A Stokes-Biot Stable H(div)-based mixed method for generalized poroelasticity

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[simula . research laboratory]

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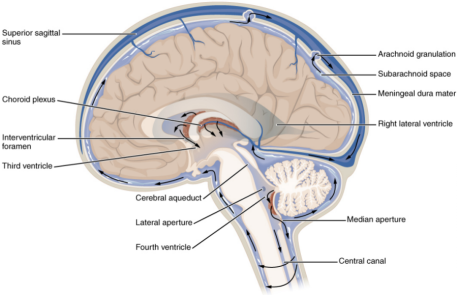
Generalized Poroelasticity

1. Biomedical Motivation
2. 3-Field Formulation for MPET with Energy Estimates
3. Stokes-Biot Stability, and Semi-Discrete MPET Estimates
4. Current Work: A modified Stokes-Biot stability definition

The Waterscape Project at Simula



waterscales

waterscape

Black Box Models Are Common In Clinical Theory

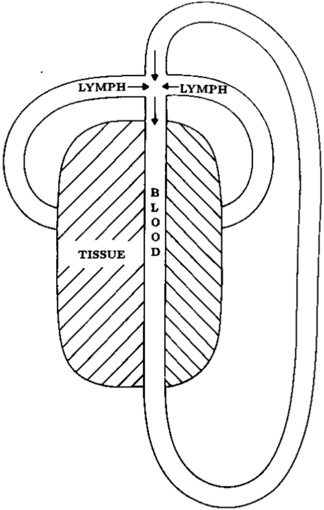


Figure: [Xie et. al. 1995]

$$\frac{\mathrm{d}}{\mathrm{d}t}V = J_V - J_L$$

$$J_V = K_f\left(P_V - p_{hyd} - \sigma(\Pi_V - \Pi_I)\right)$$
$$J_L = R_L^{-1}\left(p_{hyd} + P_p - P_L\right)$$

A Mechanical Response: Biot's Equations

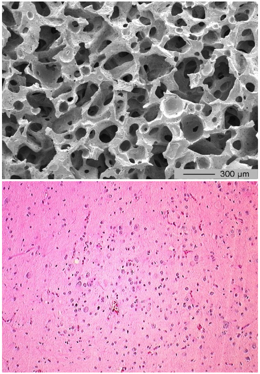


Figure: Porous Isotropic Media

Quantities

σ	effective elastic stress tensor
ϵ	linearized strain tensor
$\mathcal{C}$	linear, isotropic stiffness tensor
$\vec{q}$	fluid flux

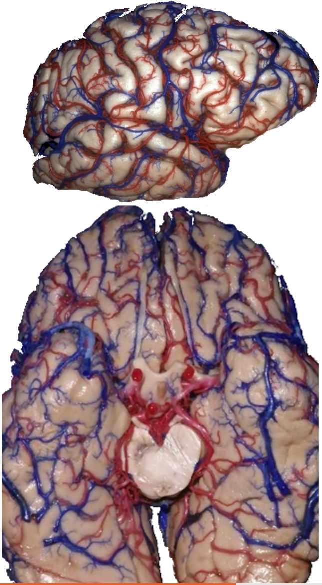
Relations

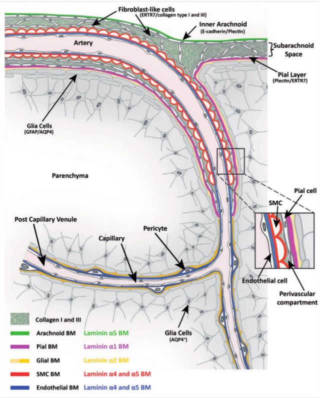
$$\sigma = \mathcal{C}\epsilon$$
$$\epsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$
$$\sigma(u) = 2\mu\epsilon + \lambda \text{tr}(\epsilon)\mathbb{I}$$
$$\vec{q} = -\kappa\nabla p$$

Biot's Equations: Single Pore Fluid

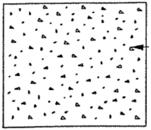
$$-\operatorname{div}\sigma(u) + \alpha\nabla p = 0,$$
$$c\dot{p} + \alpha \operatorname{div}\dot{u} - \operatorname{div}(\kappa\nabla p) = g.$$

The Brain Has Multiple Interacting Fluid Networks

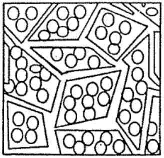




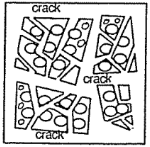
Generalized Quasi-Static Linear Poroelasticity (MPET)



