A regularity result for the incompressible Euler equation with a free interface

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Scottsdale, AZ, December 7, 2015

Mini-symposium “Deterministic and Stochastic Aspects of Fluid Dynamics”

2015 SIAM Conference on Analysis of Partial Differential Equations

We address the local existence of solutions of the free-surface Euler equations

$$u_t + u \cdot \nabla u + \nabla p = 0 \ \text{ in } \Omega(t) \times (0, T)$$
$$\operatorname{div} u = 0 \ \text{ in } \Omega(t) \times (0, T).$$

The free boundary $\Gamma_{\mathrm{f}} = \Gamma_{\mathrm{f}}(t)$ (when the surface is a graph h) evolves according to the velocity field

$$\partial_t h + v_{\mathrm{hor}} \cdot \nabla h = v_n$$

where n is the space dimension, while the boundary condition for the pressure reads

$$p(x, t) = \epsilon\sigma(x, t) \ \text{ on } \partial\Gamma_{\mathrm{f}} \times (0, T)$$

where σ represents the surface tension.

Two dimensional problem (existence & uniqueness): Nalimov 1974

local existence for analytic data: Shinbrot 1976

local existence irrotational flow finite depths (small data): Yosihara 1982

local existence linearized, stability condition $\nabla p \cdot n \leq 0$: Beale, Hou, and Lowengrub 1993

an example showing the stability condition is necessary: Ebin 1987

local existence for 2D flows for rotational flows - Taylor sign condition must hold: Wu 1997

local existence for 3D flows for rotational flows: Wu 1999

local existence with surface tension and zero surface tension limit: Ambrose and Masmoudi, 2005, 2009

a priori estimates for local existence for rotational flows: Christodoulou and Lindblad (2000)

Local existence in H^3 with the Taylor condition: Shatah and Zeng 2008, Coutand and Shkoller, 2010

2D local existence in $H^{2.5+\delta}$ with the Taylor condition: K. and Tuffaha 2012

3D local existence in $H^{2.5+\delta}$ for rotational flows, Alazard, Burq, and Zuily (2011)

Global existence results (rotational): Wu (2009, 2011), Germain, Masmoudi, and Shatah (2012), Ionesco and Pusateri (2013)

We may expect $H^{n/2+1+\delta}$ as the optimal space which would permit local existence in space dimensions $n = 2, 3$.

True for irrotational $(\mathrm{curl}\, u = 0)$ flows for two and three dimensional cases by a result due to Alazard, Burq, and Zuily (2011)
The proof was done by writing the velocity as a gradient of a potential (using irrotationality) and writing the system for the surface displacement and the restriction of the velocity on the free surface (as in Wu (1997,1999)).

Let $n \in \{2, 3\}$ denote the space dimension. We consider the Euler equation on the domain

$$\Omega = \mathbb{R}^{n-1} \times [0, 1]$$

with periodic boundary conditions along the $x_1, \dots, x_{n-1}$ directions with period 1. We assume that the moving boundary is the top

$$\Gamma_1 = \mathbb{R} \times \{x_n = 1\}$$

while the rigid bottom is

$$\Gamma_0 = \mathbb{R} \times \{x_n = 0\}.$$

Denote by $v(x, t) = u(\eta(x, t), t)$ the Lagrangian velocity of the fluid and by $q(x, t) = p(\eta(x, t), t)$ the Lagrangian pressure.

The Euler equation in the Lagrangian formulation reads

$$v_t^i + a_i^k \partial_k q = 0 \;\text{ in }\; \Omega \times (0,T), \qquad i = 1, \ldots, n$$
$$a_i^k \partial_k v^i = 0 \;\text{ in }\; \Omega \times (0,T)$$

The unknown coefficients a_j^i denote the ij entry of the $n \times n$ matrix

$$a = (\nabla \eta)^{-1}$$

where η stands for the particle map

$$\eta_t(x,t) = v(x,t) \qquad\qquad \eta(x,0) = x, \qquad x \in \Omega.$$

The Lagrangian map in turn determines the evolving domain $\Omega(t) = \eta(\Omega, t)$. The coefficients a satisfy the evolution $a_t = -a : \nabla v : a$ with $a(0) = I$.

