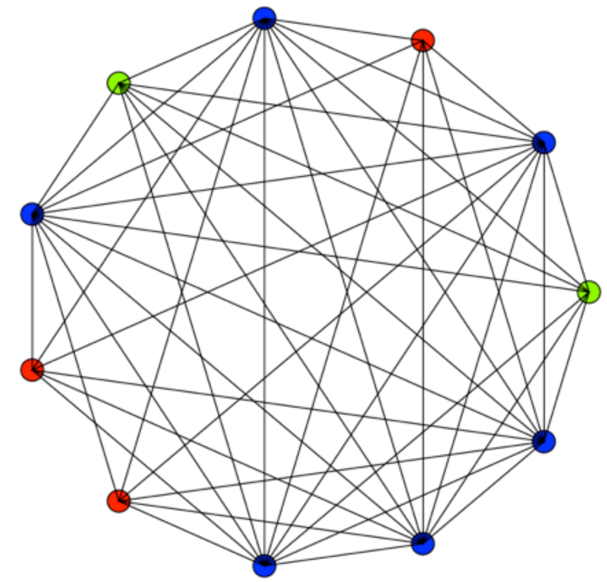
Using Symmetries and Equitable Partitions Together to Find All Synchronization Clusters and Their Stability



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MS168 Advances in Network Synchronization - Part II of II

Per Sebastian Skardal, Jie Sun, and Dane Taylor

SIAM Dynamical Systems Conference May 25, 2017

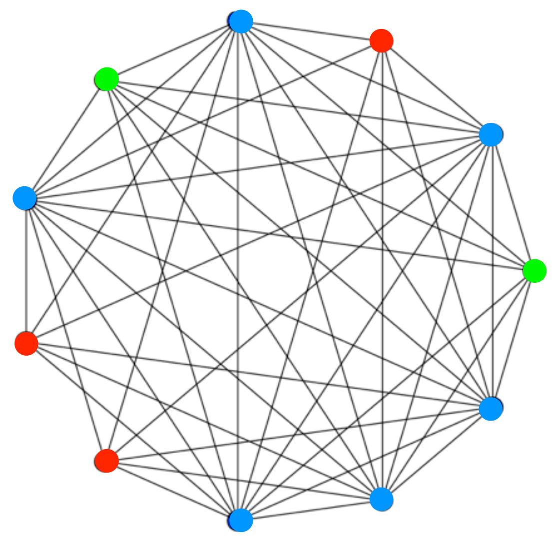
What are synchronized clusters?

Cluster Synchronization in complex networks

(various versions in the literature)

Identical nodes (oscillators), Identical edges (couplings)

Adjacency matrix

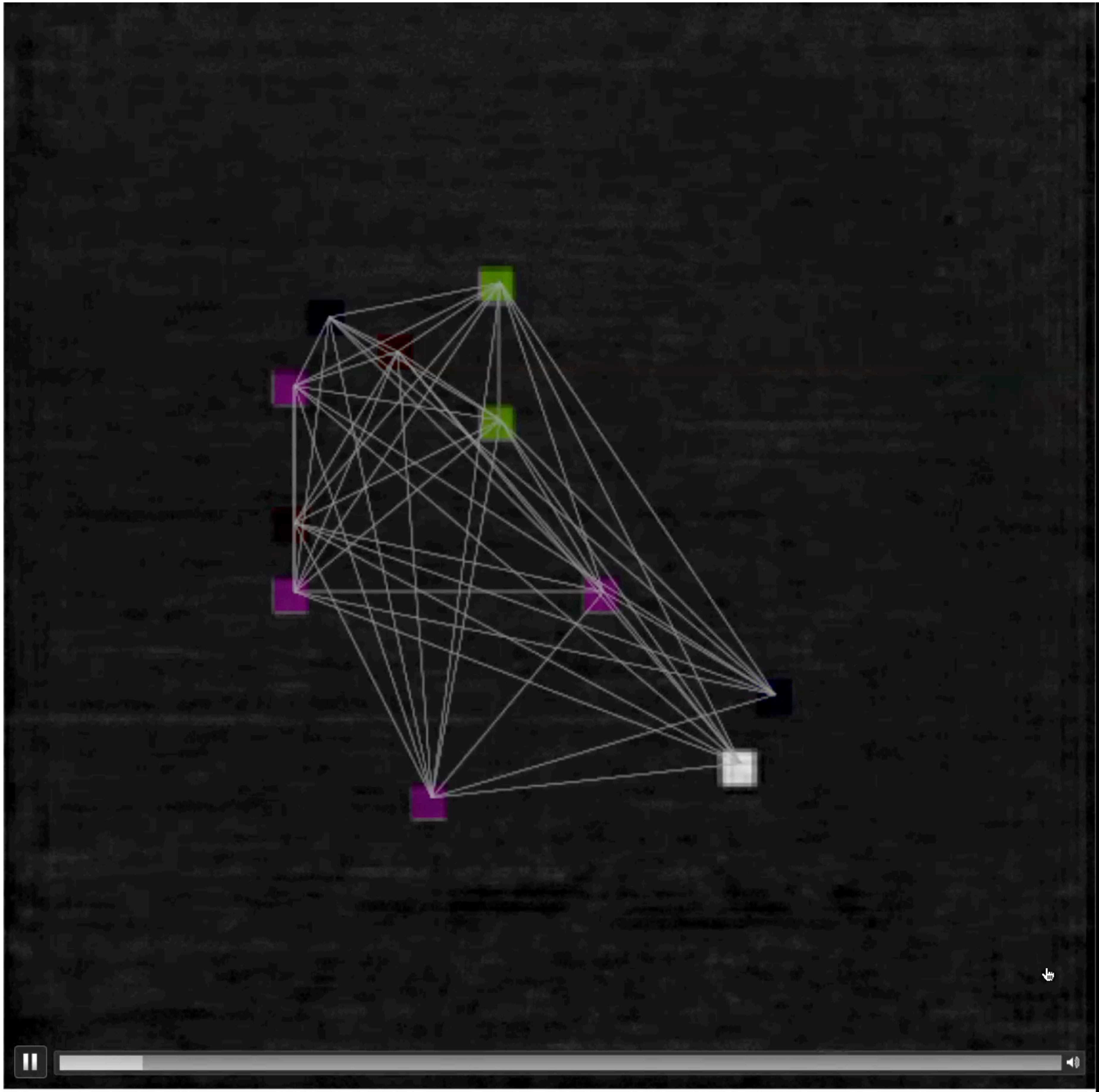


$$\frac{d\mathbf{x}_i}{dt} = F(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} C_{ij} H(\mathbf{x}_j)$$

- Identify the clusters?
- Are the clusters stable?
- Complex (large) networks
- Any dynamics (fixed pt, periodic, quasiperiodic, chaotic

Cluster synchronization, an "old" subject (early 2000's)

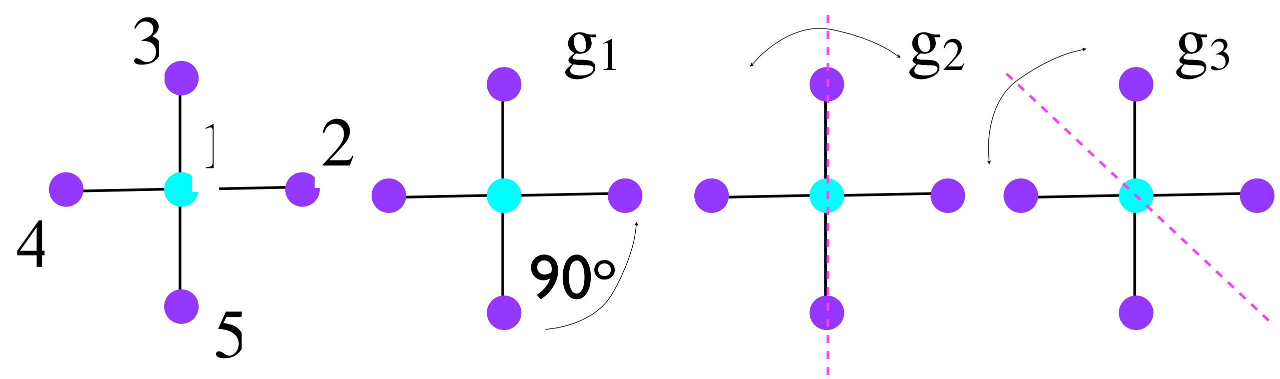
$\beta = 0.72\ \pi$



Symmetries and clusters in networks

Using symmetry to find clusters.