One Network



Two Networks



Three Networks

MPET Equations*: 'A' networks

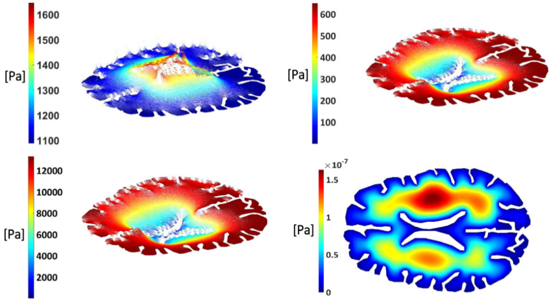
$$-\operatorname{div}\sigma(u) + \sum_{n=1}^{A} \alpha_n \nabla p_n = 0,$$

$$c_n \dot{p_n} + \alpha_n \operatorname{div}\dot{u} - \operatorname{div}(\kappa_n \nabla p_n) - \sum_{m=1}^{A} T_{m\to n} = g_n.$$

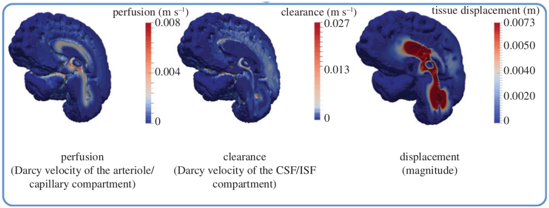
$$n = 1, 2, \ldots, A$$

* Barenblatt [1960] and Aifantiss [1980]

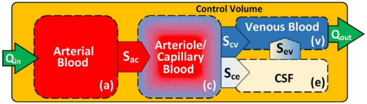
Application Of MPET To The Brain: 4 Networks



[Vardakis et. al. 2016]



[Vardakis et. al. 2017]



[Vardakis et. al. 2016]

Numerical Robustness: Limiting Parameter Values

Recent Numerical Work: Robust-In-Limit Methods

- λ: Youngs Modulus
  1. Brain Models: $\nu \approx 1/2$
  2. Implies: $\lambda \to \infty$
  3. Lee, Piersanti, Mardal, and Rognes [2018, in prep]
     Total Pressure Mixed Formulation, MPET
- κ: Hydraulic Conductivity
  1. $\approx 10^{-7} - \approx 10^{-17}$, for general tissue matrix variteties
  2. Rodrigo, et. al. [2017]
     3-Field Mixed Formulation For Biot
- C: Storativity
  1. $C \approx 0$: Nearly Incompressure solid phase, and liquid phase
  2. Associated with spurious pressure oscillations
  3. Lee [2017]
     3-Field Mixed Formulation For Biot

Search: Robust Numerical Method For MPET

- Goal: Robust Method in the $\kappa_n \to 0$ limit case
- Goal: Robust Method in the $c_n \to 0$ limit case - e.g. 'locking free'
- Assume
  $\alpha_n \neq 0$,
  $c_n > 0$ or $\kappa_n > 0$,
  $T_{m\to n} = \gamma_{mn}(p_n - p_m)$.
  $\gamma_{mm} = 0$
  $\gamma_{nm} = \gamma_{mn}$

MPET Equations*: 'A' networks

$$-\operatorname{div}\sigma(u) + \sum_{n=1}^{A} \alpha_n \nabla p_n = 0,$$

$$c_n \dot{p_n} + \alpha_n \operatorname{div}\dot{u} - \operatorname{div}(\kappa_n \nabla p_n)$$

$$-\sum_{m=1}^{A} \gamma_{mn}\,(p_n - p_m) = g_n.$$

$$n = 1, 2, \ldots, A$$

3-Field Formulation

3-Field Mixed MPET Formulation

$$
\begin{aligned}
-\operatorname{div}\sigma(u) + \sum_{n=1}^{A} \alpha_n \nabla p_n &= 0 \\
z_n + \kappa_n \nabla p_n &= 0
\end{aligned}
$$

$$
c_n \dot{p_n} + \alpha_n \operatorname{div}\dot{u} + \operatorname{div} z_n + \sum_{m=1}^{A} \gamma_{mn}(p_n - p_m) = g_n.
$$

$$
n = 1, 2, \ldots, A
$$

$$
\begin{array}{ccc}
p_n = 0, & \sigma \cdot \mathbf{n} = 0, & x \in \overline{\Gamma_t}, \\
u = 0, & z_n \cdot \mathbf{n} = 0, & x \in \overline{\Gamma_c}.
\end{array}
$$

$$
u \in H^1(0,T;\mathsf{U}) \quad z_n \in L^2(0,T;\mathsf{W}) \quad p_n \in H^1(0,T;\mathsf{Q})
$$

$$
g_n \in \mathsf{Q}
$$

Where $\mathsf{U} = H^1_{\Gamma_c}(\Omega)^d$, $\mathsf{W} = H_{\Gamma_c}(\operatorname{div};\Omega)$, $\mathsf{Q} = L^2(\Omega)$.