On the top, we assume the no surface tension boundary condition

$$q = 0 \ \text{ on } \ \Gamma_1 \times (0, T)$$

while on the stationary bottom we use

$$v^i N^i = 0 \ \text{ on } \ \Gamma_0 \times (0, T)\,;$$

where the vector

$$N = (N^1, \ldots, N^n)$$

represents the outward unit normal. In our simplified situation, we have

$$N = (0, \ldots, 0, -1) \ \text{ on } \Gamma_0$$

and

$$N = (0, \ldots, 0, 1) \ \text{ on } \Gamma_1.$$

Also, denote $H = \{v \in L^2(\Omega)^n : \partial_i v^i = 0 \text{ in } \Omega, v^i N^i|_{\Gamma_0} = 0\}$.

The following is the main result.

Theorem (K.-Tuffaha-Vicol 2015) *Let $n = 3$, let Ω be as above, and let $\delta \in (0, 0.5)$. Assume that $v(\cdot, 0) = v_0 \in H^{2.5+\delta}(\Omega) \cap H$ satisfies*

$$\operatorname{curl} v_0 \in H^{2+\delta}(\Omega)$$

and that the associated initial pressure $q(\cdot, 0)$ satisfies the Rayleigh-Taylor condition

$$\frac{\partial q}{\partial N}(x, 0) \leq -\frac{1}{C_0} < 0$$

for all $x \in \Gamma_1$.

Then there exists a unique solution (v, q, η) to the system such that

$$\eta \in L^\infty([0,T]; H^{3+\delta}(\Omega)) \cap C([0,T]; H^{2.5+\delta}(\Omega))$$

$$v \in L^\infty([0,T]; H^{2.5+\delta}(\Omega)) \cap C([0,T]; H^{2+\delta}(\Omega))$$

$$v_t \in L^\infty([0,T]; H^{2+\delta}(\Omega))$$

$$q \in L^\infty([0,T]; H^{3+\delta}(\Omega))$$

$$q_t \in L^\infty([0,T]; H^{2.5+\delta}(\Omega))$$

for some time $T > 0$ depending on the initial data.

For irrotational flows: Alazard, Burq, and Zuily 2011.

2D case: K.-Tuffaha 2014 (all exponents shifted by -0.5).

In the talk, we'll go over the a priori estimates for $n = 3$.

The a priori estimates can be made rigorous using the horizontal mollification procedure due to Coutand and Shkoller.

The first lemma gives a priori estimates on the coefficient matrix a and the particle map η. All the statements follow from

$$\eta_t = v$$

and

$$a_t = -a : \nabla v : a.$$

(Recall: $a = (\nabla \eta)^{-1}$.)

Lemma Assume that $\|\nabla v\|_{L^\infty([0,T];H^{1.5+\delta}(\Omega))} \leq M$. If

$$T \leq \frac{1}{CM}$$

where C is a sufficiently large constant, depending on ϵ, the following statements hold:

(i) $\|\nabla \eta(\cdot,t)\|_{H^{1.5+\delta}(\Omega)} \leq C$ for $t \in [0,T]$,

(ii) $\|a(\cdot,t)\|_{H^{1.5+\delta}(\Omega)} \leq C$ (and thus also $\|a(\cdot,t)\|_{L^\infty(\Omega)} \leq C$) for $t \in [0,T]$,

(iii) for every $\epsilon \in (0,1]$ we have

$$\|a - I\|^2_{H^{1.5+\delta}(\Omega)} \leq \epsilon$$

and

$$\|a : a^T - I\|^2_{H^{1.5+\delta}(\Omega)} \leq \epsilon.$$

on $[0,T]$.

Pressure estimates

The following lemma provides elliptic estimates satisfied by the pressure.

