;e^{\mathrm{i}\xi x}\,\check{g}(\xi,x).\end{aligned}$$

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From the Fourier decomposition

$$g(x) = \int_{\mathbf{R}} e^{\mathrm{i}\xi x}\;\hat{g}(\xi)d\xi.$$

Floquet-Bloch transform

$$\check{g}(\xi,x) \;:=\; \sum_{j\in\mathbf{Z}} e^{\mathrm{i}\,2j\pi x}\;\widehat{g}(\xi+2j\pi) \;=\; \sum_{k\in\mathbf{Z}} e^{-\mathrm{i}\,\xi\,(x+k)}\;g(x+k).$$

First observations.

$$h(x) \; = \; \int_{-\pi}^{\pi} e^{i\xi x} \; \check{h}(\xi, x) \; \mathrm{d}\xi \, .$$

Observations

$$\begin{aligned}
(g\, h)^{\check{}}(\xi, x) \;&=\; g(x)\; \check{h}(\xi, x)\,, \qquad && g \;\; \text{of period } 1, \\
(\partial_x h)^{\check{}}(\xi, \cdot) \;&=\; (\partial_x + \mathrm{i}\xi)\; \check{h}(\xi, \cdot)\,, && \\
\check{h}\,(\xi, x) \;&=\; \widehat{h}(\xi)\,, && h \;\; \text{low-frequency}.
\end{aligned}$$

First observations.

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\check{h}\;(\xi,x) &= \widehat{h}(\xi)\,, && h \;\text{ low-frequency.}
\end{aligned}$$

Remark : If g is of period 1 and h is low-frequency

$$(g\,h)^{\check{}}(\xi,x) \;=\; g(x)\,\widehat{h}(\xi).$$

Bloch symbols.

$\mathbf{U}(t,x) = \underline{\mathbf{U}}(\underline{k}(x-\underline{c}t))$, $\underline{\mathbf{U}}$ of period 1.

Generator

$$L := \underline{k}^2\,\partial_x^2 + \underline{k}\,\underline{c}\partial_x - \underline{k}\,\partial_x \mathrm{d}\mathbf{f}(\underline{\mathbf{U}})$$

with Bloch symbols

$$L_\xi := \underline{k}^2\,(\partial_x + \mathrm{i}\xi)^2 + \underline{k}\,(\partial_x + \mathrm{i}\xi)(\underline{c} - \mathrm{d}\mathbf{f}(\underline{\mathbf{U}})), \quad \xi \in [-\pi,\pi].$$

Bloch diagonalization

$$(L\,g)(x) \;=\; \int_{-\pi}^{\pi} e^{\mathrm{i}\xi x}\,(L_\xi\,\check{g}(\xi,\cdot))(x)\,\mathrm{d}\xi\,.$$

Each L_ξ acts on functions of period 1.

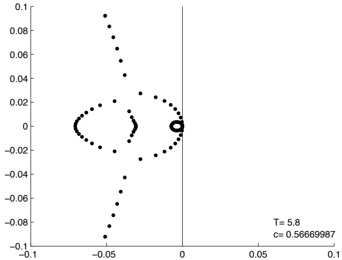
Outline.

1. Motivation
2. Structure of the spectrum
   - Floquet-Bloch theory
   - Diffusive spectral stability
3. Dynamical stability
4. Averaged dynamics
5. Conclusion

Diffusive spectral stability.

Spectral decomposition

$$\sigma(L) \;=\; \bigcup_{\xi\in[-\pi,\pi]} \sigma_{per}(L_\xi)\,.$$



Spectum of a stable wave of (SV). Barker-Johnson-Noble-LMR-Zumbrun, *Proc. Port-d'Albret* 2010.

(D1) Critical spectrum reduced to $\{0\}$.

$$\sigma(L) \subset \{\lambda \mid \mathrm{Re}\lambda < 0\} \cup \{0\}.$$

(D2) Diffusion. $\exists\theta > 0$, $\forall\xi \in [-\pi,\pi]$,

$$\sigma_{per}(L_\xi) \;\subset\; \{\lambda \mid \mathrm{Re}\lambda \leq -\theta|\xi|^2\}.$$

(D3) $\lambda = 0$ of multiplicity $d+1$ for L_0 (minimal dimension).

(H) Distinct group velocities (strict hyperbolicity).

Outline.

