Didier Auroux

University of Nice Sophia Antipolis

France

auroux@unice.fr

Data assimilation for geophysical fluids : Back and Forth Nudging (BFN) and observers

2017 SIAM Conference on Dynamical Systems

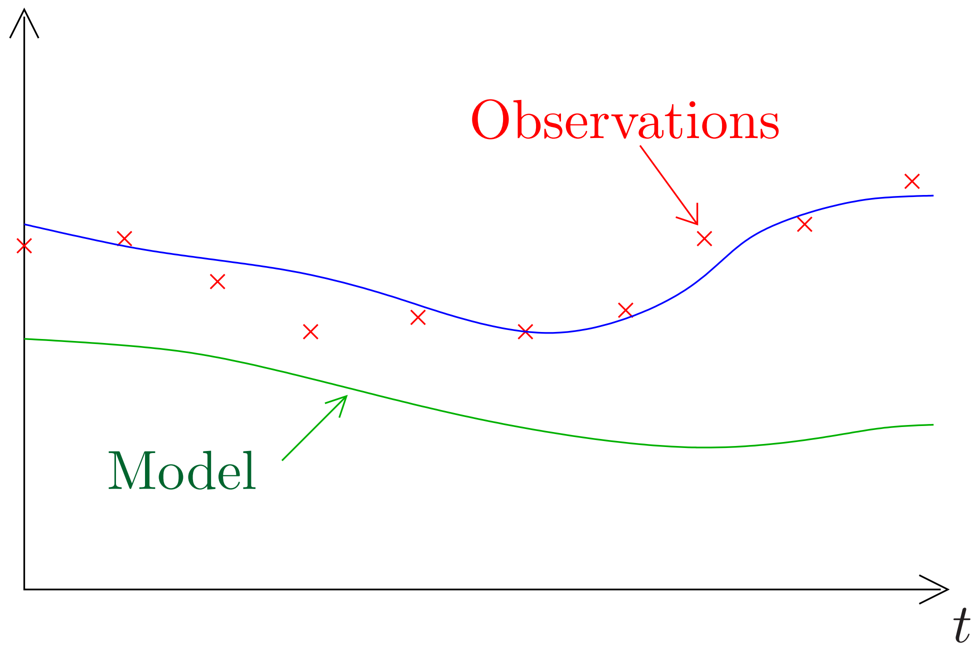
MS39 Recent Developments in Data Assimilation - Snowbird, USA

May 22, 2017

Talk overview

1. Nudging and observers
2. Back and Forth Nudging algorithm
3. Diffusive BFN algorithm

Data assimilation



combination

model + observations

$\Downarrow$

identification of the state (and/or parameters) of a geophysical system

- 4D-VAR : optimal control method, based on the minimization of the discrepancy between the model solution and the observations.
- Sequential methods : Kalman filtering, ensemble Kalman filters, ...
- Hybrid methods : En-4DVar, 4D-EnVar, ...
- Observer approach : the Back and Forth Nudging.

⇒ 1. Nudging and observers

2. Back and Forth Nudging algorithm

3. Diffusive BFN algorithm

Forward nudging

Let us consider a model governed by a system of ODE :

$$\frac{dX}{dt} = F(X), \quad 0 < t < T,$$

with an initial condition $X(0) = x_0$.

$\mathcal{Y}(t)$: observations of the system

H : observation operator.

$$\begin{cases} \dfrac{dX}{dt} = F(X) + K(\mathcal{Y} - H(X)), \quad 0 < t < T, \\ X(0) = X_0, \end{cases}$$

where K is the nudging (or gain) matrix.

In the linear case (where F is a matrix), the forward nudging is called Luenberger or asymptotic observer.

Forward nudging

- Meteorology : Hoke-Anthes (1976)
- Oceanography (QG model) : De Mey et al. (1987), Verron-Holland (1989)
- Atmosphere (meso-scale) : Stauffer-Seaman (1990)
- Optimal determination of the nudging coefficients :

  Zou-Navon-Le Dimet (1992), Stauffer-Bao (1993),

  Vidard-Le Dimet-Piacentini (2003)

  Lakshmivarahan-Lewis (2011)

Forward nudging : linear case

Luenberger observer, or asymptotic observer

[Luenberger, 1966]

$$\begin{cases} \dfrac{dX_{true}}{dt} = FX_{true}, \quad \mathcal{Y} = HX_{true}, \\ \dfrac{dX}{dt} = FX + K(\mathcal{Y} - HX). \end{cases}$$

$$\frac{d}{dt}(X - X_{true}) = (F - KH)(X - X_{true})$$

If $F - KH$ is a Hurwitz matrix, i.e. its spectrum is strictly included in the half-plane $\{\lambda \in \mathbb{C}; Re(\lambda) < 0\}$, then $X \to X_{true}$ when $t \to +\infty$.