3-Field Mixed MPET Variational Formulation

Find $u \in H^1(0,T;\mathsf{U})$, $z_n \in L^2(0,T;\mathsf{W})$, and $p_n \in L^2(0,T;\mathsf{Q})$
$\forall (v, w_n, p_n) \in \mathsf{U} \times \mathsf{W} \times \mathsf{Q}$

$$\langle \mathcal{C}\epsilon(u), \epsilon(v) \rangle - \sum_{n=1}^{A} \alpha_n \langle p_n, \operatorname{div} v \rangle = 0,$$

$$\kappa_n^{-1} \langle z_n, w_n \rangle - \langle p_n, \operatorname{div} w_n \rangle = 0,$$

$$c_n \langle \dot{p_n}, q_n \rangle + \alpha_n \langle \operatorname{div} \dot{u}, q_n \rangle + \langle \operatorname{div} z_n, q_n \rangle + \sum_{m=1}^{A} \gamma_{mn} \langle p_n - p_m, q_n \rangle = \langle g_n, q_n \rangle.$$

* We also define a new norm for $H(\operatorname{div})$ for later use; let $\kappa > 0$:

$$H_{\Gamma_c}(\operatorname{div}; \kappa^{-1}) = \left\{ z \in H_{\Gamma_c}(\operatorname{div}) \,|\, \kappa^{-1} \langle z, z \rangle + \langle \operatorname{div} z, \operatorname{div} z \rangle < \infty \right\}$$

Developing Tools

Lemma 1 (Thompson & Rognes): Let B be a banach space with dual B', and $\langle \cdot, v \rangle_{B',B}$ denote the duality pairing. Suppose $X \in C^0(0,T;B)$, $G : [0,T] \to \mathbb{R}^+ \in L^1([0,T])$, $\mathcal{C} \geq 0$ is a constant, and that $F \in L^1(0,T;B')$. Suppose $\forall t \in [0,T]$

$$||X(t)||_B^2 \leq ||X(0)||_B^2 + \int_0^t |\langle F(s), X(s) \rangle_{B',B}| + G(s)\, ds \; + \mathcal{C}$$

Then we have the following estimate

$$\begin{aligned} ||X(t)||_B &\leq ||X(0)||_B + \max\{2||F||_{L^1(0,T;B')}, \left(2\int_0^T G(s)\, ds \quad + 2\mathcal{C}\right)^{1/2}\} \\ &\leq ||X(0)||_B + \max\{2||F||_{L^1(0,T;B')}, \sqrt{2}||G||_{L^1([0,T])}^{1/2} + \sqrt{2\mathcal{C}}\}. \end{aligned}$$

Developing Tools

Corollary 1 (Thompson & Rognes)* Let $X_1, X_2, \dots, X_A$ and $F_1, F_2, \dots, F_A$ and J be continuous, non-negative functions, and $C \geq 0$. Suppose that

$$\sum_{n=1}^{A} X_n^2(t) \leq \sum_{n=1}^{A} X_n^2(0) + \sum_{n=1}^{A} \int_0^T F_n(s) X_n(s)\, ds + \int_0^T J(s)\, ds + C \qquad (1)$$

Then the following inequality holds with proportionality constant no greater than $2A$:

$$\sum_{n=1}^{A} X_n(t) \lesssim \sum_{n=1}^{A} X_n(0) + \left(\sum_{n=1}^{A} \int_0^T F_n(s)\, ds + \left(\int_0^T J(s)\, ds + C \right)^{1/2} \right). \qquad (2)$$

* The case of $A = 1$ first appears in Lee [2016].