Lemma Assume that $\|\nabla v\|_{L^\infty([0,T];H^{1.5+\delta}(\Omega))} \leq M$. Then the pressure q satisfies

$$\|q(t)\|_{H^{3+\delta}} \leq P + P\int_0^t \|q_t(s)\|_{H^{2+\delta}}\, ds, \qquad t \in [0,T]$$

where P is a polynomial in $\|v\|_{H^{2.5+\delta}}$, $\|\eta\|_{H^{3+\delta}}$, and $\|v_0\|_{H^{2.5+\delta}}$, and

$$\|q_t(t)\|_{H^{2.5+\delta}} \leq P + P\int_0^t \|q_t(s)\|_{H^{2+\delta}}\, ds, \qquad t \in [0,T].$$

Applying $a^j_i \partial_j$ to the Euler equation and summing over $i, j = 1, 2, 3$, we get

$$a^j_i \partial_j (a^k_i \partial_k q) = -a^j_i \partial_j v^i_t = \partial_t a^j_i \partial_j v^i$$

by the divergence condition. We may rewrite this as

$$\partial_{kk} q = \partial_t a^j_i \partial_j v^i + \partial_j \big((\delta_{jk} - a^j_i a^k_i) \partial_k q \big)$$

where we used the Piola identity $\partial_j a^j_i = 0$. This equation is supplemented with the boundary conditions $q = 0$ on Γ_1 and

$$\partial_i q N^i = (\delta_{ik} - a^k_i) \partial_k q N^i \ \text{ on } \ \Gamma_0 \times (0, T).$$

Applying the elliptic regularity to the equation, we get

$$\begin{aligned}\|q\|_{H^{3+\delta}} &\le C\|v\|^2_{H^{2.5+\delta}} + C(1+\|\eta\|^8_{H^{3+\delta}})\|q\|_{H^{2+\delta}} \\ &\le C\|v\|^2_{H^{2.5+\delta}} + C(1+\|\eta\|^8_{H^{3+\delta}})\int_0^t \|q_t(s)\|_{H^{2+\delta}}\,ds \\ &\quad + C(1+\|\eta\|^8_{H^{3+\delta}})\|q_t(0)\|_{H^{2+\delta}}.\end{aligned}$$

The estimate for q_t is obtained from the system

$$\begin{aligned}\partial_{kk} q_t &= \partial_{tt} a_i^j \partial_j v^i + \partial_t a_i^j \partial_j v_t^i - \partial_j(\partial_t a_i^j a_i^k \partial_k q) - \partial_j(a_i^j \partial_t a_i^k \partial_k q) \\ &\quad + \partial_j\big((\delta_{jk} - a_i^j a_i^k)\partial_k q_t\big)\end{aligned}$$

with the boundary conditions $q_t = 0$ on Γ_1 and
$\partial_i q_t N^i = -\partial_t a_i^k \partial_k q N^i + (\delta_{ik} - a_i^k)\partial_k q_t N^i$ on Γ_0 (skip).

Tangential estimates

In this section, we derive the tangential estimates on the solution (v, η, a, q). For short, we denote

$$S = \overline{\partial}^{2.5+\delta}$$

where $\overline{\partial} = (I - \Delta_2)^{1/2}$ with $\Delta_2 = \partial_{11} + \partial_{22}$.

Lemma For $t \in [0, T]$, we have

$$\begin{aligned} &\|Sv(t)\|^2_{L^2} + \|a^3_l(t)S\eta^l(t)\|^2_{L^2(\Gamma_1)} \\ &\quad \le \int_0^t P(\|v\|_{H^{2.5+\delta}}, \|v_t\|_{H^{2+\delta}}, \|q\|_{H^{3+\delta}}, \|q_t\|_{H^{2.5+\delta}}, \|\eta\|_{H^{3+\delta}})\, ds \\ &\qquad\qquad + Q(\|v_0\|_{H^{2.5+\delta}}) \end{aligned}$$

where P and Q are polynomials in indicated arguments.