Network with identical nodes (oscillators) and identical edges



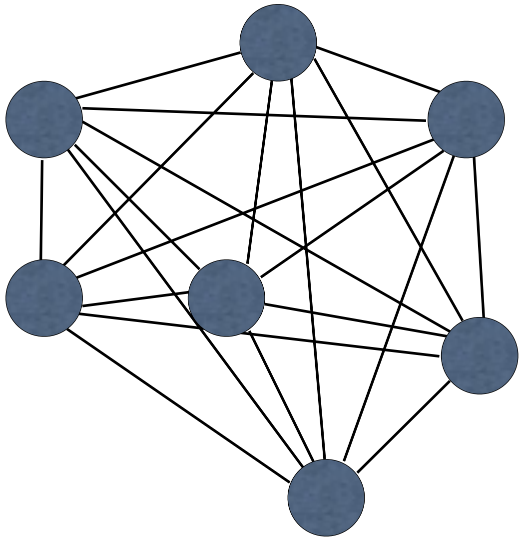
group $\mathcal{G}$ $=\{g_i\}$

$$R_{g_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad R_{g_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad R_{g_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

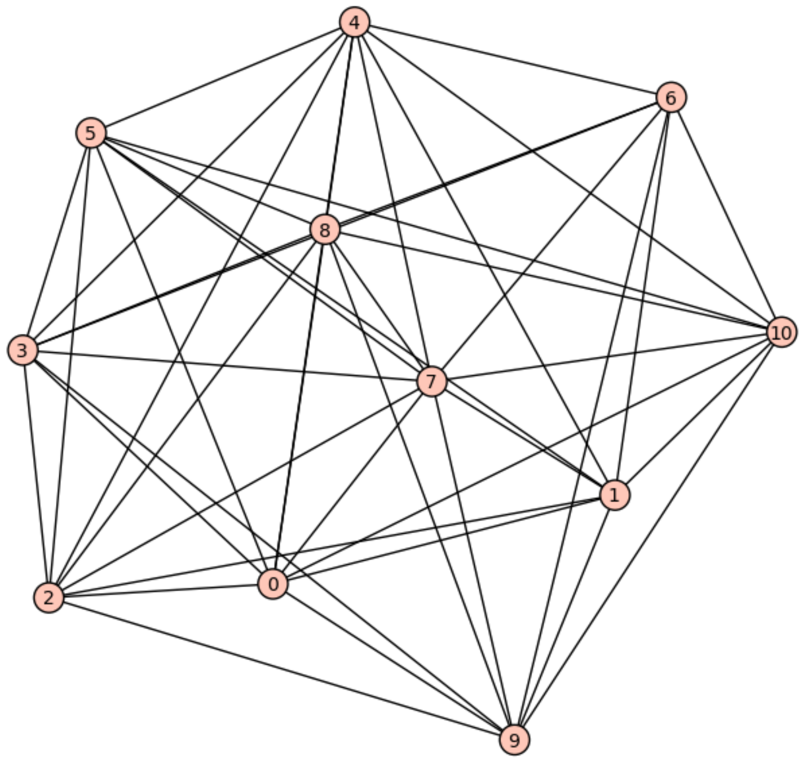
$\{R_{g_i}\}$ Representation of the group $\mathcal{G}$ e.g. $g_2 g_1^{-1} = g_3 \Rightarrow R_{g_2} R_{g_1}^{-1} = R_{g_3}$

$$\frac{d\mathbf{x}_i}{dt} = F(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} C_{ij} H(\mathbf{x}_j) \quad \Rightarrow \quad R_g C = C R_g \quad \Rightarrow \quad C = R_g^{-1} C R_g$$

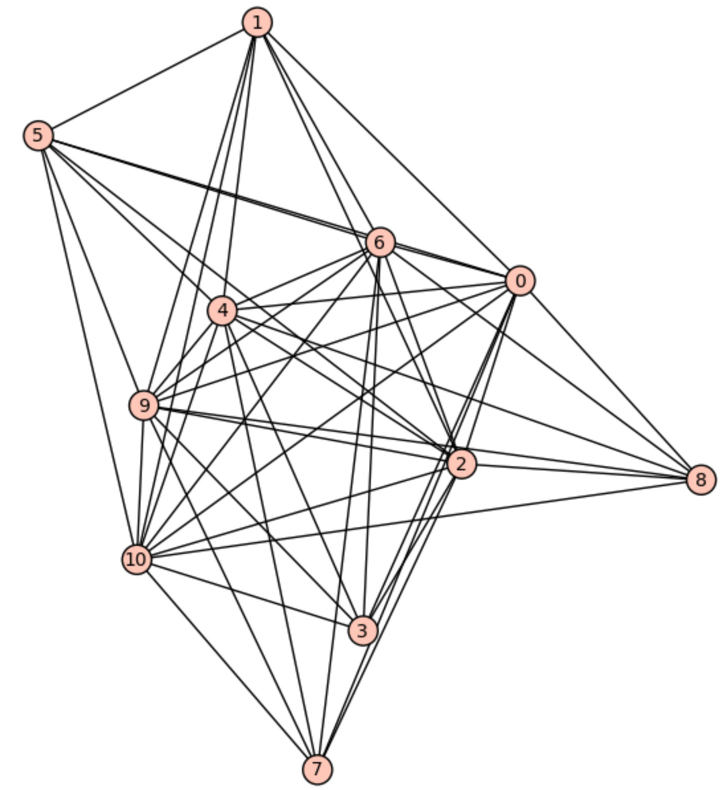
Random Graphs



Random networks 11 nodes - 9 random edges



0 symmetries



8640 symmetries

G.gens()= [(7,10), (6,7), (5,6), (4,8), (2,4)(8,9), (1,5), (1,11)]

$\{g_i\}$

Computational Group Theory

GAP

Groups, Algorithms, Programming -
a System for Computational Discrete Algebra

Sage

http://www.sagemath.org/

```
sage: x = var('x')
sage: solve(x^2 + 3*x + 2, x)
[x == -2, x == -1]
```

Python

```
for i,row in
enumerate(Adjmat):
   rsum= row.sum()
   Cplmat[i,i]= -rsum
print Cplmat
```

Free ! (open source)

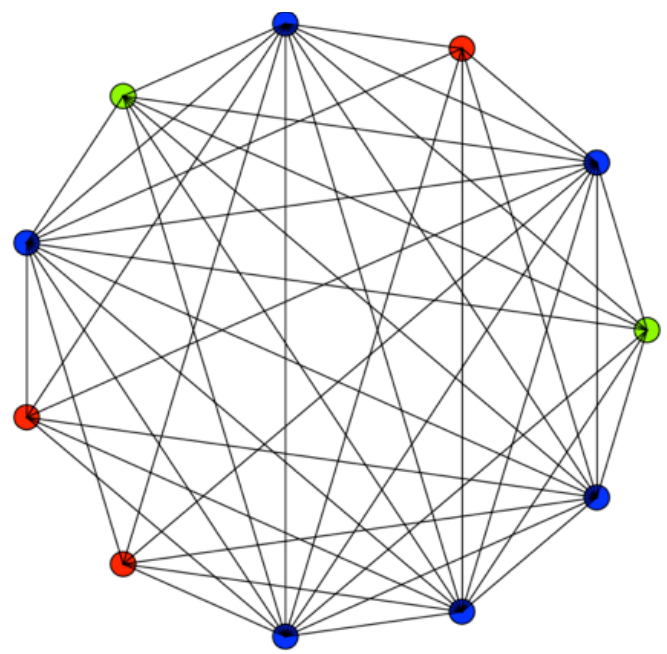
```
G.order(), G.gens()= 8640 [(9,10), (7,8), (6,9),
(4,6), (3,7), (2,4), (2,11), (1,5)]

node sync vectors:
  Node 2

orb= [1, 5]
nodeSyncvec  [0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0]
cycleSyncvec  [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
  Node 1

orb= [2, 4, 11, 6, 9, 10]
nodeSyncvec  [1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1]
cycleSyncvec  [0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1]
  Node 4

orb= [3, 7, 8]
nodeSyncvec  [0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0]
cycleSyncvec  [0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0]
```



M. Golubitsky, I. Stewart, and D.G. Schaeffer, Singularities and groups in bifurcation theory, Vols. I & II (Springer-Verlag, New York, NY, 1985).

(1) No computer code for large networks   (2) Did not examine stability of clusters.

Stability: Coupling Matrix

-7.	1.	0.	1.	1.	1.	0.	0.	1.	1.	1.
1.	-10.	1.	1.	1.	1.	1.	1.	1.	1.	1.
0.	1.	-6.	1.	0.	1.	0.	0.	1.	1.	1.
1.	1.	1.	-10.	1.	1.	1.	1.	1.	1.	1.
1.	1.	0.	1.	-7.	1.	0.	0.	1.	1.	1.
1.	1.	1.	1.	1.	-10.	1.	1.	1.	1.	1.
0.	1.	0.	1.	0.	1.	-6.	0.	1.	1.	1.
0.	1.	0.	1.	0.	1.	0.	-6.	1.	1.	1.
1.	1.	1.	1.	1.	1.	1.	1.	-10.	1.	1.
1.	1.	1.	1.	1.	1.	1.	1.	1.	-10.	1.
1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	-10.

$$\frac{d\,\delta \mathbf{x}_i}{dt} = D\mathbf{F}(\mathbf{x}_i)\delta \mathbf{x}_i + \sigma \sum_{j=1}^{N} C_{ij} D\mathbf{H}(\mathbf{x}_j)\delta \mathbf{x}_j$$

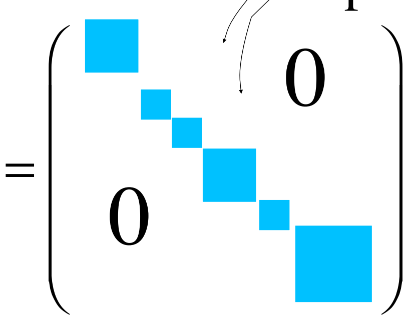
$$\delta \mathbf{x}_i \rightarrow \delta \mathbf{x}_i + \mathbf{b} \quad \forall i \in Cluster$$

Simplifying the variational matrix

$\forall g \in \mathcal{G}$ = group of all symmetries $\{R_g\}$ is a representation of $\mathcal{G}$

Irreducible Representations(IR

$\{R_g\}$ It is reducible if $\exists\, T\ \forall\, g$ $TR_gT^{-1} =$



$$C = R_g^{-1} C R_g$$

T block diagonalizes C

The software does not supply *T*

-7.	1.	0.	1.	1.	1.	0.	0.	1.	1.	1.
1.	-10.	1.	1.	1.	1.	1.	1.	1.	1.	1.
0.	1.	-6.	1.	0.	1.	0.	0.	1.	1.	1.
1.	1.	1.	-10.	1.	1.	1.	1.	1.	1.	1.
1.	1.	0.	1.	-7.	1.	0.	0.	1.	1.	1.
1.	1.	1.	1.	1.	-10.	1.	1.	1.	1.	1.
0.	1.	0.	1.	0.	1.	-6.	0.	1.	1.	1.
0.	1.	0.	1.	0.	1.	0.	-6.	1.	1.	1.
1.	1.	1.	1.	1.	1.	1.	1.	-10.	1.	1.
1.	1.	1.	1.	1.	1.	1.	1.	1.	-10.	1.
1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	-10.

