Using Symmetries and Equitable Partitions Together to Find All Synchronization Clusters and Their Stability

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MS168 Advances in Network Synchronization - Part II of II
Per Sebastian Skardal, Jie Sun, and Dane Taylor

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What are synchronized clusters?
Cluster Synchronization in complex networks
(various versions in the literature)

Identical nodes (oscillators), Identical edges (couplings)

Adjacency matrix

\[
\frac{dx_i}{dt} = F(x_i) + \sigma \sum_{j=1}^{N} C_{ij} H(x_j)
\]

- Identify the clusters?
- Are the clusters stable?
- Complex (large) networks
- Any dynamics (fixed pt, periodic, quasiperiodic, chaotic)

Cluster synchronization, an “old” subject (early 2000’s)
$\beta = 0.72 \pi$
Symmetries and clusters in networks
Using symmetry to find clusters.

Network with identical nodes (oscillators) and identical edges

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
R_{g_1} \quad R_{g_2} \quad R_{g_3}
\]

\[R_{g_i} \}

\{R_{g_i}\} \text{ Representation of the group } \quad G \quad \text{ e.g. } \quad g_2 g_1^{-1} = g_3 \Rightarrow R_{g_2} R_{g_1}^{-1} = R_{g_3}

\[
\frac{dx_i}{dt} = F(x_i) + \sigma \sum_{j=1}^{N} C_{ij} H(x_j) \quad \Rightarrow \quad R_g C = CR_g \quad \Rightarrow \quad C = R_g^{-1} CR_g
\]
Random Graphs
Random networks 11 nodes – 9 random edges

G.gens() = [(7, 10), (6, 7), (5, 6), (4, 8), (2, 4)(8, 9), (1, 5), (1 11)^1]

{g_t}
Computational Group Theory

GAP

Groups, Algorithms, Programming - a System for Computational Discrete Algebra

Sage

http://www.sagemath.org/

```
sage: x = var('x')
sage: solve(x^2 + 3*x + 2, x)
[x == -2, x == -1]
```

Python

```
for i, row in enumerate(Adjmat):
    rsum = row.sum()
    Cplmat[i,i] = -rsum
print Cplmat
```

Free! (open source)
G.order(), G.gens() = 8640 [(9,10), (7,8), (6,9), (4,6), (3,7), (2,4), (2,11), (1,5)]

node sync vectors:

Node 2
orb = [1, 5]
nodexsyncvec = [0, 1, 0, 0, 1, 0, 0, 0, 0, 0]
cyclesyncvec = [1, 0, 0, 1, 0, 0, 0, 0, 0, 0]

Node 1
orb = [2, 4, 11, 6, 9, 10]
nodexsyncvec = [1, 0, 1, 0, 1, 0, 1, 0, 1, 1]
cyclesyncvec = [0, 1, 0, 1, 0, 1, 0, 0, 1, 1]

Node 4
orb = [3, 7, 8]
nodexsyncvec = [0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0]
cyclesyncvec = [0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1]


(1) No computer code for large networks  (2) Did not examine stability of clusters.
Simplifying the variational matrix

\( \forall g \in G = \text{group of } \{ R_g \} \) is a representation of \( G \)

all symmetries

\( \{ R_g \} \) It is reducible if \( \exists T \ \forall g \ TR_gT^{-1} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \)

\( C = R_g^{-1}CR_g \)

\( T \) block diagonalizes \( C \)

The software does not supply \( T \)


\( = I \) Trivial Representation

\( \frac{d}{dt} \delta x = DF(x)\delta x + \sigma \sum C_D H(x)\delta x \)

\( \frac{d}{dt} = (DF(s) + C_N DH(s))\delta \xi \)

Synchronization\(^1\)

Manifold

All other

Representations

Transverse Manifold (sub-blocks associated with each cluster)
Experimental confirmation of symmetry clusters

Aaron Hagerstrom
Rajarashi Roy
Thomas Murphy
Francesco Sorrentino
How to implement maps in experiment

Phase written to SLM

Coupling strength

Adjacency matrix

Add phase shift to destabilize fixed point $\varphi=0$

$$\varphi_{i}^{t+1} = \beta I(\varphi_{i}^{t}) + \sigma \sum_{j} A_{ij} I(\varphi_{j}^{t}) + \delta$$

