A New Frame for Phase Space Analysis: Using Differential Geometry to Reveal Local Dynamics

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Overview

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where $a, b, k, \alpha, \beta, \gamma, \delta > 0$ and $n \geq 12$.

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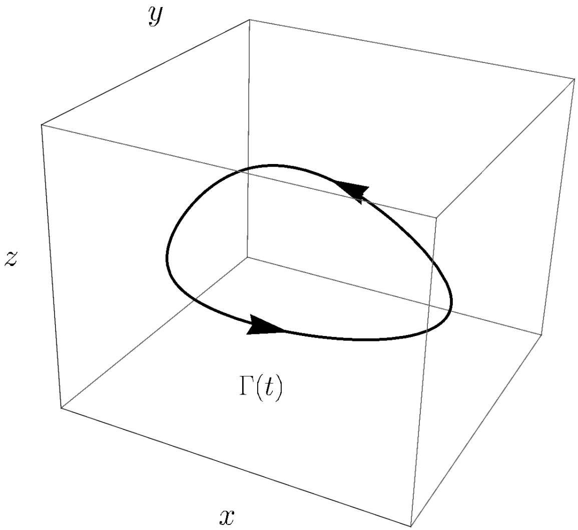
  where $a, b, k, \alpha, \beta, \gamma, \delta > 0$ and $n \geq 12$.
- The system has a unique, globally stable periodic trajectory across a wide range of parameter values.

Dynamics near the Limit Cycle



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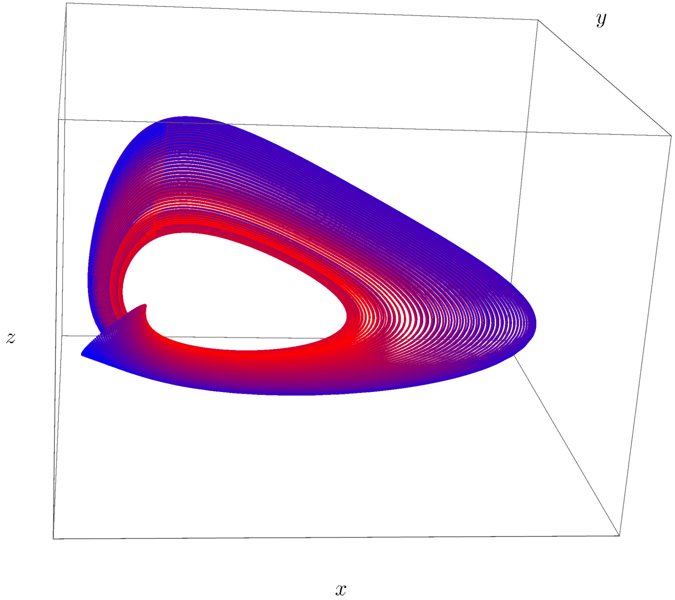

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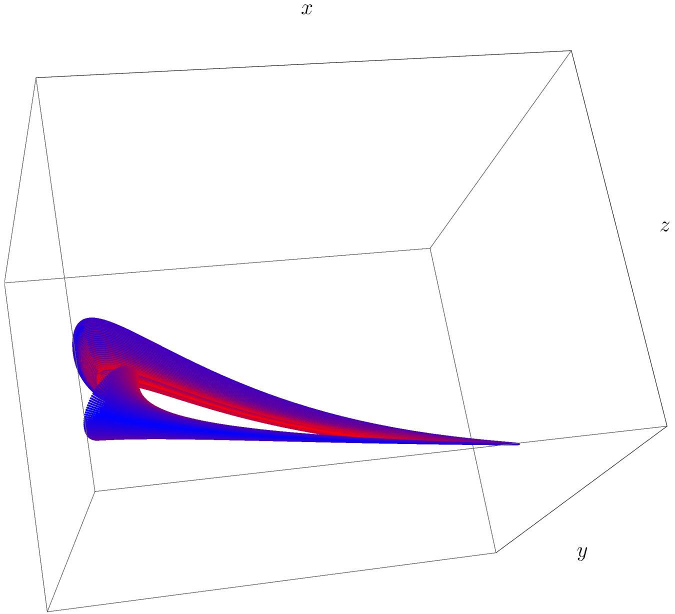

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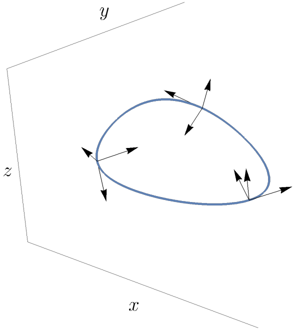
- Furthermore, is there a stable invariant 2-manifold upon which the limit cycle lies? If so, can it be easily identified in order to reduce the dimension of our system?
- In order to study the transient dynamics near the limit cycle, we need to change our geometry.