More complex observers

If the model has more than one variable (or if all components are not observed), the standard nudging only corrects the observed variables with themselves.
$\Rightarrow$ extension to more complex observers, in which non observed variables are controlled by observed ones.

Example on a 2D shallow water model :

$$\begin{cases} \dfrac{\partial h}{\partial t} = -\nabla \cdot (hv), \\ \dfrac{\partial v}{\partial t} = -(v \cdot \nabla)v - g\nabla h \end{cases}$$

on a square domain with rigid boundaries and no-slip lateral boundary conditions. These equations are derived from Navier-Stokes equations, assuming the horizontal scale is much greater than the vertical one $\Rightarrow$ conservation of mass and of momentum.

Can we identify/correct both variables (height and velocity) if only the water height h is observed ?

Observer design

Any non-linear observer for this model writes :

$$\begin{cases} \dfrac{\partial h}{\partial t} = -\nabla \cdot (hv) + F_h(h_{obs}, v, h), \\ \dfrac{\partial v}{\partial t} = -(v \cdot \nabla)v - g\nabla h + F_v(h_{obs}, v, h), \end{cases}$$

where $F = 0$ when the estimated height h is equal to the observed height h_{obs}.

Formal requirements : symmetry preservation (invariance to translations and rotations of the model, and then of the observer), smoothing by convolution (noisy data), local stability (strong asymptotic convergence of the linearized error system)

Most simple observer that should work : (smallest order of derivative)

$$F_h = \varphi_h * (h - h_{obs}), \qquad F_v = \varphi_v * \nabla(h - h_{obs})$$

with simple invariant kernels :

$$\varphi(x, y) = \beta \exp(-\alpha(x^2 + y^2)).$$

Convergence study

Convergence on the linearized system : let δh and δv be the perturbations around the reference state, and let $\tilde{h} = \delta h - \delta h_{true}$ and $\tilde{v} = \delta v - \delta v_{true}$ be the estimation errors, solutions of

$$\frac{\partial \tilde{h}}{\partial t} = -\bar{h}\ \nabla \cdot \tilde{v} - \varphi_h * \tilde{h},$$

$$\frac{\partial \tilde{v}}{\partial t} = -g\nabla \tilde{h} - \varphi_v * \nabla \tilde{h}.$$

Eliminating $\tilde{v}$ yields a modified damped wave equation with external viscous damping :

$$\frac{\partial^2 \tilde{h}}{\partial t^2} = g\bar{h}\Delta\tilde{h} + \bar{h}\,\varphi_v * \Delta\tilde{h} - \varphi_h * \frac{\partial \tilde{h}}{\partial t}.$$

Convergence study

Theorem : If φ_v and φ_h are defined by $\varphi(x,y) = \beta \exp(-\alpha(x^2+y^2))$ with $\beta_v, \beta_h, \alpha_v, \alpha_h > 0$, then the first order approximation of the error system around the equilibrium $(h,v) = (\bar{h}, 0)$ is strongly asymptotically convergent. Indeed if we consider the following Hilbert space and norm : $\mathcal{H} = H^1(\Omega) \times L^2(\Omega)$,

$$\|(u,w)\|_{\mathcal{H}} = \left(\int_\Omega \|\nabla u\|^2 + |w|^2 \right)^{1/2},$$