Intensity measured by camera

Self feedback strength

Spatial Light Modulator

computer

LED

QWP

PBS

camera

SLM
11 node random network (video)
32 Symmetries
4 nontrivial clusters
1 trivial cluster
5 x 5 Sync block
$\beta = 0.72 \pi$
\beta = 1.4 \pi
Chimera in identical synchronized network
(Not just for phase oscillators)

Chimeras, Cluster States, and Symmetries: Experiments
on the Smallest Chimera - Joseph Hart
MS 122

Not transients
Constructing non-symmetry clusters

Beyond symmetry clusters
Forming Equitable and Laplacian Clusters from Symmetry (orbital) Clusters

Def: An Equitable Partition of a graph is a collection of disjoint subsets $C_i$ ($C_i \cap C_j = \emptyset$) of all the nodes such that, if node $v \in C_i$ has $d$ connections to nodes in $C_j$ then for any other node $u \in C_i$ has $d$ connections to nodes in $C_j$.

Example: McKay’s graph

Besides being an Equitable Partition (EP) this is also an Orbital Partition (OP) from the symmetries.

Every OP is an EP, but the converse is not true. There are EPs that are not OPs.

Similar to groupoid synchronization criterion of Golubitsky, Stewart and Török (SIAM J. APPLIED DYNAMICAL SYSTEMS Vol. 4, No. 1, pp. 78–100 (2005)

An algorithm from Igor Belykh* provides an efficient method for finding the coarsest EP. However finding finer EPs is a hard problem. This is important since there may be finer EPs that are still not OPs. An example coming up.

Problem 1: Find finer EPs that are not OPs.

* I. Belykh and M. Hasler, CHAOS 21, 016106 (2011)
Problem 2: Stability calculations. Reducing the transverse manifold in the variational equations.

\[ \frac{d}{dt} \delta x_i = DF x_i (\delta x) + \sigma C_{ij} DH x_j (\delta x^j) \]

Each block is associated with particular clusters. Master Stability Function for each cluster.

- Stability calculations scale badly \((N^2)\)
- Stability of individual clusters is unknown.
- Type of desynchronization bifurcation is unknown.

Ref: Schaub et al.*

Is it even possible to block diagonalize the transverse manifold?

*M. Schaub, et al., Chaos 26, 094821 (2016)
Constructing an EP from an OP

Symmetries -> OP orbitals = [1,2,3,4], [5,7,8,10], [6,9]

Make the EP by merging clusters from its OP refinement

[5,7,8,10] U [6,9]

Observation: We can always construct an EP from an OP refinement.
There is always a subgroup of the original symmetry group that will be a refinement of any EP.
Construction of a block diagonalized variational matrix for the EP

\[
\begin{pmatrix}
-4.00 & -1.41 & -1.41 \\
-1.41 & -3.00 & 2.00 \\
-1.41 & 2.00 & -4.00 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\begin{pmatrix}
-5.00 \\
-3.00 \\
-4.00 \\
\end{pmatrix}
\begin{pmatrix}
-4.00 & -1.41 & -1.41 \\
-1.41 & -5.00 & 2.00 \\
-1.41 & 2.00 & -4.00 \\
\end{pmatrix}
\begin{pmatrix}
-4.00 \\
-4.00 \\
\end{pmatrix}
\]

\[T_{\text{merged clusters}}\]

\[u_1 = Ts\]

for the new EP

\[
\begin{pmatrix}
-4.00 & -2.45 \\
-2.45 & -2.00 \\
\end{pmatrix}
\begin{pmatrix}
-5.00 \\
-5.00 \\
\end{pmatrix}
\begin{pmatrix}
-4.00 & -1.41 & -1.41 \\
-1.41 & -5.00 & 2.00 \\
-1.41 & 2.00 & -4.00 \\
\end{pmatrix}
\begin{pmatrix}
-4.00 \\
-4.00 \\
\end{pmatrix}
\]

\[u_1 = T_{\text{merged clusters}}v_1\]

\[v_2, v_3\]

new transverse direction

new sync direction
Laplacian coupling - \( \text{diag} = -\text{row sum} \)

\[
\begin{pmatrix}
-3 & 1 & 0 & 1 & 1 \\
1 & -3 & 1 & 0 & 1 \\
0 & 1 & -3 & 1 & 1 \\
1 & 0 & 1 & -3 & 1 \\
1 & 1 & 1 & 1 & -4
\end{pmatrix}
\]