The Frenet Frame

- For each value of $t \in [0, T)$ we can construct a tangent, normal, and binormal vector to $\Gamma(t)$, which we denote $T\Gamma(t), N_1\Gamma(t), N_2\Gamma(t)$ respectively



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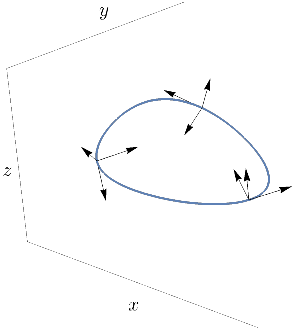

- Together $T\Gamma, N_1\Gamma, N_2\Gamma$ form the Frenet frame to Γ. We will use this frame to simplify the dynamics.

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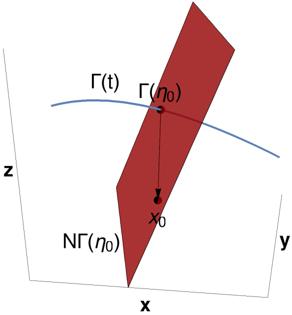
for $\eta_0 \in [0, t), \xi_0 := (\xi_{1,0}, \xi_{2,0}) \in \mathbb{R}^2$



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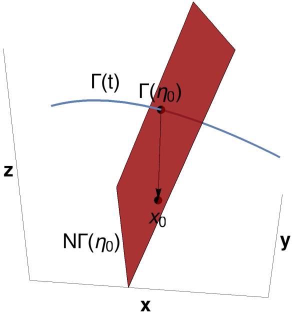
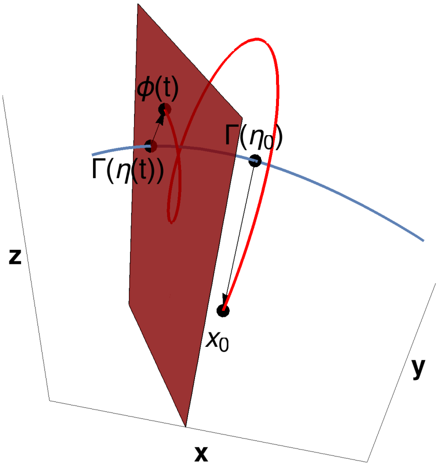


- In other words, x_0 lies in the normal plane to Γ at $t = \eta_0$.

Local Orthogonal Rectification

- Can we continue to track $\phi(t)$ in this manner?



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- *Lemma*
If $x_0 = \Psi(\eta_0, \xi_0)$ and $||\xi_0||$ is sufficiently small, then there exist $\epsilon > 0$ and smooth functions $\eta(t), \xi(t)$ such that
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- Taking time derivatives of the above, we can derive a system of ODEs governing $\eta(t), \xi(t)$.

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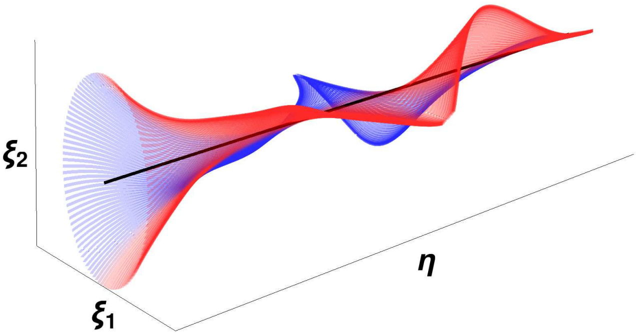
- Note also, that $\Psi(\eta + T, \xi) = \Psi(\eta, \xi)$, as Γ is T-periodic.