Synchronization Manifold

$=1$ Trivial Representation

$$\frac{d\,\delta \mathbf{x}_i}{dt} = D\mathbf{F}(\mathbf{x}_i)\delta\mathbf{x}_i + \sigma\sum_{j=1}^{N} C_{ij} D\mathbf{H}(\mathbf{x}_j)\delta\mathbf{x}_j$$

$$\frac{d\xi_\lambda}{dt} = \left(DF(\mathbf{s}) + \Lambda_\lambda DH(\mathbf{s})\right)\cdot\xi_\lambda$$

All other Representations

Transverse Manifold (sub-blocks associated with each cluster)

Experimental confirmation of symmetry clusters

Aaron Hagerstrom
Rajarashi Roy
Thomas Murphy
Francesco Sorrentino

How to implement maps in experiment

Self feedback strength

Coupling strength

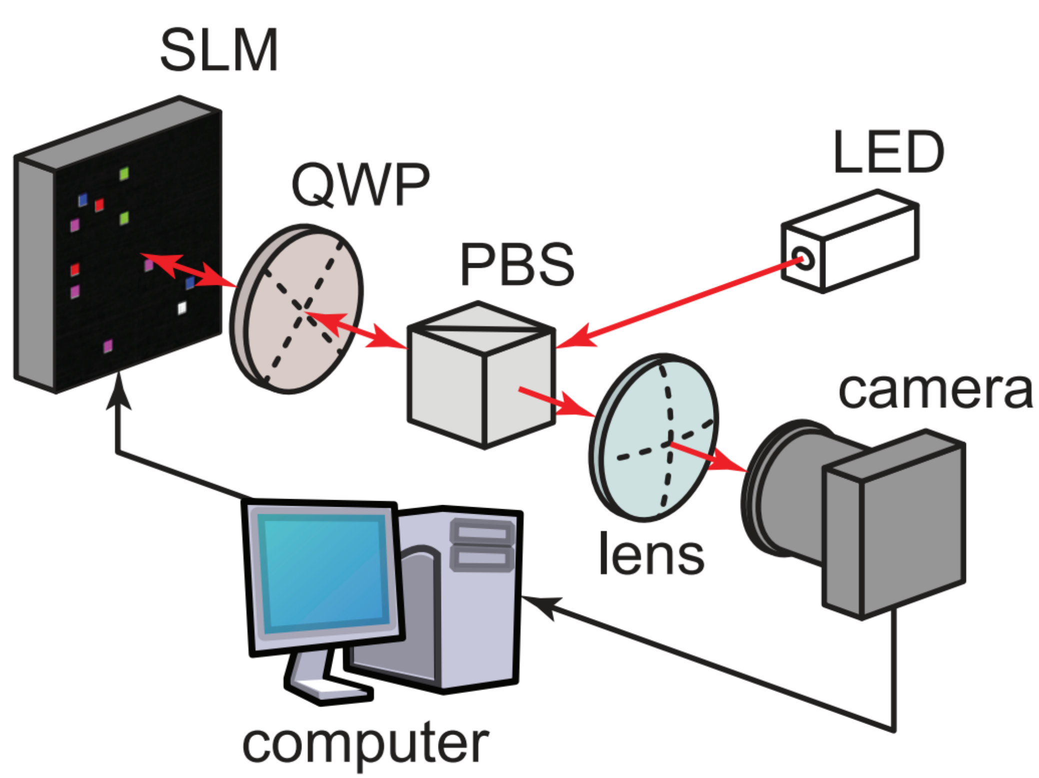
Adjacency matrix

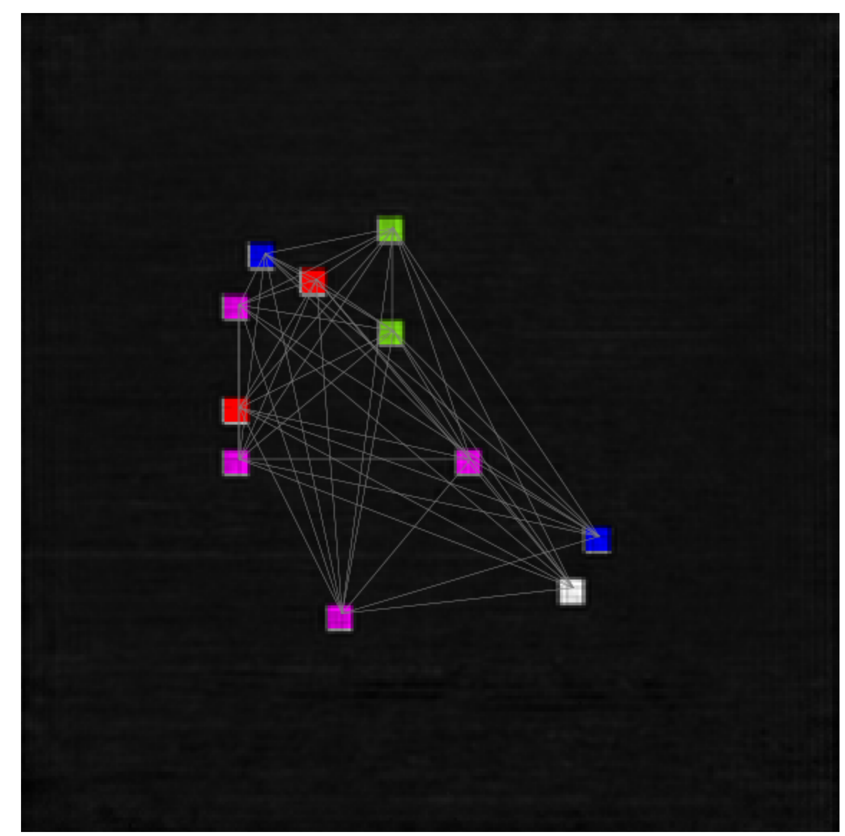
Add phase shift to destabilize fixed point $\varphi=0$

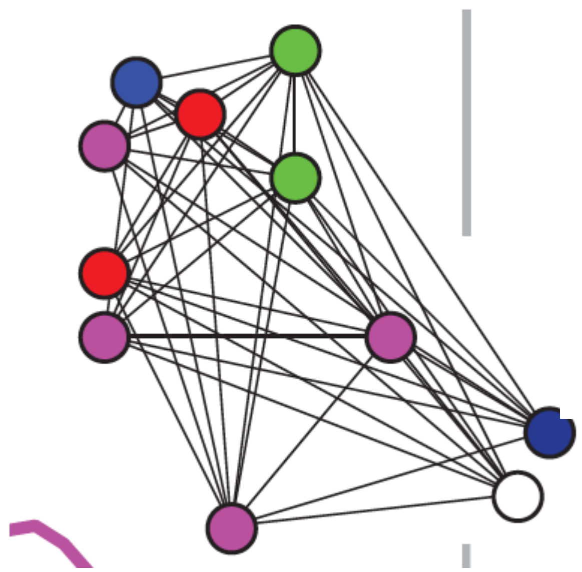
$$\phi_i^{t+1} = \beta I(\phi_i^t) + \sigma \sum_j A_{ij} I(\phi_j^t) + \delta$$