then

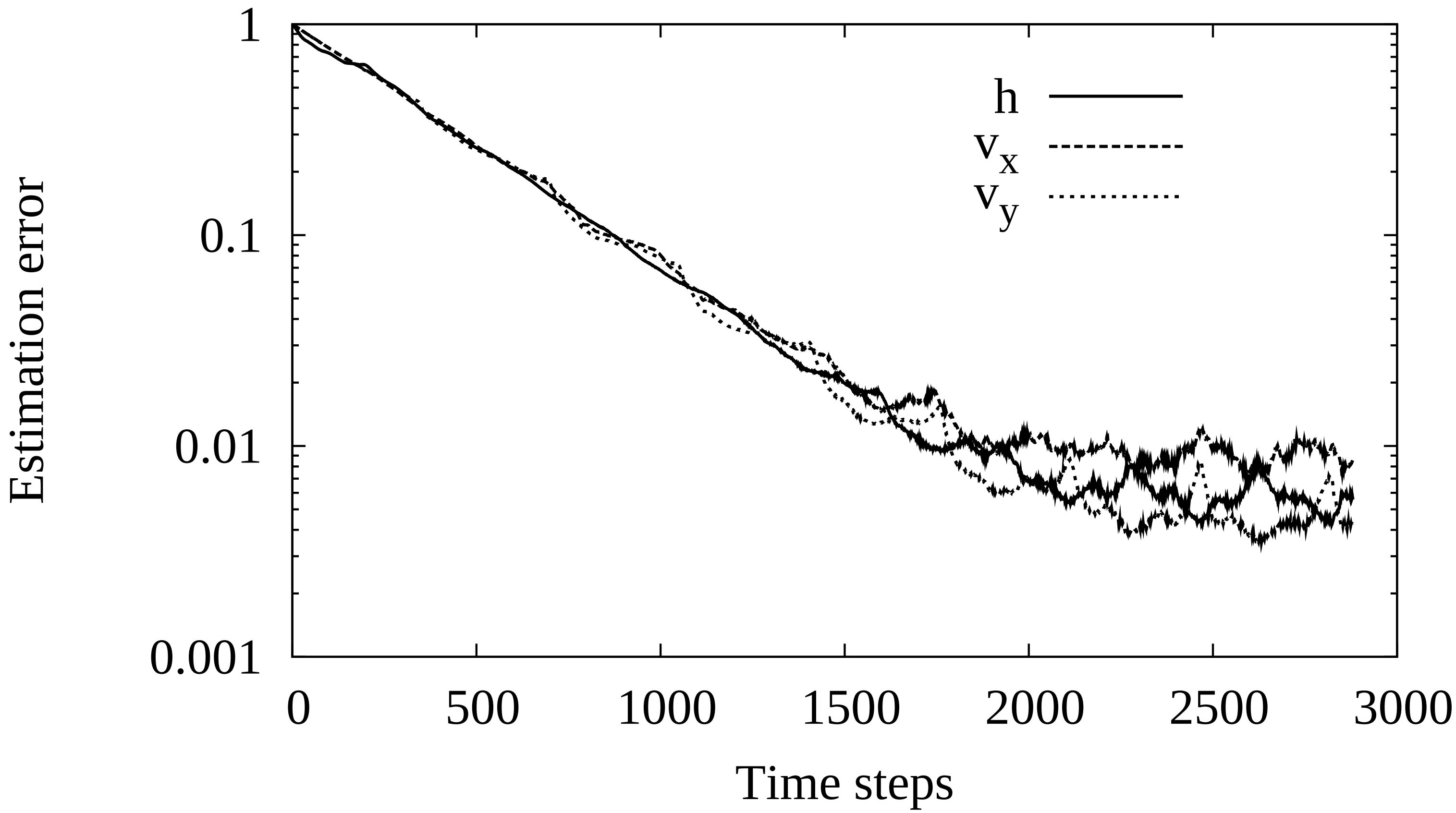
$$\lim_{t\to\infty} \left\| \left(\tilde{h}(t), \frac{\partial \tilde{h}}{\partial t}(t) \right) \right\|_{\mathcal{H}} = 0\,.$$

This theorem proves the strong and asymptotic convergence of the error $\tilde{h}$ towards 0, and then it also gives the same convergence for $\tilde{v}$. We deduce that the observer tends to the true state when time goes to infinity.

Proof : based on Fourier decomposition of the solution.

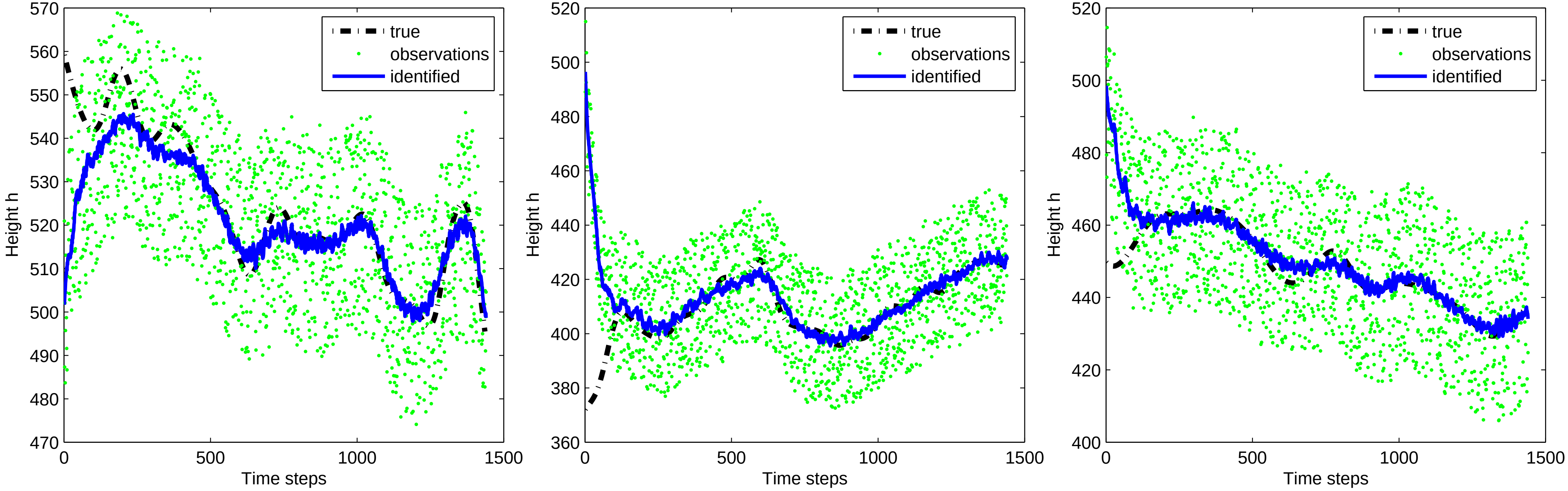
[Auroux et al, IEEE TAC 2011]