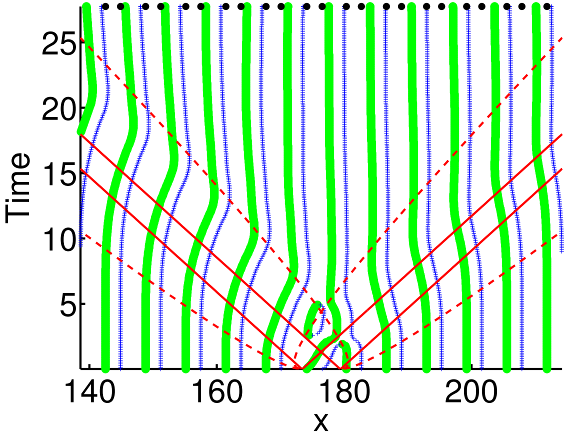
Outline.

Direct simulation : space-time diagramm.

About a stable wave of (KdV-KS).

Barker-Johnson-Noble-LMR-Zumbrun, *Phys. D* 2013.

Peaks.
Troughs.
Theory.



Space-modulation of distances.

Allow for resynchronization by comparing functions with

$$\delta_{\mathcal{H}}(u,v) \;=\; \inf_{\Psi \text{ one-to-one}} \|u\circ\Psi - v\|_{\mathcal{H}} + \|\partial_x(\Psi - \mathrm{Id}_{\mathbf{R}})\|_{\mathcal{H}}\,.$$

$\underline{\mathbf{U}}$ reference wave.

$$\|\mathbf{U}(t,\cdot) - \underline{\mathbf{U}}\|_{L^p(\mathbf{R})} \quad \overset{t\to\infty}{\sim} \quad \sigma\ t^{\frac{1}{p}}$$

while, in a diffusive context,

$$\delta_{L^p(\mathbf{R})}(\,\mathbf{U}(t,\cdot)\,,\,\underline{\mathbf{U}}\,) \qquad \overset{t\to\infty}{\sim} \quad \sigma\ t^{-\frac{1}{2}\,(1-1/p)}\,.$$

Outline.

Nonlinear diffusive stability.

Theorem (Johnson-Noble-LMR-Zumbrun, *Inventiones Math.* 2014.)

A diffusively spectrally stable periodic wave of (P) is

$$\delta_{L^1(\mathbf{R})\cap H^K(\mathbf{R})} \qquad \text{to} \qquad \delta_{H^K(\mathbf{R})}$$

asymptotically stable when $K \geq 3$, with algebraic decay rates.

Elements of proof : introducing the phase.

Seek for $(\mathbf{V}, \psi)$ with $(\mathbf{V}, \partial_x \psi)$ small and such that

$$\mathbf{V}(t,\cdot) \; = \; \mathbf{U}(t,\cdot) \circ (\mathrm{Id}_{\mathbf{R}} - \psi(t,\cdot)) - \underline{\mathbf{U}}.$$

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Equation

$$(\partial_t - L)\,(\mathbf{V}(t) + \underline{\mathbf{U}}_x \psi(t)) \;=\; \mathcal{N}(t)\,.$$

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Equation

$$(\partial_t - L)\,(\mathbf{V}(t) + \underline{\mathbf{U}}_x \psi(t)) \;=\; \mathcal{N}(t)\,.$$

Integral formulation

$$\mathbf{V}(t) + \underline{\mathbf{U}}_x \psi(t) = S(t)\,(\mathbf{V}(0) + \underline{\mathbf{U}}_x \psi_0) + \int_0^t S(t-s)\mathcal{N}(s)\mathrm{d}s\,,$$

with $S(t) := e^{t\,L}$.

Isolating the phase.

Separation

$$S(t) \;=\; \underline{\mathbf{U}}_x \, s^{\phi}(t) + \tilde{S}(t)$$

$$\psi(t) \;=\; s^{\phi}(t)\,(\mathbf{V}_0 + \underline{\mathbf{U}}_x\psi_0) + \int_0^t s^{\phi}(t-s)\mathcal{N}(s)\mathrm{d}s$$

$$\mathbf{V}(t) \;=\; \tilde{S}(t)\,(\mathbf{V}_0 + \underline{\mathbf{U}}_x\psi_0) + \int_0^t \tilde{S}(t-s)\mathcal{N}(s)\mathrm{d}s$$

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with χ cutting off large times.

Isolating the phase.

Separation

$$S(t) \;=\; \underline{\mathbf{U}}_x\ s^{\phi}(t) + \tilde{S}(t)$$

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with χ cutting off large times.