The proof is obtained by applying the differential operator S to the Euler equation leading to

$$Sv_t^i + S(a_i^k \partial_k q) = 0.$$

We then multiply by Sv^i, integrate, and sum over $i = 1, 2, 3$ in order to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Sv\|_{L^2}^2 &= -\int_\Omega S(a_i^k \partial_k q) Sv^i \, dx \\ &= -\int_\Omega Sa_i^k \partial_k q Sv^i \, dx - \int_\Omega a_i^k \partial_k Sq Sv^i \, dx \\ &\quad - \int_\Omega \big(S(a_i^k \partial_k q) - Sa_i^k \partial_k q - a_i^k \partial_k Sq \big) Sv^i \, dx = I_1 + I_2 + I_3.\end{aligned}$$

Then we estimate the three terms I_1, I_2, and I_3. The Taylor condition appears when treating the first term. The last term is handled by using Kato-Ponce type (double) commutators.

First, we write

$$S = \sum_{m=1}^{2} S_m \partial_m + S_0$$

where $S_m = -(-\Delta_2)^{0.25+\delta/2}\partial_m$ for $m = 1, 2$ and $S_0 = (I - \Delta_2)^{0.25+\delta/2}$. Using

$$\partial_m a_i^k = -a_l^k \partial_s \partial_m \eta^l a_i^s, \qquad m = 1, 2$$

which follows by differentiating $a : \nabla\eta = I$, we may rewrite the term $I_1 = -\int_\Omega S a_i^k \partial_k q S v^i \, dx$ as

$$\begin{aligned} I_1 &= -\sum_{m=1}^{2} \int_\Omega S_m \partial_m a_i^k \partial_k q S v^i \, dx - \int_\Omega S_0 a_i^k \partial_k q S v^i \, dx \\ &= -\sum_{m=1}^{2} \int_\Omega S_m (a_l^k \partial_s \partial_m \eta^l a_i^s) \partial_k q S v^i \, dx - \int_\Omega S_0 a_i^k \partial_k q S v^i \, dx \end{aligned}$$

In the leading term $-\sum_{m=1}^{2} \int_\Omega a_l^k S_m \partial_s \partial_m \eta^l a_i^s \partial_k q S v^i \, dx$, we integrate by parts in x_s. The boundary term uses Taylor (skip the rest).

Divergence-curl estimates for η and v

Finally we need divergence and curl estimates for both, η and v.

For any given matrix function $a(x)$, introduce the variable curl operator B_a acting on the vector function $f = (f^1, f^2, f^3)$ according to

$$B_a f = \begin{bmatrix} a_2^k \partial_k f^3 - a_3^k \partial_k f^2 \\ a_3^k \partial_k f^1 - a_1^k \partial_k f^3 \\ a_1^k \partial_k f^2 - a_2^k \partial_k f^1 \end{bmatrix}.$$

Similarly, we introduce the variable divergence operator

$$A_a f = a_i^k \partial_k v^i.$$

(If $a = I$, then B_I and A_I agree with the usual curl and divergence operators.)

We start by differentiating in space the Cauchy invariance

$$\epsilon_{ijk}\partial_j v^m \partial_k \eta^m = \omega_0^i, \qquad t \geq 0, \qquad i = 1,2,3$$

obtaining

$$\epsilon_{ijk}\partial_j v^m \nabla \partial_k \eta^m + \epsilon_{ijk}\partial_k \eta^m \partial_j \nabla v^m = \nabla \omega_0^i, \qquad t \geq 0, \qquad i = 1,2,3.$$

Then we use the fundamental theorem of calculus:

$$\epsilon_{ijk}\partial_k \eta^m \partial_j \nabla \eta^m = \int_0^t \left(\epsilon_{ijk}\partial_k \eta^m \partial_j \nabla \eta_t^m + \epsilon_{ijk}\partial_k \eta_t^m \partial_j \nabla \eta^m \right) ds, \qquad i = 1,2,3.$$