Energy Estimates

Lemma 2 (Thompson & Rognes): Suppose the exact solutions satisfy $u \in H^1(0,T;\mathsf{U})$, $z_n \in L^2(0,T;\mathsf{W})$, and $p_n \in H^1(0,T;\mathsf{Q})$, with sources $g_n \in L^1(0,T;\mathsf{Q})$. Suppose there exists $c > 0$ such that κ_n, c_n, $\alpha_n > c$ $\forall n = 1,2,\dots,A$ then

$$\int_0^T \left(\sum_{m,n=1}^{A} \gamma_{mn} ||p_n - p_m||_{L^2}^2 + \sum_{n=1}^{A} \kappa_n^{-1} ||z_n||_{L^2}^2 \, \mathrm{d}s \right)^{\frac{1}{2}}$$

$$+ ||u(t)||_1 + \sum_{n=1}^{A} \sqrt{c_n} ||p_n(t)||_{L^2} \lesssim ||u(0)||_1 + \sum_{n=1}^{A} ||p_n(0)||_{L^2}$$

$$+ \sum_{n=1}^{A} \frac{1}{\sqrt{c_n}} ||g_n||_{L^1(0,T;L^2)}$$

We Control:

$$\sup_{t\in[0,T]} ||u(t)||_1, \quad \sup_{t\in[0,T]} ||p_n(t)||_{L^2}, \quad ||p_n - p_m||_{L^2(0,T;L^2)}, \quad \kappa_n^{-1} ||z_n||_{L^2(0,T;L^2)}.$$

Energy Estimates

Proof Outline.

Select $v = \dot{u}$, $w_j = z_j$, and $q_j = p_j$. Use γ_{mn} symmetric:

$$\langle \mathcal{C}\epsilon(u), \epsilon(\dot{u})\rangle + \sum_{n=1}^{A} \kappa_n^{-1} ||z_n||_{L^2}^2 + \sum_{n=1}^{A} \frac{d}{dt} c_n ||p_n||_{L^2}^2 + \frac{1}{2} \sum_{n,m=1}^{A} \gamma_{mn} ||p_n - p_m||_{L^2}^2 = \sum_{n=1}^{A} \langle g_n, p_n\rangle$$

Integrating from 0 to T and using standard inequalities gives

$$||u(t)||_{H^1}^2 + \sum_{n=1}^{A} c_n ||p(t)||_{L^2}^2 + \int_0^T \sum_{n,m=1}^{A} \gamma_{mn} ||p_n - p_m||_{L^2}^2 \, ds + \int_0^T \sum_{n=1}^{A} \kappa_n^{-1} ||z_n||_{L^2}^2 \, ds \lesssim ||u(0)||_{H^1}^2 + \sum_{n=1}^{A} c_n ||p_n(0)||_{L^2}^2 + \sum_{n=1}^{A} \int_0^T (1/\sqrt{c_n}) ||g_n(s)|| \sqrt{c_n} ||p_n(s)|| \, ds$$

Apply the generalized Gronwall-type inequality to the above. □

Energy Estimates

Remark If in addition $z_n \in H^1(0,T;\mathrm{W})$ and $g_n \in L^1(0,T;\mathrm{Q}) \cap L^2(0,T;\mathrm{Q})$

$$
\begin{aligned}
&||u(t)||_1 + \sum_{n=1}^{A} \sqrt{c_n}||p_n(t)||_{L^2} + \sum_{n=1}^{A} \sqrt{\kappa_n^{-1}}||z_n(t)||_{L^2} + \sum_{m,n=1}^{A} \sqrt{\gamma_{mn}}||p_n(t) - p_m(t)||_{L^2} \\
&+ \left(\int_0^T ||\dot{u}||_1^2 + \sum_{n=1}^{A} c_n||\dot{p}_n||_{L^2}^2 + \sum_{n=1}^{A} ||z_n||^2_{H(\mathrm{div};\kappa_n^{-1})} + \sum_{m,n=1}^{A} \gamma_{mn}||p_n - p_m||_{L^2}^2 \,\mathrm{ds} \right)^{1/2} \\
&\lesssim ||u(0)||_1 + \sum_{n=1}^{A} \sqrt{c_n}||p_n(0)||_{L^2} + \sum_{n=1}^{A} ||z_n(0)||_{H(\mathrm{div};k_n^{-1})} \\
&+ \sum_{m,n=1}^{A} \sqrt{\gamma_{mn}}||p_n(0) - p_m(0)||_{L^2} + \sum_{n=1}^{A} \frac{1}{\sqrt{c_n}}||g_n||_{L^1(0,T;L^2)} \\
&+ \left(\sum_{n=1}^{A} \frac{1}{c_n}||g_n||^2_{L^2(0,T;L^2)} \right)^{1/2}
\end{aligned}
$$