Critical evolution.

Expand

$$\frac{1}{\mathrm{i}\underline{k}\xi}\sum_{\alpha=1}^{d+1} e^{\lambda_\alpha(\xi)t}\ \psi_\alpha(\xi,\cdot)\ \langle\tilde{\phi}_\alpha(\xi,\cdot)|\cdot\rangle$$

with

$$\psi_\alpha(\xi,\cdot) \;=\; \underline{\mathbf{U}}_x(\cdot)\,\beta^{(\alpha)}(\xi) \;+\; \mathcal{O}(\xi)$$

Critical evolution.

Expand

$$\frac{1}{\mathrm{i}\underline{k}\xi}\sum_{\alpha=1}^{d+1} e^{\lambda_\alpha(\xi)t}\ \psi_\alpha(\xi,\cdot)\ \langle\tilde{\phi}_\alpha(\xi,\cdot)|\cdot\rangle$$

with

$$\psi_\alpha(\xi,\cdot)\ =\ \underline{\mathbf{U}}_x(\cdot)\,\beta^{(\alpha)}(\xi)\ +\ \mathcal{O}(\xi)$$

to

$$\underline{\mathbf{U}}_x(\cdot)\quad\sum_{\alpha=1}^{d+1}\left\langle\frac{1}{\mathrm{i}\underline{k}\xi}\ e^{\lambda_\alpha(\xi)t}\ \beta^{(\alpha)}(\xi)\ \tilde{\phi}_\alpha(\xi,\cdot)\middle|\ \cdot\ \right\rangle\ +\ \mathcal{O}(e^{-\theta'\,|\xi|^2\,t})\,.$$

Decomposition

$$S(t)\ =\ \underline{\mathbf{U}}_x\ s^\phi(t)+\tilde{S}(t)$$

with $s^\phi(t)=\sum_{\alpha=1}^{d+1}s_\alpha^\phi(t)$.

Bounds.

1. Bloch analysis.

Hausdorff-Young inequalities

For $2 \leq p \leq \infty$,

$$\begin{aligned} \|g\|_{L^p(\mathbf{R})} &\lesssim \|\check{g}\|_{L^{p'}([-\pi,\pi],L^p([0,1]))}\,, \\ \|\check{g}\|_{L^p([-\pi,\pi],L^{p'}([0,1]))} &\lesssim \|g\|_{L^{p'}(\mathbf{R})}\,. \end{aligned}$$

One estimate

$\psi_0(-\infty) = -\psi_0(\infty)$. For $t \geq 0$, $2 \leq p \leq \infty$,

$$\left\|\partial_x s_\alpha^\phi(t)(\psi_0\,\underline{\mathbf{U}}_x)\right\|_{L^p(\mathbf{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)}\|\partial_x\psi_0\|_{L^1(\mathbf{R})}\,.$$

2. Energy estimates
and resolvent estimates in a Hilbertian framework.

Out of time ?

Outline.

Outline.

1. Motivation
2. Structure of the spectrum
3. Dynamical stability
4. Averaged dynamics
   - Formal derivation
   - Nonlinear validation
5. Conclusion

Parametrization.

At fixed k wavenumber,

$$\mathbf{U}(t,x) = \underline{\mathbf{U}}(k(x - ct))$$

with $\underline{\mathbf{U}}$ of period 1 determined by averaged values $\mathbf{M} := \langle \underline{\mathbf{U}} \rangle$.

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Profile : $\underline{\mathbf{U}} = \underline{\mathbf{U}}^{(\mathbf{M},k)}$. Phase velocity and time frequency

$$c = c(\mathbf{M},k)\,, \qquad \omega(\mathbf{M},k) = -k\,c(\mathbf{M},k)\,.$$

Two-scale *ansatz*.

'Two-timing' method. Fast oscillation/slow variation.

$$\mathbf{U}(t,x) = \left(\mathbf{U}_0 + \varepsilon\mathbf{U}_1 + \varepsilon^2\mathbf{U}_2\right)\left(\underbrace{\varepsilon t}_{T}, \underbrace{\varepsilon x}_{X}, \underbrace{\frac{(\Psi_0 + \varepsilon\Psi_1)(\varepsilon t, \varepsilon x)}{\varepsilon}}_{\theta}\right) + o(\varepsilon^2)$$

with $\mathbf{U}_0$, $\mathbf{U}_1$ and $\mathbf{U}_2$ of period 1 in θ.