LOR Dynamics

- The same trajectories in the LOR frame



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- The $\Theta(\eta,\theta)$ term describes how trajectories rotate around Γ, and the $R(\eta,\theta)$ term measures radial contraction/expansion near Γ.
- The invariant set $r = 0$ corresponds to the limit cycle.

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- On the set $r = 0$, we have dynamics

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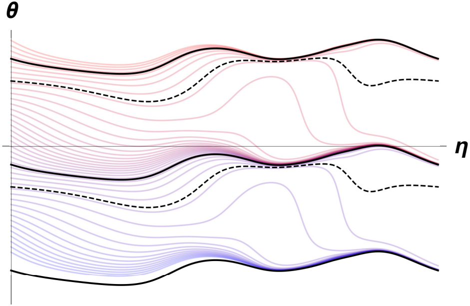
- Intuitively, for r sufficiently small, the above dynamics should be dominant, hence these are the *angular dynamics near the limit cycle*.
- By studying this flow on $S^1 \times S^1$ we can identify the organizing features of our original flow.

Periodic Angular Solutions

- We find there are four organizing angular trajectories



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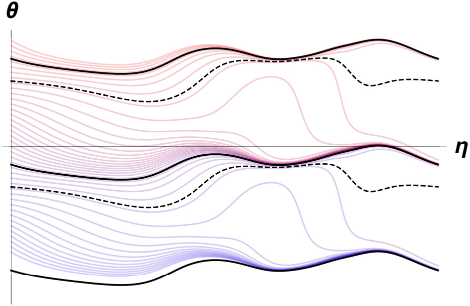

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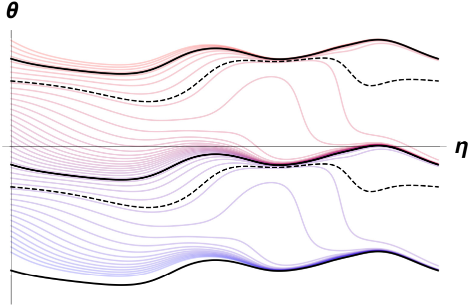

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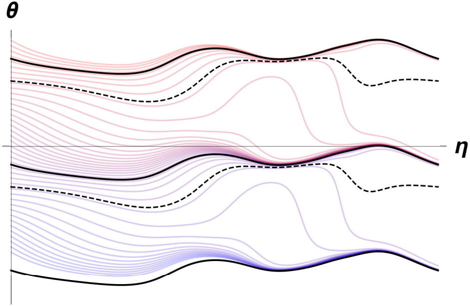
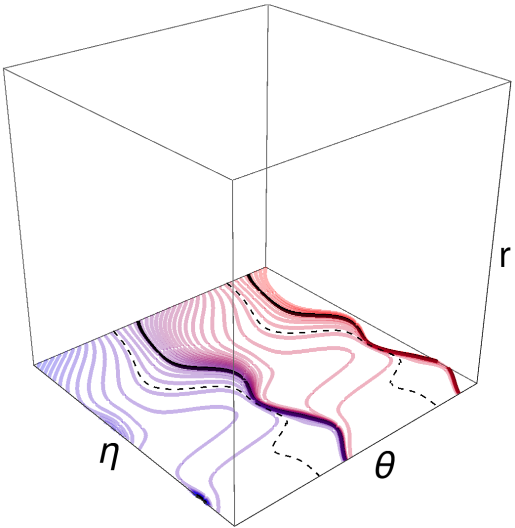


- The two (with one 2π shifted copy) solid black curves are stable, T-periodic angular trajectories.
- The two dashed black curves are unstable, T-periodic angular trajectories.
- Note that, near $\eta = 2/3T$ the stable and unstable periodic solutions lie near one another, hence the system is extremely sensitive to small angular perturbations.