Phase written to SLM

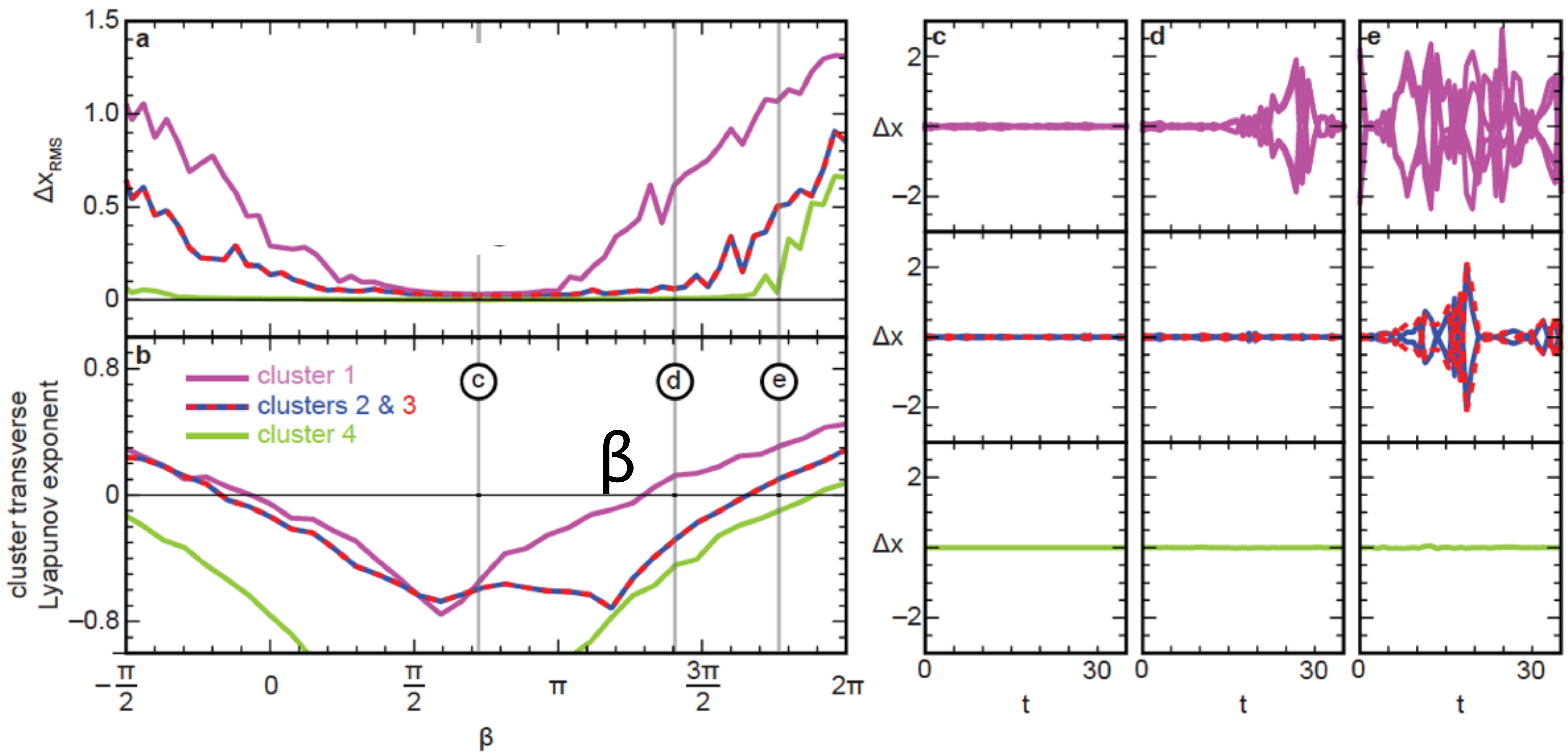
Intensity measured by camera



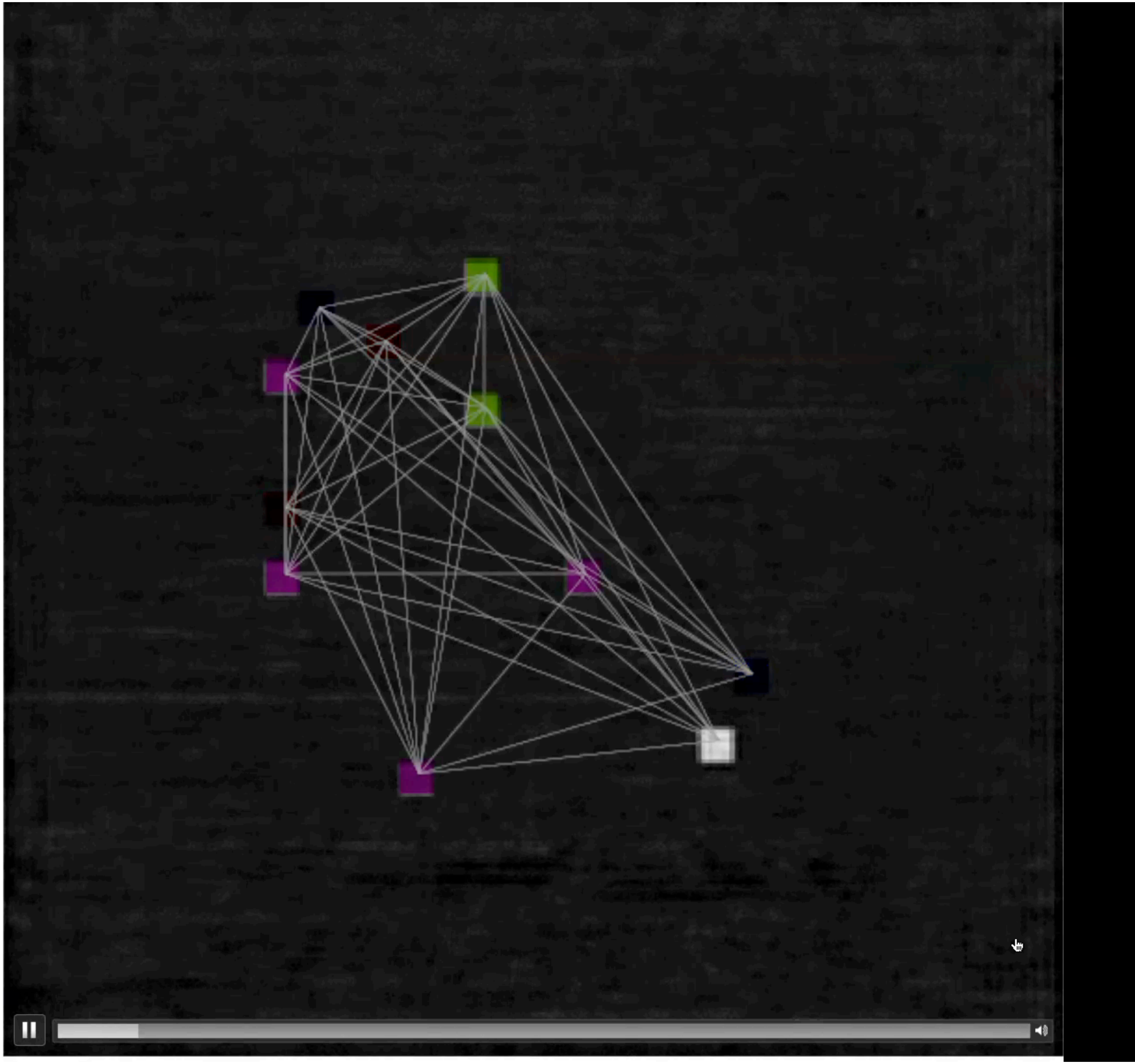




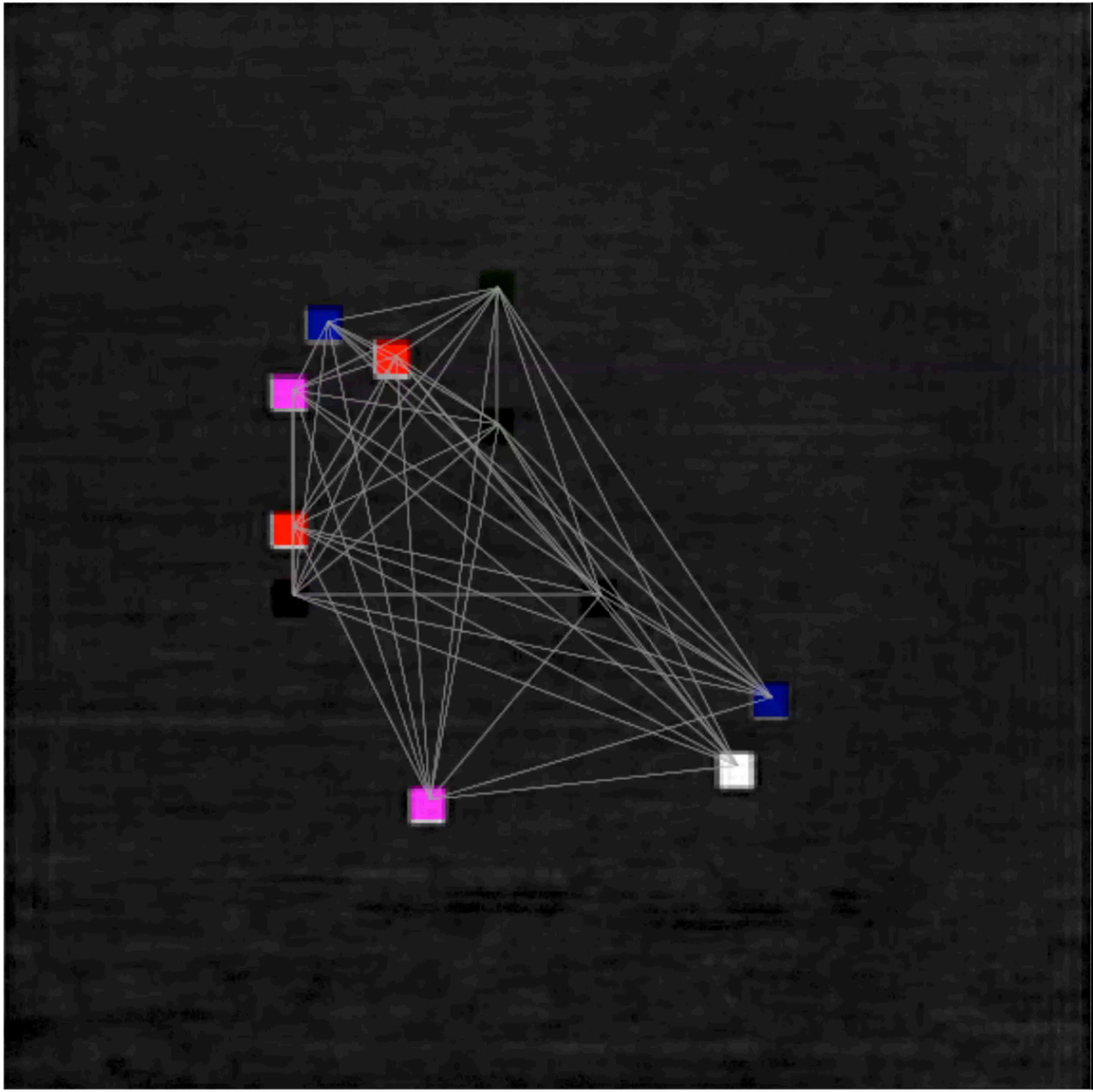
11 node random network (video)
32 Symmetries
4 nontrivial clusters
1 trivial cluster
5 x 5 Sync block