Non-symmetry, Non-equitable clusters

\[
\frac{dx_i}{dt} = F(x_i) + \sigma \sum_{j=1}^{N} C_{ij} H(x_j)
\]

Complete cluster

\( x_i = s \quad \forall i \)

\[
\frac{ds}{dt} = F(s) + \sigma \sum_{j=1}^{N} C_{ij} H(s) = F(s) + \sigma H(s) \sum_{i=1}^{N} C_{ij} = F(s)
\]

Combine clusters \( x_i = s \quad i = 1, 3, 5 \)

\[
\frac{ds}{dt} = F(s) + \sigma \left[ H(x_2) + H(x_4) + H(s) - 3H(s) \right]
\]

\[
= F(s) + \sigma \left[ H(x_2) + H(x_4) - 2H(s) \right]
\]
Hierarchy of Adjacency and Laplacian Clustering

Orbital Partitions
Symmetry Clusters

Equitable Partitions
[Symmetry Clusters]

External Equitable Partitions
[Symmetry Clusters]

Adjacency and Laplacian

Building Blocks

...and simplify the variational equation.
Conclusions and remarks

- We can analyze all potential cluster formations in Adjacency and Laplacian networks
- Encompasses or overlaps other "phenomena"
  
  Cluster sync, Partial sync, Remote sync, Some Chimera states
- Weighted edges/couplings
- Different oscillators
- Bifurcation forms

  Normal forms & symmetry

1. Cluster Synchronization and Isolated Desynchronization in Complex Networks with Symmetries, Pecora, Sorrentino, Hagerstrom, Murphy, and Roy, *Nature Communications*, 5, 4079 (13 June 2014)


3. Francesco Sorrentino and Louis Pecora, Approximate cluster synchronization in networks with symmetries and parameter mismatches," *CHAOS* 26, 094823 (2016);


$\beta = 0.72 \pi$

Questions?
Learning Group Theory and Group Representation Theory

representation theory in the first three chapters (~ 37 pages).

online:  http://www.learnerstv.com/Free-Maths-Video-lectures-ltv759-Page1.htm

Learner's TV


Michael T. Vaughn, Introduction to Mathematical Physics, Chapter 9  (Wiley-VCH)


Geometric decomposition into subgroups.

<table>
<thead>
<tr>
<th>Network</th>
<th>Number of Nodes</th>
<th>Number of Edges</th>
<th>Number of Symmetries</th>
</tr>
</thead>
<tbody>
<tr>
<td>Human B Cell Genetic Interactions[3]</td>
<td>5,930</td>
<td>64,645</td>
<td>$5.9374 \times 10^{13}$</td>
</tr>
<tr>
<td>C. elegans Genetic Interactions[26]</td>
<td>2,060</td>
<td>18,000</td>
<td>$6.9985 \times 10^{161}$</td>
</tr>
<tr>
<td>BioGRID datasets[23]:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Human</td>
<td>7,013</td>
<td>20,587</td>
<td>$1.2607 \times 10^{485}$</td>
</tr>
<tr>
<td>S. cerevisiae</td>
<td>5,295</td>
<td>50,723</td>
<td>$6.8622 \times 10^{64}$</td>
</tr>
<tr>
<td>Drosophila</td>
<td>7,371</td>
<td>25,043</td>
<td>$3.0687 \times 10^{493}$</td>
</tr>
<tr>
<td>Mus musculus</td>
<td>209</td>
<td>393</td>
<td>$5.3481 \times 10^{125}$</td>
</tr>
<tr>
<td>Internet (Autonomous Systems Level)[12]</td>
<td>22,332</td>
<td>45,392</td>
<td>$1.2822 \times 10^{11.208}$</td>
</tr>
<tr>
<td>US Power Grid[25]</td>
<td>4,941</td>
<td>6,594</td>
<td>$5.1851 \times 10^{152}$</td>
</tr>
</tbody>
</table>

> 88% of nodes are in clusters in all above networks
Other networks
Symmetries and clusters in networks with different topologies

\[ N = 100 \text{ nodes (oscillators)} \quad 10,000 \text{ realizations of each type} \]

\[ 10^{60} \text{ symmetries} \]

Random
\[ n_{\text{delete}} = 50 \]

Scale-free Tree

Random Bipartite


Sage routine RandomBipartite().
Electric power grid of Nepal

$N=15$

Cluster synchronization?

86,400 symmetries
15 Nodes
3 clusters
2 trivial clusters
3 subgroups