Numerical tests



Evolution of the estimation error in relative norm versus the number of time steps, in the case of noisy observations (20% noise), with $\alpha_h = \alpha_v = 1\ m^{-2}$ and $\beta_h = 2.10^{-7}\ s^{-1}$, and with a 100% error on the initial conditions, for the height h, longitudinal velocity v_x and transversal velocity v_y.

Numerical tests : non-linear model



Evolution of the true height, the observed (noisy) height, and the identified (observer) height versus time, for three different points of the domain, located along the energetic current in the middle of the domain.

Backward nudging

Another issue of standard nudging are : we get no information about the initial condition ; what can we do on a small time window ?

$\Rightarrow$ Can we recover the initial state from the final solution ?

Backward model :

$$\begin{cases} \dfrac{d\tilde{X}}{dt} = F(\tilde{X}), \quad T > t > 0, \\ \tilde{X}(T) = \tilde{X}_T. \end{cases}$$

If we apply nudging to this backward model :

$$\begin{cases} \dfrac{d\tilde{X}}{dt} = F(\tilde{X}) - K(\mathcal{Y} - H\tilde{X}), \quad T > t > 0, \\ \tilde{X}(T) = \tilde{X}_T. \end{cases}$$

BFN : Back and Forth Nudging algorithm

Iterative algorithm (forward and backward resolutions) :

$$\tilde{X}_0(0) = X_b \text{ (first guess)}$$

$$\begin{cases} \dfrac{dX_k}{dt} = F(X_k) + K(\mathcal{Y} - H(X_k)) \\ X_k(0) = \tilde{X}_{k-1}(0) \end{cases}$$

$$\begin{cases} \dfrac{d\tilde{X}_k}{dt} = F(\tilde{X}_k) - K'(\mathcal{Y} - H(\tilde{X}_k)) \\ \tilde{X}_k(T) = X_k(T) \end{cases}$$

[Auroux and Blum, C. R. Acad. Sci. Math. 2005]

If X_k and $\tilde{X}_k$ converge towards the same limit X, and if $K = K'$, then X satisfies the state equation and fits to the observations.

Choice of the direct nudging matrix K

Implicit discretization of the direct model equation with nudging :

$$\frac{X^{n+1} - X^n}{\Delta t} = FX^{n+1} + K(\mathcal{Y} - HX^{n+1}).$$

Variational interpretation : direct nudging is a compromise between the minimization of the energy of the system and the quadratic distance to the observations :

$$\min_X \left[\frac{1}{2}\langle X - X^n, X - X^n\rangle - \frac{\Delta t}{2}\langle FX, X\rangle + \frac{\Delta t}{2}\langle R^{-1}(\mathcal{Y} - HX), \mathcal{Y} - HX\rangle \right],$$

by chosing

$$K = kH^T R^{-1}$$

where R is the covariance matrix of the errors of observation, and k is a scalar.

[Auroux and Blum, Nonlin. Proc. Geophys. 2008]

Choice of the backward nudging matrix K'

The feedback term has a double role :

- stabilization of the backward resolution of the model (irreversible system)
- feedback to the observations

If the system is observable, i.e. $rank[H, HF, \dots, HF^{N-1}] = N$, then there exists a matrix K' such that $-F - K'H$ is a Hurwitz matrix (pole assignment method).

Simpler solution : one can define $K' = k'H^T R^{-1}$, where k' is e.g. the smallest value making the backward numerical integration stable.