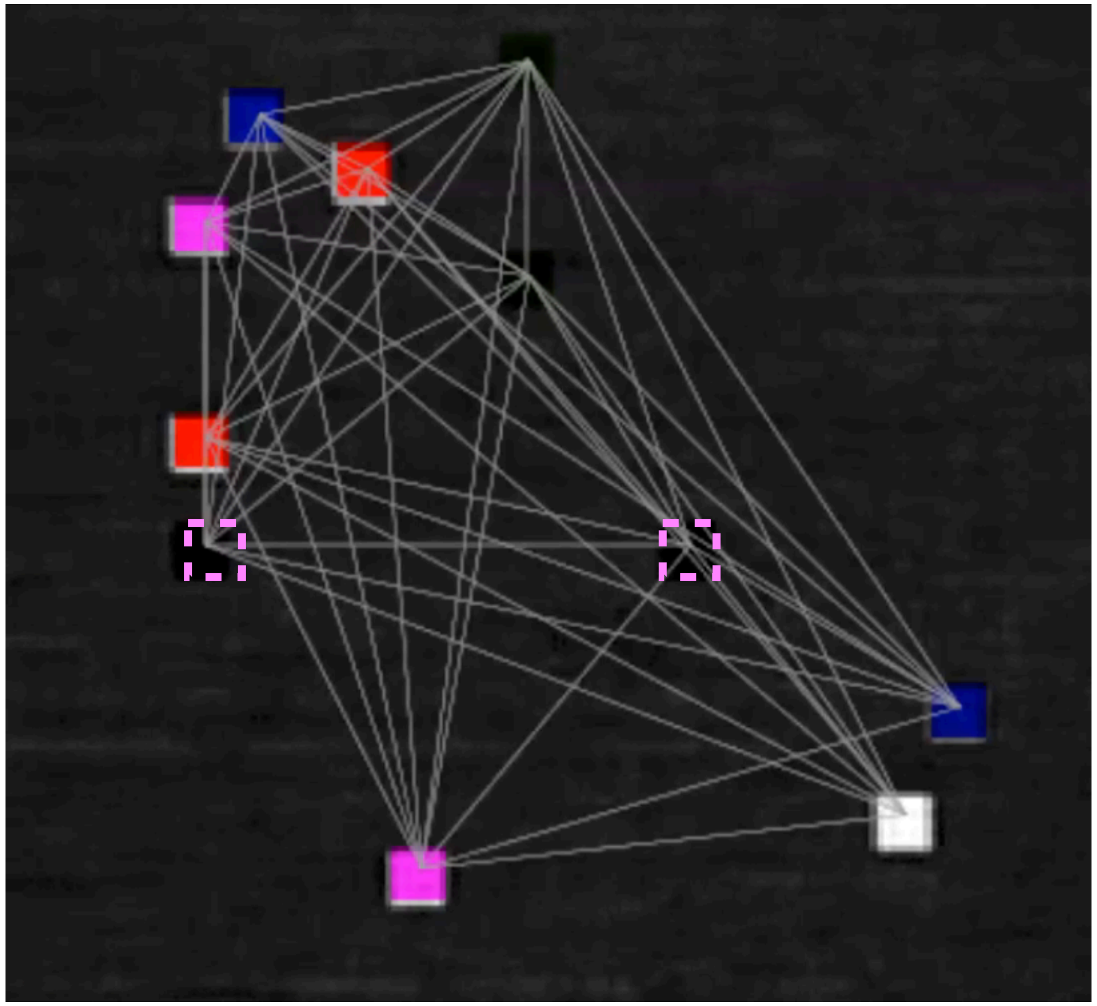


$\beta = 0.72\ \pi$



$\beta = 1.4\,\pi$





Chimera in identical synchronized network

(Not just for phase oscillators)

Chimeras, Cluster States, and Symmetries: Experiments on the Smallest Chimera - Joseph Hart

MS 122

Not transients

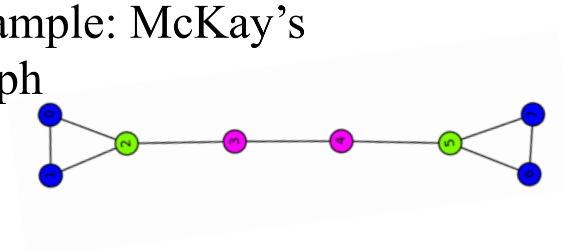
Constructing non-symmetry clusters

Beyond symmetry clusters

Forming Equitable and Laplacian Clusters from Symmetry (orbital) Clusters

Def: An Equitable Partition of a graph is a collection of disjoint subsets C_i ($C_i \cap C_j = \varnothing$) of all the nodes such that, if node $v \in C_i$ has d connections to nodes in C_j then for any other node $u \in C_i$ has d connections to nodes in C_j.

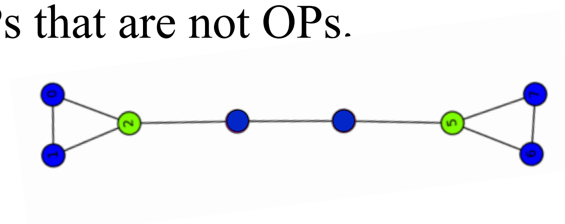
Example: McKay's graph



C_1
C_2
C_3

Besides being an Equitable Partition (EP) this is also an Orbital Partition (OP) from the symmetries.

Every OP is an EP, but the converse is not true. There are EPs that are not OPs.



C_1
C_2

Similar to groupoid synchronization criterion of Golubitsky, Stewart and Török (SIAM J. APPLIED DYNAMICAL SYSTEMS Vol. 4, No. 1, pp. 78–100 (2005)

An algorithm from Igor Belykh* provides an efficient method for finding the coarsest EP. However finding finer EPs is a hard problem. This is important since there may be finer EPs that are still not OPs. An example coming up.

Problem 1: Find finer EPs that are not OPs.

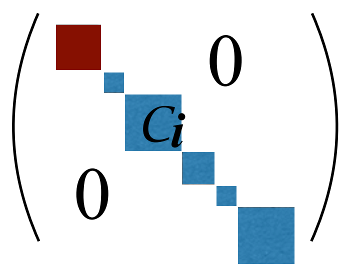
* I. Belykh and M. Hasler, CHAOS 21, 016106 (2011)

Problem 2: Stability calculations. Reducing the transverse manifold in the variational equations.

$$\frac{d\,\delta\mathbf{x}_i}{dt} = D\mathbf{F}(\mathbf{x}_i)\delta\mathbf{x}_i + \sigma\sum_{j=1}^{N} A_{ij}\, D\mathbf{H}(\mathbf{x}_j)\,\delta\mathbf{x}_j \qquad \frac{d\xi_\lambda}{dt}$$

OPs (Symmetry Clusters)

C_i IRR



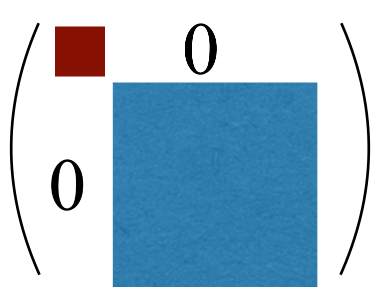
each block is associated with particular clusters. Master Stability Function for each cluster.

EPs (Equitable Clusters)

Ref: Schaub et al.*

Make synchronization manifold basis and find the orthogonal compliment (the transverse space, V , this will separate out the synchronization manifold, but will not block diagonalize the transverse manifold.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ & & & & V_\perp & & & & & \end{pmatrix}$$



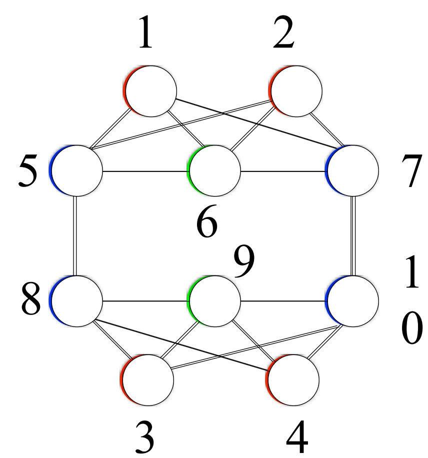
- Stability calculations scale badly (N^2)
- Stability of individual clusters is unknown.
- Type of desynchronization bifurcation is unknown.