Angular Manifolds

- Recall that these dynamics lie in the $r = 0$ invariant plane



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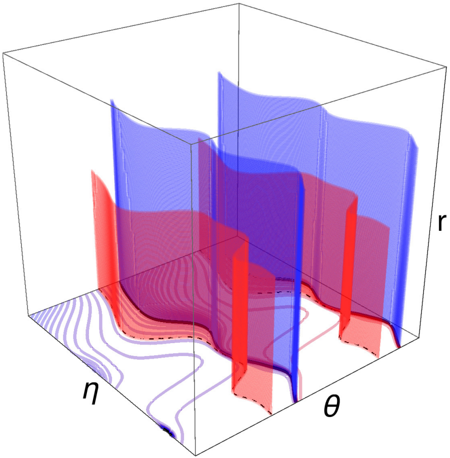
- We call these the angular invariant manifolds



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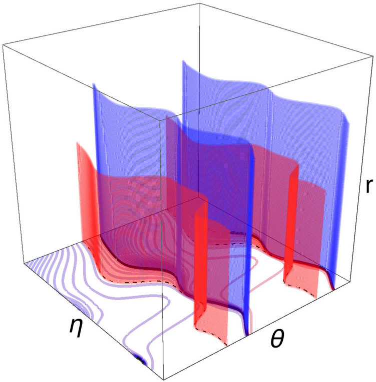

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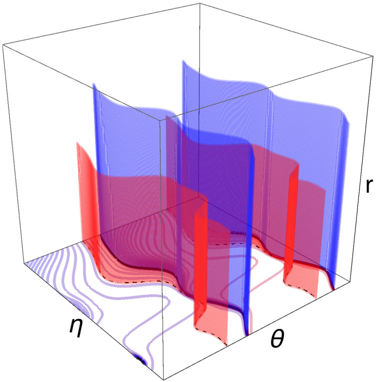
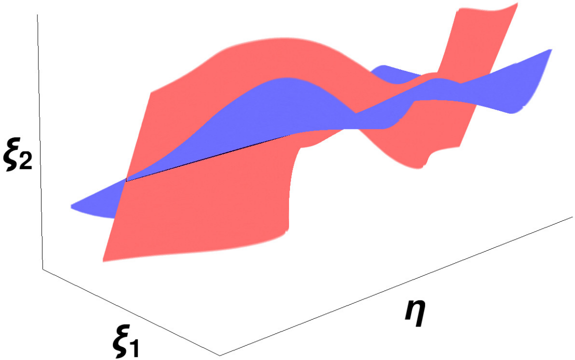


- The red manifolds, attendant to the unstable periodic solutions, are radially stable and angularly unstable

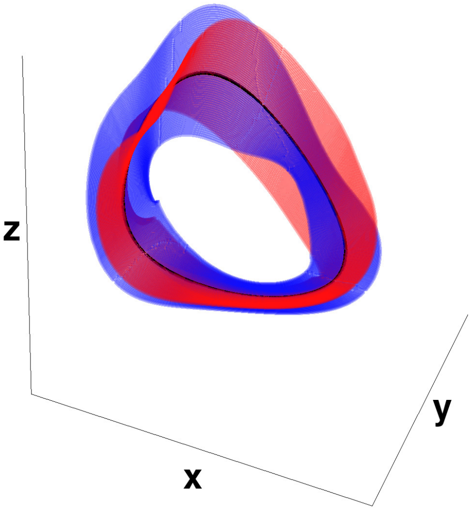
Angular Manifolds

- Transforming back to our LOR coordinates, the angular manifolds remain invariant



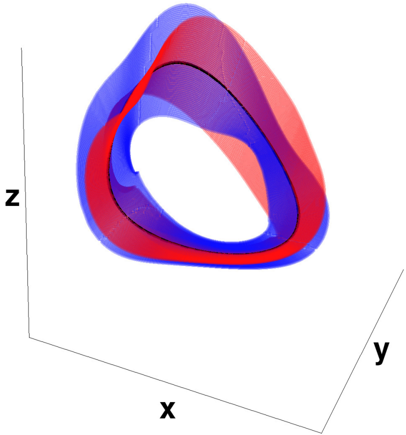
Angular Manifolds

- Finally, transforming back to Cartesian coordinates, we find that we have identified the desired manifold



Angular Manifolds

- As a bonus, we have identified a second 2-manifold which is unstable. This surface is a separatrix, which displays high sensitivity to initial conditions.



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- Indeed, the same analysis can be done for any periodic trajectory in $\mathbb{R}^n$.
- The key step in this analysis is Local Orthogonal Rectification, which allows us to flatten out complicated curvilinear geometries.

Acknowledgements

- Thanks to Jonathan Rubin for his guidance and support
- This work was partially funded by the Andrew Mellon Foundation.