We Control:

$$
||\dot{u}||_{L^2(0,T;H^1)} \quad ||\dot{p_n}||_{L^2(0,T;L^2)} \quad ||z_n||_{L^2(0,T;H(\mathrm{div};\kappa_n^{-1}))} \quad ||p_n - p_m||_{L^2(0,T;L^2)}
$$

$$
\sup_{t\in[0,T]} ||u||_1, \quad \sup_{t\in[0,T]} ||p_n||_{L^2}, \quad \sup_{t\in[0,T]} ||p_n - p_m||_{L^2}, \quad \sup_{t\in[0,T]} \kappa_n^{-1}||z_n||_{L^2}
$$

Semi-discrete formulation

Let $\mathsf{U}_h \subset \mathsf{U}$, $\mathsf{W}_h \subset \mathsf{W}$, and $\mathsf{Q}_h \subset \mathsf{Q}$ be finite dimensional.
suppose the existence and uniqueness of semi-discrete solutions $\mathbf{u}_h(t,x) \in C^2(0,T;\mathcal{X}_h)$, $\mathbf{z}_{h,n}(t,x) \in C^1(0,T;W_h)$, and $p_{h,n}(t,x) \in C^1(0,T;Q_h)$, for each $n \in \{1,2,\ldots,N\}$, such that for every $(\mathbf{v}_h, \mathbf{w}_h, q_h) \in \mathcal{X}_h \times W_h \times Q_h$:

$$\begin{aligned}
\langle \mathcal{C}\epsilon\left(u_h\right), \epsilon\left(v_h\right)\rangle + \sum_{n=1}^{A} \alpha_n \mathsf{b}(v_h, p_{h,n}) &= 0 \\
\kappa_n^{-1}\left(z_{h,n}, w_h\right) + \mathsf{b}(w_h, p_{h,n}) &= 0 \\
c_n\left(\dot{p}_{h,n}, q_h\right) - \alpha_n \mathsf{b}(\dot{u}_h, q_h) - \mathsf{b}(z_{h,n}, q_h) + \sum_{m=1}^{N} \gamma_{mn}\left(p_{h,n} - p_{h,m}, q_h\right) &= \left(g_n, q_h\right)
\end{aligned}$$

where $\mathsf{b}(w,q) = -\langle \nabla \cdot w, q\rangle$

Stokes-Biot Stability

Stokes-Biot Stability

Definition The triple $\mathsf{U}_h \times \mathsf{W}_h \times \mathsf{Q}_h \subset \mathsf{U} \times \mathsf{W} \times \mathsf{Q}$ is called *Stokes-Biot stable* if

1. The bilinear form $\langle \mathcal{C}\epsilon(u), \epsilon(v)\rangle$ is bounded on U and coercive on U_h,
2. $(\mathsf{W}_h, \mathsf{Q}_h)$ is a Poisson-Stable pair of discrete spaces,
3. $(\mathsf{U}_h, \mathsf{Q}_h)$ is a Stokes-Stable pair of discrete spaces.

Define the Poisson Galerkin projection for (z_n, p_n) by

$$\kappa_n^{-1}(\Pi_{W_h}\mathbf{z}^n, \mathbf{w}_h) + \mathsf{b}(\mathbf{w}_h, \Pi_{Q_h}p^n) = \kappa^{-1}(\mathbf{z}^n, \mathbf{w}_h) + \mathsf{b}(\mathbf{w}_h, p^n), \quad \forall \mathbf{w}_h \in W_h$$
$$\mathsf{b}(\Pi_{W_h}\mathbf{z}^n, q_h) = \mathsf{b}(\mathbf{z}^n, q_h), \quad \forall q_h \in Q_h$$

Define the augmented elastic projection for u by

$$\langle \mathcal{C}\epsilon(\Pi_{\mathsf{U}_h}u), \epsilon(v_h)\rangle = \langle \mathcal{C}\epsilon(u), \epsilon(v_h)\rangle - \sum_{n=1}^{n} \alpha_n \mathsf{b}(v_h, I_{\mathsf{Q}_h}^{p_n}), \quad \forall v_h \in v_h$$

As a first step: assume the material parameters are uniformly bounded below and derive interpolation error estimates. c.f. [Rodrigo et. al 2016]

Interpolation Error Estimate

Claim: Define $I^u_{U_h} = \Pi_{U_h} u - u$, and likewise for p_n, and z_n. Then

$$||I_h^{z_n}||_W + ||I_h^{p_n}||_Q \lesssim \inf_{w_h \in W_h} ||z_n - w_h||_W + \inf_{q_h \in Q_h} ||p_n - q_h||_Q \quad (3)$$

$$||I_h^u||_U \lesssim \inf_{v_h \in U_h} ||u - v_h||_U + \sum_{n=1}^{A} \inf_{w_h \in W_h} ||z_n - w_h||_W \quad (4)$$
$$+ \sum_{n=1}^{A} \inf_{q_h \in Q_h} ||p_n - q_h||_Q$$

Proof Outline.