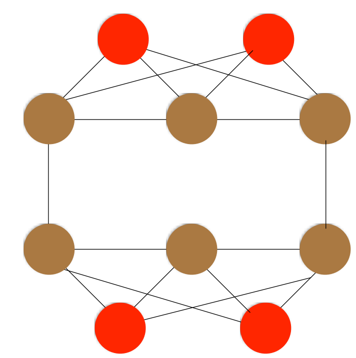
Is it even possible to block diagonalize the transverse manifold?

*M. Schaub, et al., Chaos 26 , 094821 (2016)

Constructing an EP from an OP

The Abu network.





Symmetries->OP orbitals= [1,2,3,4], [5,7,8,10], [6,9]

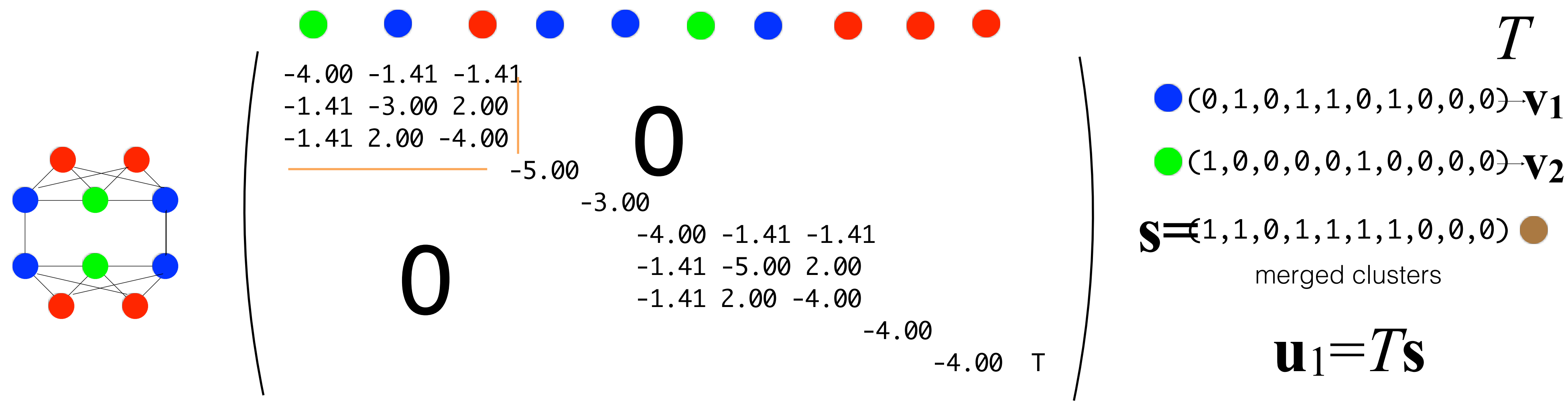
EP ->Equitable Clusters=
[1,2,3,4], [5,7,8,10,6,9]

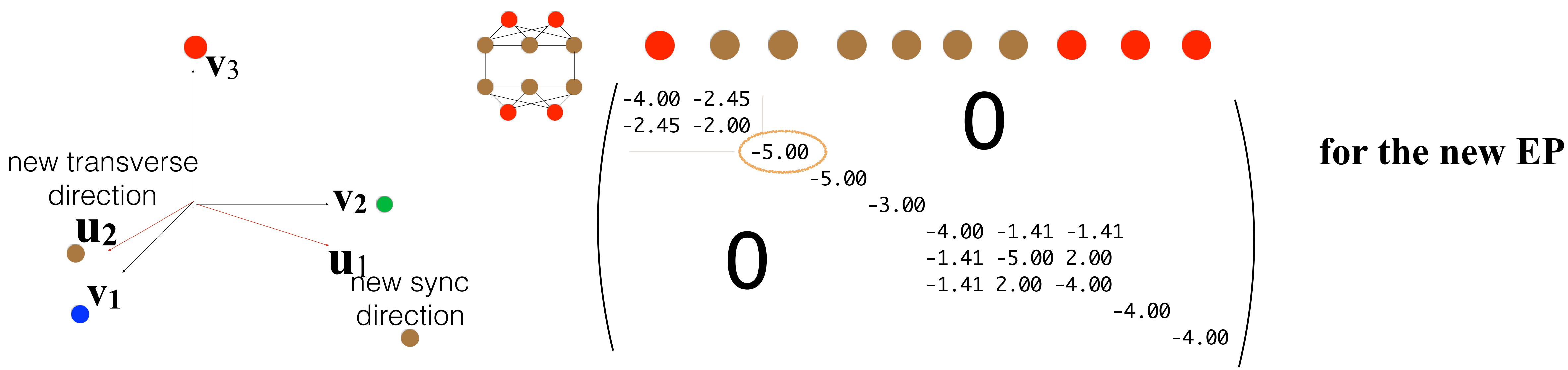
Make the EP by merging clusters
from its OP refinement

[5,7,8,10] U [6,9]

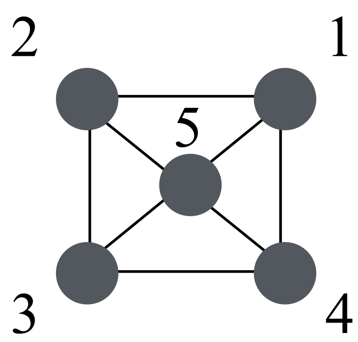
Observation: We can always construct an EP from an OP refinement.
There is always a subgroup of the original symmetry group that will be a refinement of any EP

Construction of a block diagonalized variational matrix for the EP



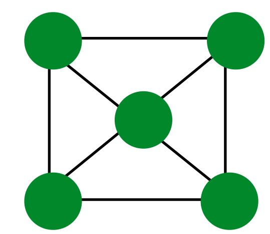


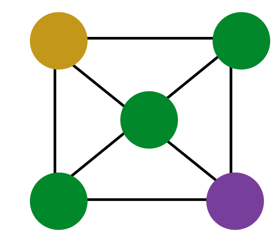
- Laplacian coupling - diag = – row sum



$$\begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$$

Non-symmetry, Non-equitable clusters





$$\frac{d\mathbf{x}_i}{dt} = F(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} C_{ij} H(\mathbf{x}_j)$$

Complete cluster

$\mathbf{x}_i = \mathbf{s} \quad \forall i$

0

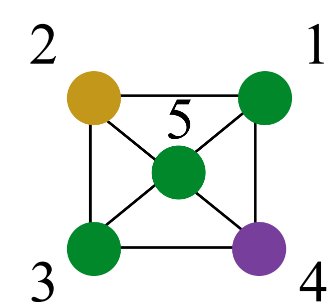
$$\frac{d\mathbf{s}}{dt} = F(\mathbf{s}) + \sigma \sum_{j=1}^{N} C_{ij} H(\mathbf{s}) = F(\mathbf{s}) + \sigma H(\mathbf{s}) \sum_{j=1}^{N} C_{ij} = F(\mathbf{s})$$

Combine clusters

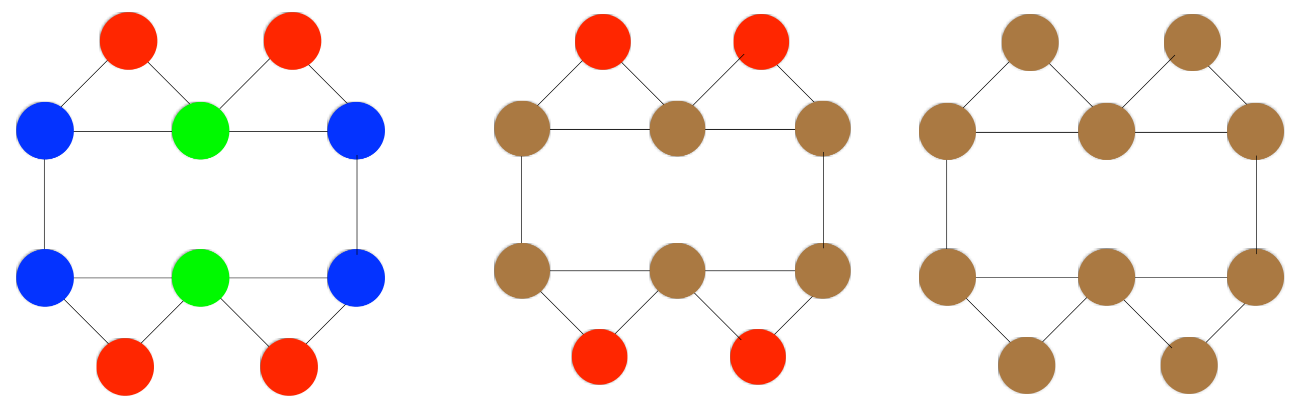
$\mathbf{x}_i = \mathbf{s} \quad i = 1,3,5$

$$\frac{d\mathbf{s}}{dt} = F(\mathbf{s}) + \sigma\left[H(\mathbf{x}_2) + H(\mathbf{x}_4) + H(\mathbf{s}) - 3H(\mathbf{s})\right]$$

$$= F(\mathbf{s}) + \sigma\left[H(\mathbf{x}_2) + H(\mathbf{x}_4) - 2H(\mathbf{s})\right]$$