Slow modulation behavior.

$$(\mathbf{U}_0 + \varepsilon \mathbf{U}_1)(T, X, \theta) \;=\; \underline{\mathbf{U}}^{((\mathcal{M}_0, \kappa_0) + \epsilon(\mathcal{M}_1, \kappa_1))(T,X)}\Big(\theta\Big) \;+\; o(\varepsilon)$$

with

$$\kappa_0 + \epsilon\kappa_1 = (\Psi_0 + \epsilon\Psi_1)_X \quad \text{local wave number,}$$

$$\mathcal{M}_0 + \epsilon\mathcal{M}_1 \qquad \text{local averages}\,.$$

Averaged dynamics : matching with slow evolution.

Evolution of $(\mathcal{M}_0, \kappa_0)$, $(\mathcal{M}_1, \kappa_1)$ coincide with expansion of slow *ansatz*

$$(\mathcal{M}, \kappa)(x, t) \ = \ \Big((\mathcal{M}_0, \kappa_0) + \varepsilon(\mathcal{M}_1, \kappa_1)\Big)\Big(\underbrace{\varepsilon t}_{T}, \underbrace{\varepsilon x}_{X}\Big) + o(\varepsilon^2)$$

into

$$(W) \qquad \begin{cases} \mathcal{M}_t \ + \ (F(\mathcal{M}, \kappa))_x & = \ (d_{11}(\mathcal{M}, \kappa)\mathcal{M}_x + d_{12}(\mathcal{M}, \kappa)\kappa_x)_x, \\ \kappa_t \ - \ (\omega(\mathcal{M}, \kappa))_x & = \ (d_{21}(\mathcal{M}, \kappa)\mathcal{M}_x + d_{22}(\mathcal{M}, \kappa)\kappa_x)_x. \end{cases}$$

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$$(W)_{phase}\qquad \Psi_t \;-\; \omega(\mathcal{M},\kappa) \;=\; d_{21}(\mathcal{M},\kappa)\mathcal{M}_x+d_{22}(\mathcal{M},\kappa)\kappa_x\,.$$

I shall hide a choice leading to a canonical artificial viscosity system.

▸ Out of time ?

Outline.

Asymptotic behavior, refined description.

Theorem (Johnson-Noble-LMR-Zumbrun, *Inventiones Math.* 2014.)

Let $\eta > 0$ and $K \geq 4$. There exists $\varepsilon > 0$ and $C > 0$ such that if

$$E_0 := \|\mathbf{U}_0 \circ (\mathrm{Id}_{\mathbf{R}} - \psi_0) - \underline{\mathbf{U}}\|_{L^1(\mathbf{R}) \cap H^K(\mathbf{R})} + \|\partial_x \psi_0\|_{L^1(\mathbf{R}) \cap H^K(\mathbf{R})} \;\leq\; \varepsilon$$

for some ψ_0,

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for some ψ_0, then, there exist $(\mathbf{U}, \psi)$ with initial data $(\mathbf{U}_0, \psi_0)$ and $\mathbf{M}$ such that, for $t > 0$ and $2 \leq p \leq \infty$,

$$\begin{aligned}
\|\mathbf{U}(t, \cdot - \psi(t,\cdot)) &- \underline{\mathbf{U}}^{\left(\underline{\mathbf{M}} + \mathbf{M}(t,\cdot), \frac{\underline{k}}{(1-\psi_x(t,\cdot))}\right)}(\,\cdot\,)\|_{L^p(\mathbf{R})} \\
&\leq C\, E_0 \, \ln(2+t)\, (1+t)^{-\frac{3}{4}} \\
\|(\mathbf{M}, \underline{k}\, \psi_x)(t,\cdot)\|_{L^p(\mathbf{R})} &\leq C\, E_0\, (1+t)^{-\frac{1}{2}(1-1/p)}\,.
\end{aligned}$$

Asymptotic behavior, validation of (W).

Theorem (Johnson-Noble-LMR-Zumbrun, *Inventiones Math.* 2014.)

Moreover, setting $\Psi(t,\cdot) = (\mathrm{Id}_{\mathbf{R}} - \psi(t,\cdot))^{-1}$, $\kappa = \underline{k}\,\partial_x \Psi$,

$$\mathcal{M}(t,\cdot) = (\underline{\mathbf{M}} + \mathbf{M}(t,\cdot)) \circ \Psi(t,\cdot),$$

Asymptotic behavior, validation of (W).