The estimate for the Galerkin projector of the Poisson problem, e.g. (3), is well known; for κ_n bounded below, the right-hand side is a fixed constant. We now establish the estimate of (4) □

Interpolation Error Estimate

Proof Outline (cont.)

Define $\hat{u}_h$ by: $\forall v_h \in U_h$ $\langle \mathcal{C}\epsilon(\hat{u}_h), \epsilon(v_h)\rangle = \langle \mathcal{C}\epsilon(u), \epsilon(v_h)\rangle$. It is well known that $||u - \hat{u}_h||_1 \lesssim \inf_{v_h \in U_h} ||u - v_h||_{U_h}$. Coercivity of $\langle \mathcal{C}\epsilon(\cdot), \epsilon(\cdot)\rangle$ on U_h gives

$$||\hat{u}_h - \Pi_{U_h} u||_U^2 \lesssim \langle \mathcal{C}\epsilon(\hat{u}_h - \Pi_{U_h} u), \epsilon(\hat{u}_h - \Pi_{U_h} u)\rangle$$

Using the definition of the augmented elastic projector:

$$\langle \mathcal{C}\epsilon(\hat{u}_h - \Pi_{U_h} u), \epsilon(\hat{u}_h - \Pi_{U_h} u)\rangle = \langle \mathcal{C}\epsilon(\hat{u}_h - u), \epsilon(\hat{u}_h - \Pi_{U_h} u)\rangle + \sum_{n=1}^{A} \alpha_n \mathsf{b}(\hat{u}_h - \Pi_{U_h} u, I_{\mathsf{Q}_h}^{p_n})$$

Cauchy Schwarz and boundedness gives

$$||\hat{u}_h - \Pi_{U_h} u||_U \lesssim \sum_{n=1}^{A} ||I_h^{p_n}||_Q$$

The result follows from $||u - \Pi_{U_h} u|| \leq ||u - \hat{u}_h|| + ||\hat{u}_h - \Pi_{U_h} u||$ and the interpolation estimates established for $I_h^{p_n}$. □

Interpolation Error Estimate

Recall: $I^u_{\mathsf{U}_h} = \Pi_{\mathsf{U}_h} u - u$, and $E^u_{\mathsf{U}_h} = \Pi_{\mathsf{U}_h} u - u_h$ and likewise for p_n, and z_n

Proposition (Thompson & Rognes)

$$||E^u_{\mathsf{U}_h}(t)||_1 + \sum_{n=1}^{A} \sqrt{c_n} ||E^{p_n}_{\mathsf{Q}_h}(t)||_{L^2} + \sum_{n=1}^{A} \sqrt{\kappa_n^{-1}} ||E^{z_n}_{\mathsf{W}_h}||_{L^2(0,T;L^2)}$$

$$+ \sum_{m,n=1}^{A} \sqrt{\gamma_{m,n}} ||E^{p_n,p_m}_{\mathsf{Q}_h}||_{L^2(0,T;L^2)} \lesssim ||E^u_{\mathsf{U}_h}(0)||_1 + \sum_{n=1}^{A} \sqrt{c_n} ||E^{p_n}_{\mathsf{Q}_h}(0)||_{L^2}$$

$$+ \inf_{v_h \in \mathsf{U}_h} ||\dot{u} - v_h||_1 + \sum_{n=1}^{A} \inf_{w_h \in \mathsf{W}_h} (||\dot{z}_n - w_h||_W + ||z_n - w_h||_W)$$

$$+ \sum_{n=1}^{A} \inf_{q_h \in \mathsf{Q}_h} (||\dot{p}_n - q_h||_{L^2} + ||p_n - q_h||_{L^2})$$

Current Work: Improved Estimates

κ-dependence In The Galerkin Projector

- The previous estimates work for material coefficients bounded below; namely $\kappa_n \geq \overline{\kappa} > 0 \; \forall n = 1, 2, \ldots, A$
- The projection strategy discussed is based on the Galerkin Projection for the Poisson problem:

$$a(\mathbf{z}^n, \mathbf{w}) + \mathsf{b}(\mathbf{w}, p^n) = \mathcal{F}(\mathbf{w}), \quad \forall \mathbf{w}_h \in W_h$$
$$\mathsf{b}(\mathbf{z}^n, q) = \mathcal{G}(q), \quad \forall q \in Q$$