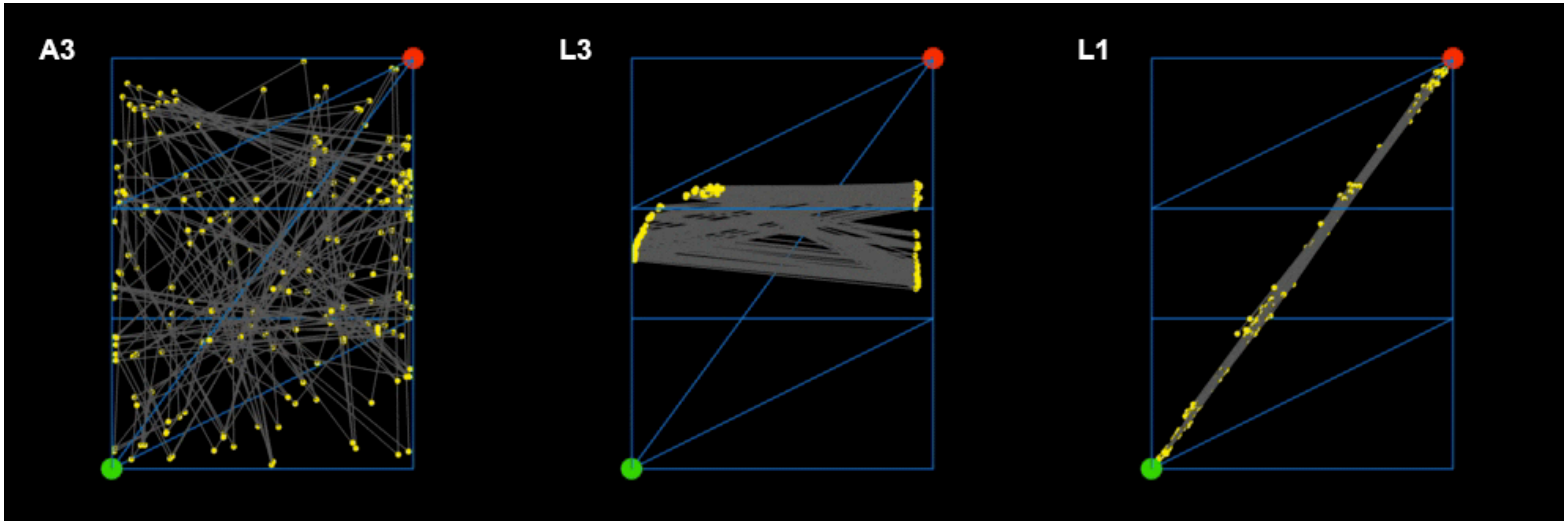


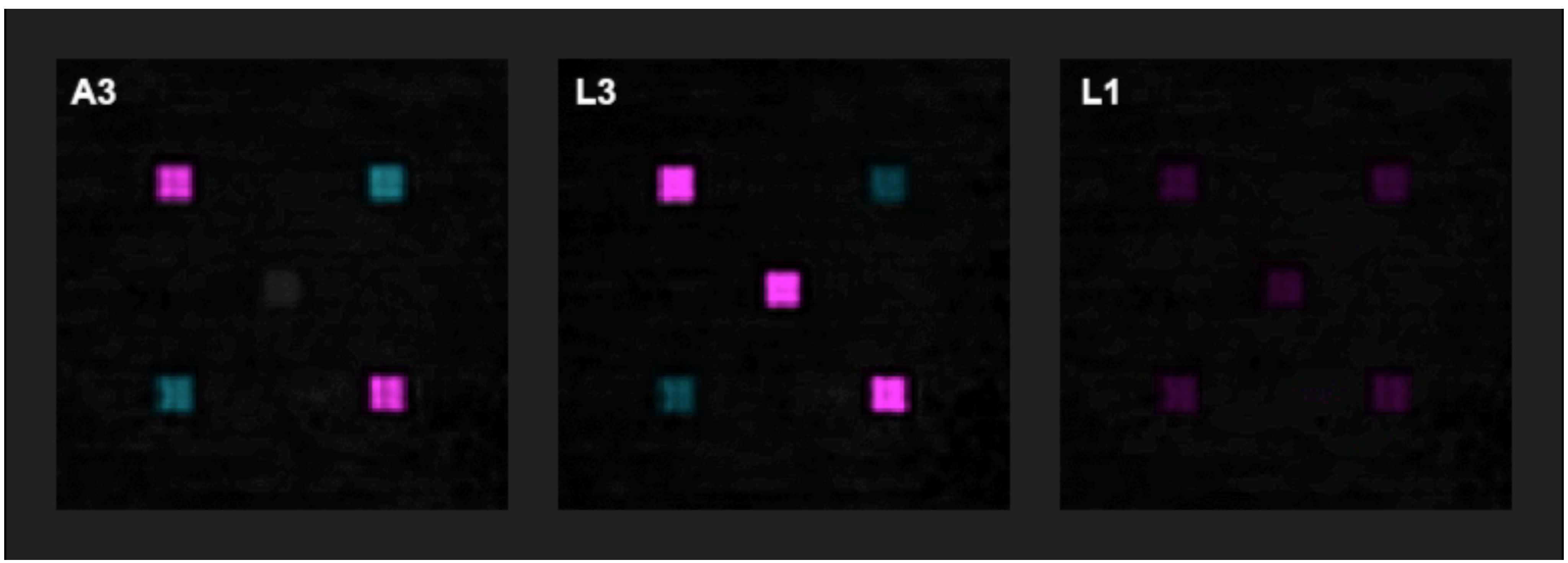
Hierarchy of Adjacency and Laplacian Clustering



Orbital Partitions Symmetry Clusters	Equitable Partitions [Symmetry Clusters]	External Equitable Partitions [Symmetry Clusters]
Adjacency and Laplacian	Adjacency and Laplacian	Laplacian
Building Blocks		

…and simplify the variational equation.


A3
L3
L1


A3
L3
L1

Conclusions and remarks

- We can analyze all potential cluster formations in Adjacency and Laplacian networks

- Encompasses or overlaps other "phenomena"

  Cluster sync, Partial sync, Remote sync, Some Chimera states

- Weighted edges/couplings

- Different oscillators

- Bifurcation forms

  Normal forms & symmetry

M. Golubitsky, I. Stewart, and D.G. Schaeffer, Singularities and groups in bifurcation theory, Vols. I & II (Springer-Verlag, New York, NY, 1985).

1. Cluster Synchronization and Isolated Desynchronization in Complex Networks with Symmetries, Pecora, Sorrentino, Hagerstrom, Murphy, and Roy, *Nature Communications*, 5, 4079 (13 June 2014)

2. Complete Characterization of Stability of cluster Synchronization in Complex Dynamical Systems, Sorrentino, Pecora, Hagerstrom, Murphy, and Roy, *Science Advances **5**, 011005–1–17 (2015).*

3. Francesco Sorrentino and Louis Pecora, Approximate cluster synchronization in networks with symmetries and parameter mismatches," *CHAOS* 26, 094823 (2016);

4. Joseph D. Hart, Kanika Bansal, Thomas E. Murphy, and Rajarshi Roy, Experimental observation of chimera and cluster states in a minimal globally coupled network, *CHAOS* 26, 094801 (2016)