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Moreover, setting $\Psi(t,\cdot) = (\mathrm{Id}_{\mathbf{R}} - \psi(t,\cdot))^{-1}$, $\kappa = \underline{k}\,\partial_x \Psi$,

$$\mathcal{M}(t,\cdot) = (\underline{\mathbf{M}} + \mathbf{M}(t,\cdot)) \circ \Psi(t,\cdot),$$

and letting $(\mathcal{M}_W, \kappa_W)$ and Ψ_W solve (W) and $(W)_{phase}$ with initial data

$$\begin{aligned}
\kappa_W(0,\cdot) &= \underline{k}\,\partial_x \Psi(0,\cdot), \qquad \Psi_W(0,\cdot) = \Psi(0,\cdot),\\
\mathcal{M}_W(0,\cdot) &= \underline{\mathbf{M}} + \mathbf{U}_0 - \underline{\mathbf{U}} \circ \Psi(0,\cdot)\\
&+ \left(\frac{1}{\partial_x \Psi(0,\cdot)} - 1\right)(\underline{\mathbf{U}} \circ \Psi(0,\cdot) - \underline{\mathbf{M}}),
\end{aligned}$$

Asymptotic behavior, validation of (W).

Theorem (Johnson-Noble-LMR-Zumbrun, *Inventiones Math.* 2014.)

Moreover, setting $\Psi(t,\cdot) = (\mathrm{Id}_{\mathbf{R}} - \psi(t,\cdot))^{-1}$, $\kappa = \underline{k}\,\partial_x \Psi$,

$$\mathcal{M}(t,\cdot) = (\underline{\mathbf{M}} + \mathbf{M}(t,\cdot)) \circ \Psi(t,\cdot),$$

and letting $(\mathcal{M}_W, \kappa_W)$ and Ψ_W solve (W) and $(W)_{phase}$ with initial data

$$\begin{aligned}
\kappa_W(0,\cdot) &= \underline{k}\,\partial_x \Psi(0,\cdot), \qquad \Psi_W(0,\cdot) = \Psi(0,\cdot),\\
\mathcal{M}_W(0,\cdot) &= \underline{\mathbf{M}} + \mathbf{U}_0 - \underline{\mathbf{U}} \circ \Psi(0,\cdot)\\
&+ \left(\frac{1}{\partial_x \Psi(0,\cdot)} - 1\right)(\underline{\mathbf{U}} \circ \Psi(0,\cdot) - \underline{\mathbf{M}}),
\end{aligned}$$

we have, for $t \geq 0$, $2 \leq p \leq \infty$,

$$\begin{aligned}
\|(\mathcal{M},\kappa)(t,\cdot) - (\mathcal{M}_W,\kappa_W)(t,\cdot)\|_{L^p(\mathbf{R})} &\leq C\,E_0\,(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta},\\
\|\Psi(t,\cdot) - \Psi_W(t,\cdot)\|_{L^p(\mathbf{R})} &\leq C\,E_0\,(1+t)^{-\frac{1}{2}(1-1/p)+\eta}.
\end{aligned}$$

Corollaries when $\psi_0 \equiv 0$.

Question : Could we get usual asymptotic stability in some special cases ?

Roadmap : look at uncoupling of

$$\kappa_t \;-\; (\omega(\mathcal{M},\kappa))_x \;=\; (d_{21}(\mathcal{M},\kappa)\mathcal{M}_x + d_{22}(\mathcal{M},\kappa)\kappa_x)_x.$$

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Quadratic phase uncoupling

$$\mathrm{d}_{\mathbf{M}}\omega(\underline{\mathbf{M}},\underline{k}) \equiv 0 \qquad \text{and} \qquad \mathrm{d}^2_{\mathbf{M}}\omega(\underline{\mathbf{M}},\underline{k}) \equiv 0.$$

Corollary (Johnson-Noble-LMR-Zumbrun, *Inventiones Math.* 2014.)

Jointly with previous assumptions this implies asymptotic stability

$$\text{from} \qquad \|\cdot\|_{L^1(\mathbf{R})\cap H^K(\mathbf{R})} \qquad \text{to} \qquad \|\cdot\|_{H^K(\mathbf{R})}.$$

with decay $(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}$ in $L^p(\mathbf{R})$, $2 \leq p \leq \infty$.

Corollaries when $\psi_0 \equiv 0$.