Where $a(\mathbf{z}, \mathbf{q}) = \kappa^{-1}\langle \mathbf{z}, \mathbf{w} \rangle$, $\mathsf{b}(\mathbf{w}, q) = -\langle \operatorname{div} \mathbf{w}, q \rangle$

- The Galerkin Projection estimates are:

$$||\mathbf{z}^n - \mathbf{z}_h^n||_W \leq c_1 \inf_{\mathbf{w}_h \in W_h} ||\mathbf{z}^n - \mathbf{w}_h||_W + c_2 \inf_{q_h \in Q_h} ||p^n - q_h||_Q$$
$$||p^n - p_h^n||_W \leq c_3 \inf_{\mathbf{w}_h \in W_h} ||\mathbf{z}^n - \mathbf{w}_h||_W + c_4 \inf_{q_h \in Q_h} ||p^n - q_h||_Q$$

$c_1 = 2\left(1 + \beta_h^{-1}\right) \quad c_2 = \kappa \quad c_3 = 2\kappa^{-1}\beta_h^{-1}\left(1 + \beta_h^{-1}\right) \quad \left(1 + 2\beta_h^{-1}\right)$

*β_h is a constant depending only on $c(\Omega) > 0$ via Poincare

Numerical Tests: Stokes-Biot Stability

$$\boldsymbol{u}(x,y,t) = \operatorname{curl}\varphi = \begin{pmatrix} \partial_y \varphi \\ -\partial_x \varphi \end{pmatrix}, \quad \varphi(x,y) = [xy(1-x)(1-y)]^2,$$
$$p(x,y,t) = 1.$$

$$P_1 \times RT_1 \times P_0$$

κ \ N	8	16	32	64	128	
10^{-4}	0.0187	0.0040	0.0009	0.0002	5.66×10^{-5}	$\|\Pi_1 \boldsymbol{u} - \boldsymbol{u}_h\|_A$
	0.0590	0.0090	0.0016	0.0003	8.33×10^{-5}	$\|\Pi_0 p - p_h\|_{L^2}$
10^{-6}	0.0547	0.0302	0.0050	0.0005	8.93×10^{-5}	$\|\Pi_1 \boldsymbol{u} - \boldsymbol{u}_h\|_A$
	0.3187	0.3098	0.0741	0.0097	0.0012	$\|\Pi_0 p - p_h\|_{L^2}$
10^{-8}	0.0578	0.0567	0.0476	0.0165	0.0018	$\|\Pi_1 \boldsymbol{u} - \boldsymbol{u}_h\|_A$
	0.3388	0.7067	1.1418	0.6450	0.1142	$\|\Pi_0 p - p_h\|_{L^2}$
10^{-10}	0.0578	0.0574	0.0570	0.0550	0.0502	$\|\Pi_1 \boldsymbol{u} - \boldsymbol{u}_h\|_A$
	0.3372	0.7176	1.4527	2.7790	3.4403	$\|\Pi_0 p - p_h\|_{L^2}$

* MMS test problem from [Rodrigo et. al 2016]

Numerical Tests: $\kappa = 1 \times 10^{-8}$

$P_2 \times RT_0 \times P_0$

N	4	8	16	32	64
$\|\|u - u_h\|\|_{L^2}$	$5.251e-04$	$5.734e-05$	$6.001e-06$	$6.879e-07$	$8.366e-08$
$\|\|u - u_h\|\|_1$	$1.002e-02$	$2.650e-03$	$6.629e-04$	$1.651e-04$	$4.120e-05$
$\|\|p - p_h\|\|_{L^2}$	$4.462e-03$	$5.834e-04$	$7.058e-05$	$8.634e-06$	$1.074e-06$

$||u - u_h||_{L^2} \approx O(h^3)$ $\quad ||u - u_h||_1 \approx O(h^2)$ $\quad ||p - p_h||_{L^2} \approx O(h^3)^*$

$P_4 \times RT_3 \times DG_3^{**}$

N	4	8	16	32
$\|\|u - u_h\|\|_{L^2}$	$4.853e-06$	$1.580e-07$	$4.950e-09$	$1.543e-10$
$\|\|u - u_h\|\|_1$	$3.117e-04$	$2.057e-05$	$1.301e-06$	$8.155e-08$
$\|\|p - p_h\|\|_{L^2}$	$1.180e-03$	$7.747e-05$	$4.865e-06$	$3.037e-07$