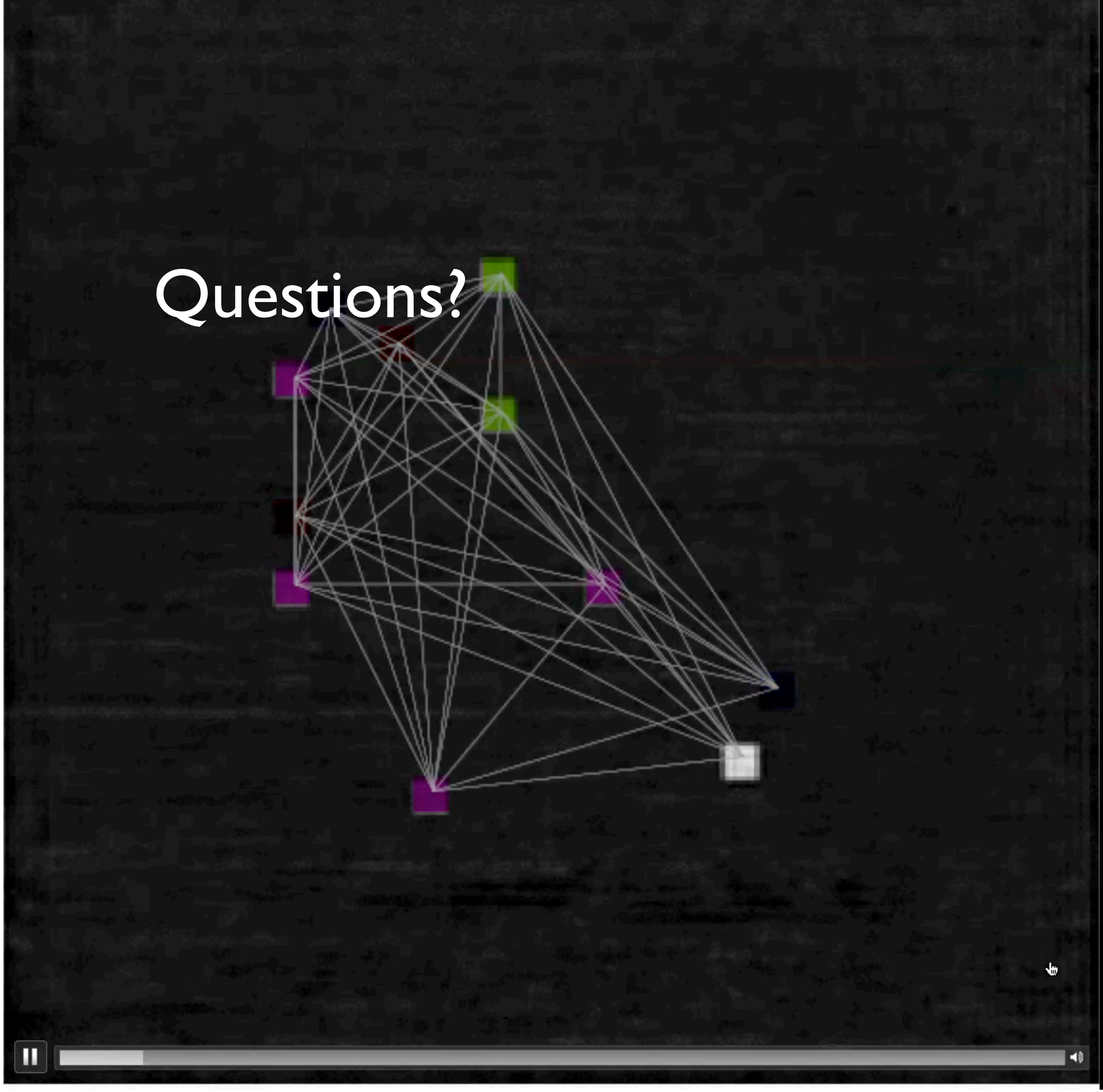
Software Hagerstrom, A. Network Symmetries and Synchronization (https://sourceforge.net/projects/networksym/, 2014).

Thanks to:

B.D. MacArthur and R.J. Sanchez-Garcia, Spectral characteristics of network redundancy," Physical Review E 80, 026117 (2009).

B.D. MacArthur, R.J. Sanchez-Garcia, and J.W. Anderson, \On automorphism groups of networks," Discrete Appl. Math. 156, 3525 (2008).

$$\beta = 0.72\,\pi$$



Learning Group Theory and Group Representation Theory

M. Tinkham, Group Theory and Quantum Mechanics (McGraw-Hill, New York, NY, 1964)
representation theory in the first three chapters (~ 37 pages).

online: http://www.learnerstv.com/Free-Maths-Video-lectures-ltv759-Page1.htm
Learner's TV

Nadir Jeevanjee, An Introduction to Tensors and Group Theory for Physicists, Birkhäuser (New York, NY, 2010)

Michael T. Vaughn, Introduction to Mathematical Physics, Chapter 9 (Wiley-VCH)

M. Golubitsky, I. Stewart, and D.G. Schaeffer, Singularities and groups in bifurcation theory, Vols. I & II (Springer-Verlag, New York, NY, 1985).

B.D. MacArthur, R.J. Sanchez-Garcia, and J.W. Anderson, \On automorphism groups of networks," Discrete Appl. Math. 156, 3525 (2008).

Geometric decomposition into subgroups.

Network	Number of Nodes $N_{\mathscr{G}}$	Number of Edges $M_{\mathscr{G}}$	Number of Symmetries $a_{\mathscr{G}}$
Human B Cell Genetic Interactions[3]	5,930	64,645	5.9374×10^{13}
C. elegans Genetic Interactions[26]	2,060	18,000	6.9985×10^{161}
BioGRID datasets[23]:			
Human	7,013	20,587	1.2607×10^{485}
S. cerevisiae	5,295	50,723	6.8622×10^{64}
Drosophila	7,371	25,043	3.0687×10^{493}
Mus musculus	209	393	5.3481×10^{125}
Internet (Autonomous Systems Level)[12]	22,332	45,392	$1.2822 \times 10^{11,298}$
US Power Grid[25]	4,941	6,594	5.1851×10^{152}

> 88% of nodes are in clusters in all above networks

Other networks

Symmetries and clusters in networks with different topologies

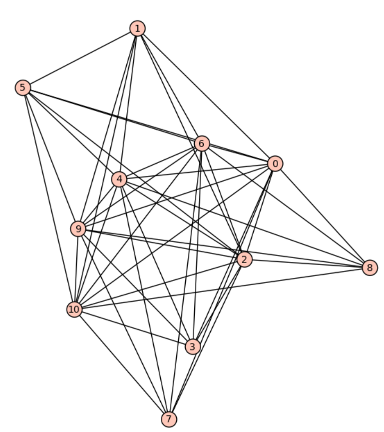
N= 100 nodes (oscillators)

10,000 realizations of each type

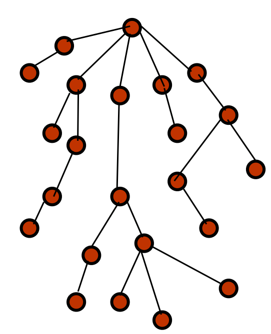
10^{60} symmetries

Random

n_{delete}= 50

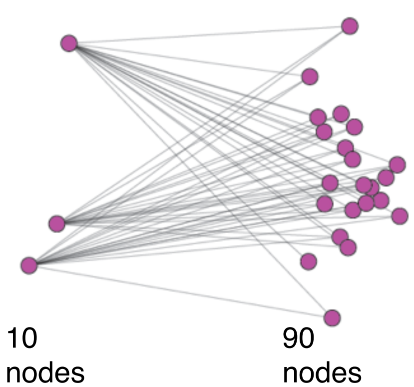


Scale-free Tree



A.-L. Barabasi and R. Albert, \Emergence of scaling in random networks," Science 286, 509{512 (1999).

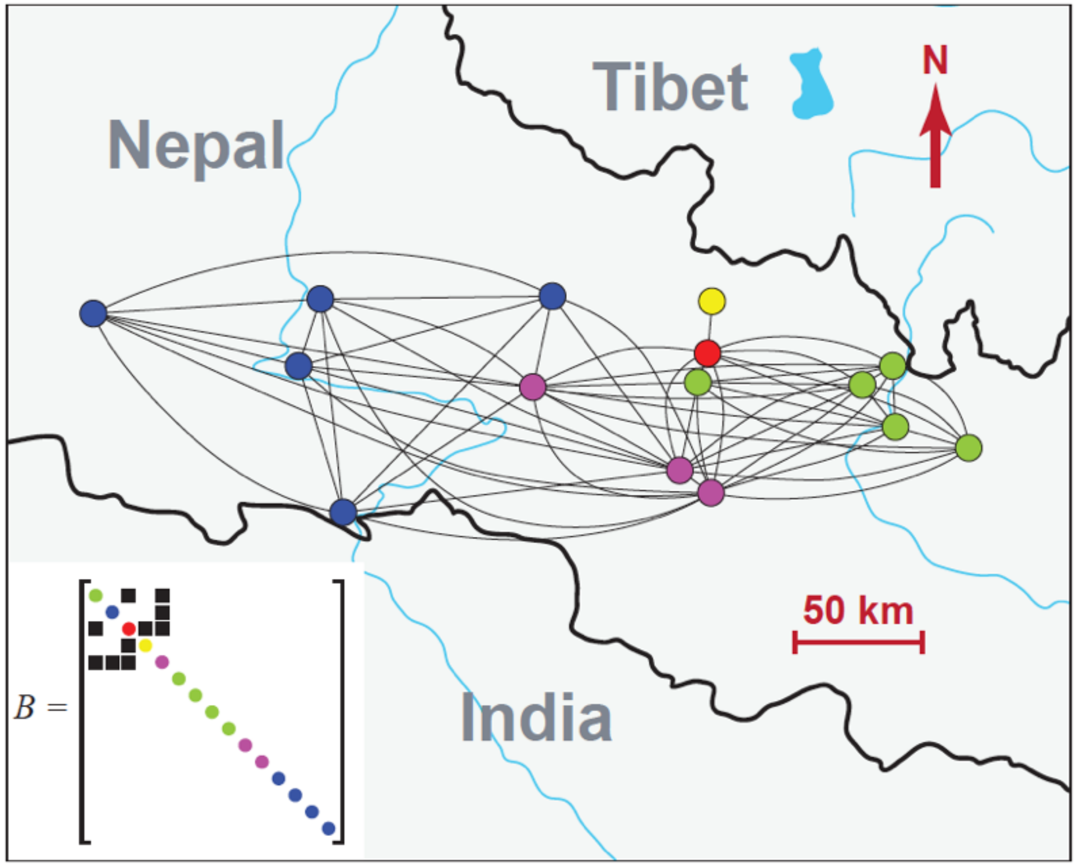
Random Bipartite



Sage routine RandomBipartite().

Electric power grid of Nepal N=15

Cluster synchronization?



86,400 symmetries
15 Nodes
3 clusters
2 trivial clusters
3 subgroups