Question : Could we get usual asymptotic stability in some special cases?

Roadmap : look at uncoupling of

$$\kappa_t \ - \ (\omega(\mathcal{M},\kappa))_x \ = \ (d_{21}(\mathcal{M},\kappa)\mathcal{M}_x + d_{22}(\mathcal{M},\kappa)\kappa_x)_x.$$

Linear phase uncoupling

$$\mathrm{d}_{\mathbf{M}}\omega(\underline{\mathbf{M}},\underline{k}) \equiv 0.$$

Corollary (Johnson-Noble-LMR-Zumbrun, *Inventiones Math.* 2014.)

Jointly with previous assumptions this implies asymptotic stability

$$\text{from} \qquad \|\cdot\|_{L^1(\mathbf{R};(1+|\cdot|))\cap H^K(\mathbf{R})} \qquad \text{to} \qquad \|\cdot\|_{H^K(\mathbf{R})}.$$

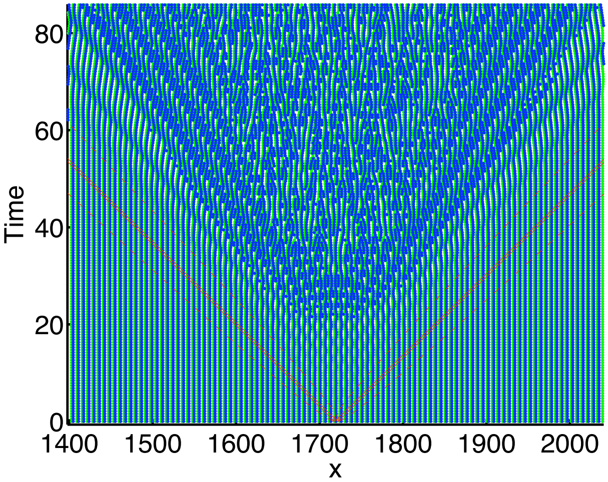
with decay $(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}+\eta}$ in $L^p(\mathbf{R})$, $2 \leq p \leq \infty$.

Outline.

Open questions.

- Verification of **spectral assumptions** : case-by-case.
- Space-modulated **instability**.
- **Genuinely multidimensional** periodic waves.
- **Dispersive nonlinear stability** (not on a torus).
- **Composed patterns**.

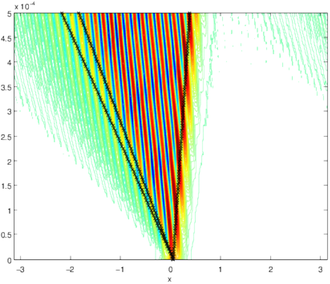
Bonus 1 : diffusive instability.

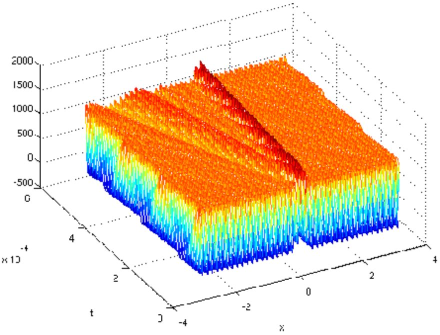


Failure of diffusivity in (KdV-KS).

Barker-Johnson-Noble-LMR-Zumbrun, *Phys. D* 2013.

Bonus 2 : nonlinear dynamics of (KdV).

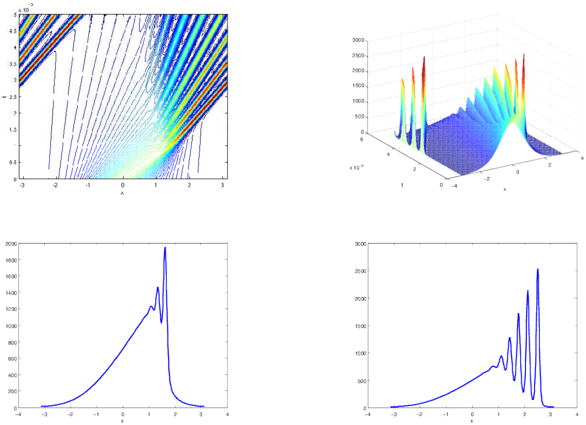




LMR, *in preparation*.

Right : graph of the full solution.
Left : perturbation seen from above.

Bonus 3 : a dispersive shock in (KdV).



A 2-rarefaction wave
of the averaged system.

LMR, ***in preparation***.