$||u - u_h||_{L^2} \approx O(h^5)$ $\quad ||u - u_h||_1 \approx O(h^4)$ $\quad ||p - p_h||_{L^4} \approx O(h^3)$

** $P_4 \times DG_3$ is the Scott-Vogelius (Stokes-Stable) element

An Alternative Projection: 1 Network (Biot)

∘ Norms yielding estimates free of κ-dependent constants (Weakly Robust) ∘ Utilize Stokes Stability

Consider the following model problem:
Find $(\mathbf{u}, \mathbf{z}, p) \in [H^1]^d \times H(\text{div}; \kappa^{-1}) \times L^2$ such that for every $(\mathbf{v}, \mathbf{w}, q)$

$$\begin{bmatrix} A_S & 0 & b^T \\ 0 & \kappa^{-1} A_P & b^T \\ b & b & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{z} \\ p \end{bmatrix} = \begin{bmatrix} \mathcal{W}(\mathbf{v}) \\ \mathcal{X}(\mathbf{w}) \\ \mathcal{Y}(q) \end{bmatrix}$$

$$\langle A_S \mathbf{u}, \mathbf{v} \rangle = \langle \mathcal{C}\epsilon(\mathbf{u}), \epsilon(\mathbf{v}) \rangle \quad \langle A_P \mathbf{z}, \mathbf{w} \rangle = \langle \mathbf{z}, \mathbf{w} \rangle \quad \langle b\mathbf{u}, \mathbf{v} \rangle = -\langle \text{div}\, u, \mathbf{v} \rangle$$

This problem can be re-cast into a saddle point problem as:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \sigma \\ p \end{bmatrix} \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \end{bmatrix}$$

where $\sigma = (\mathbf{u}, \mathbf{z}) \in [H^1_{\Gamma_c}]^d \times H(\text{div}; \kappa^{-1})$, $p \in L^2$

An Alternative Projection: Well Posedness

The Following results hold

- $\mathcal{A}$ is bounded
- $\mathcal{A}$ is coercive on $\text{Ker}(\mathcal{B}) = \left\{\sigma = (\mathbf{u}, \mathbf{z}) \in [H^1_{\Gamma_c}]^d \times H(\text{div}; \kappa^{-1}) \mid \text{div}\, \mathbf{u} + \text{div}\, \mathbf{z} = 0\right\}$ provided $\text{div}(W_h) \subset Q_h$
- $\mathcal{B}$ is bounded
- The inf-sup condition holds for $\mathcal{B}$ and β coincides with the Stokes stability constant

The associated Galerkin projection based on this problem provides:

- Exact cancellation of 'problematic' terms in the semi-discrete error equation for the MPET equation
- Error estimates for $||\mathbf{u} - \mathbf{u}_h||_{H_1}$, $||\mathbf{z}^n - \mathbf{z}_h^n||_{H(\text{div};\kappa^{-1})}$, and $||p^n - p_h^n||_{L^2}$ with κ-independent constants

An Alternative Projection: Well Posedness

Redefining Stokes-Biot stability

Definition The triple $\mathrm{U}_h \times \mathrm{W}_h \times \mathrm{Q}_h \subset \mathrm{U} \times \mathrm{W} \times \mathrm{Q}$ is called *Stokes-Biot stable* if

1. The bilinear form $\langle \mathcal{C}\epsilon(u), \epsilon(v) \rangle$ is bounded on U and coercive on U_h,
2. $(\mathrm{U}_h, \mathrm{Q}_h)$ is a Stokes-Stable pair of discrete spaces.
3. $\mathrm{div}(W_h) \subset Q_h$

$$||I_h^u||_{H^1} + \sum_{n=1}^{A} ||I_h^{zn}||_{H(\mathrm{div};\kappa^{-1})} + ||I_h^{pn}||_{L^2} \lesssim c_1 \inf_{v_h \in U_h} ||u - v_h||_{H^1} + c_2 \sum_{n=1}^{A} \inf_{w_h \in W_h} ||z_{h,n} - q_h||_{H(\mathrm{div};\kappa^{-1})} + \inf_{q_h \in Q_h} ||p_n - q_h||_{L^2} \quad (5)$$

- c_1 and c_2 free of all material parameters
- Only stability constant is Stokes
- Same error equation as previous approach, same discrete error estimates

Ongoing Work:

- Generalizing to multiple fluid networks (Complete)
- Fully discrete a-priori error estimates
- Redefining Stokes-Biot stability, and locking study (Gaspar, Rodrigo, Mardal, Rognes, Thompson)

Further Reading

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Thank You For Your Attention.

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