The first term inside the integral sign may be rewritten as

$$\begin{aligned} \epsilon_{ijk}\partial_k \eta^m \partial_j \nabla \eta_t^m &= \epsilon_{ijk}\partial_k \eta^m \partial_j \nabla v^m \\ &= -\epsilon_{ijk}\partial_j v^m \nabla \partial_k \eta^m + \nabla \omega_0^i, \qquad i = 1,2,3 \end{aligned}$$

and we get

$$\epsilon_{ijk}\partial_k \eta^m \partial_j \nabla \eta^m = \int_0^t \left(-\epsilon_{ijk}\partial_j v^m \partial_k \nabla \eta^m + \epsilon_{ijk}\partial_k v^m \partial_j \nabla \eta^m \right) ds + t\nabla \omega_0^i, \qquad i =$$

i.e.,

$$\epsilon_{ijk}\partial_k\eta^m\partial_j\nabla\eta^m = 2\epsilon_{ijk}\int_0^t \partial_k v^m\partial_j\nabla\eta^m\, ds + t\nabla\omega_0^i, \qquad i = 1, 2, 3.$$

Applying the $H^{1+\delta}$ norms of both sides we have

$$\begin{aligned}\|\nabla \operatorname{curl}\eta\|_{H^{1+\delta}} \leq{}& C\|\eta\|_{H^{3+\delta}}\|I - \nabla\eta\|_{H^{1.5+\delta}} \\ &+ C\int_0^t \|v\|_{H^{2.5+\delta}}\|\eta\|_{H^{3+\delta}}\, ds + C\|\omega_0\|_{H^{2+\delta}}.\end{aligned}$$

Then write $I - \nabla\eta$ as a time integral of its derivative, which is $-\nabla v$.

For divergence, we also use the fundamental theorem of calculus to obtain

$$\begin{aligned}\|\operatorname{div}\eta\|_{H^{2+\delta}} \leq{}& C\int_0^t \|v\|_{H^{2.5+\delta}}\|\eta\|_{H^{3+\delta}}\, ds + C\int_0^t \|\eta\|^2_{H^{3+\delta}}\|v\|_{H^{2.5+\delta}}\, ds \\ &+ C\|\eta\|_{H^{3+\delta}}\int_0^t \|v\|_{H^{2.5+\delta}}\, ds + C\int_0^t \|v\|_{H^{2+\delta}}\, ds + C.\end{aligned}$$

Using the inequality

$$\|f\|_{H^s(\Omega)} \leq C\|f\|_{L^2(\Omega)} + C\|\operatorname{curl} f\|_{H^{s-1}(\Omega)} + C\|\operatorname{div} f\|_{H^{s-1}(\Omega)} + C\|(\nabla_2 f) \cdot N\|_{H^{s-1.5}(\partial\Omega)}$$

we get

$$\|\eta\|_{H^{3+\delta}} \leq C\|\eta\|_{L^2} + C\|\operatorname{curl} \eta\|_{H^{2+\delta}} + C\|\operatorname{div} \eta\|_{H^{2+\delta}} + C\|S\eta^3\|_{L^2(\Gamma_1)}.$$

The bound on the last terms comes from the tangential estimate.

In order to obtain an estimate for $\operatorname{curl} v$, we use the Cauchy invariance again and write

$$(\operatorname{curl} v)^i = \epsilon_{ijk}\partial_j v^m = \epsilon_{ijk}\partial_j v^m(\delta_{km} - \partial_k \eta^m) + \omega_0^i, \qquad i = 1, 2, 3$$

from where, using the algebra property of $H^{1.5+\delta}$,

$$\begin{aligned} \|\operatorname{curl} v\|_{H^{1.5+\delta}} &\leq C\|\nabla v\|_{H^{1.5+\delta}} \sum_{k,m=1}^{3} \|\delta_{km} - \partial_k \eta^m\|_{H^{1.5+\delta}} + \|\omega_0\|_{H^{1.5+\delta}} \\ &\leq C\|v\|_{H^{2.5+\delta}} \int_0^t \|v\|_{H^{2.5+\delta}}\, ds + \|\omega_0\|_{H^{1.5+\delta}}. \end{aligned}$$

For $\operatorname{div} v$ we also use the fundamental theorem of calculus, while the trace values of $v \cdot n$ $(= v_3)$, needed in div-curl, are bounded using the tangential estimates.