Example of convergence results

Viscous linear transport equation :

$$\begin{cases} \partial_t u - \nu\partial_{xx}u + a(x)\partial_x u = -K(u - u_{obs}), & u(x, t = 0) = u_0(x) \\ \partial_t \tilde{u} - \nu\partial_{xx}\tilde{u} + a(x)\partial_x \tilde{u} = K'(\tilde{u} - u_{obs}), & \tilde{u}(x, t = T) = u_T(x) \end{cases}$$

We set $w(t) = u(t) - u_{obs}(t)$ and $\tilde{w}(t) = \tilde{u}(t) - u_{obs}(t)$ the errors.

• If K and K' are constant, then $\forall t \in [0, T]$: $\widetilde{w}(t) = e^{(-K-K')(T-t)}w(t)$

(still true if the observation period does not cover $[0, T]$)

• If the domain is not fully observed, then the problem is ill-posed.

Error after k iterations : $w_k(0) = e^{-[(K+K')kT]}w_0(0)$

$\rightsquigarrow$ exponential decrease of the error, thanks to :

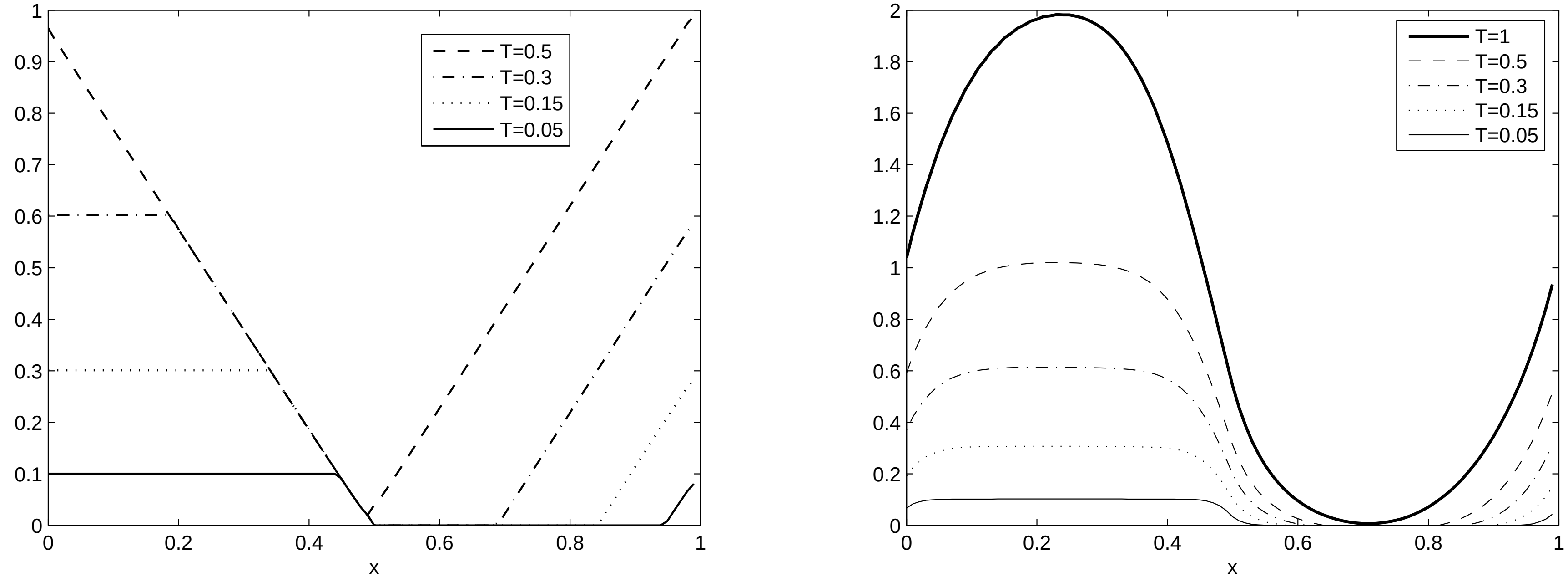
- $K + K'$: infinite feedback to the observations (not physical)
- T : asymptotic observer (Luenberger)
- k : infinite number of iterations (BFN)

[Auroux and Nodet, COCV 2012]

Observability condition

Let $\chi(x)$ be the time during which the characteristic curve with foot x lies in the support of K. Then the system is observable if and only if $\min_x \chi(x) > 0$.

Partial observations in space : half of the domain is observed.



Decrease rate of the error after one iteration of BFN as a function of the space variable x, for various final times T.

Linear case (left) : theoretical observability condition $= T > 0.5$

Nonlinear case (right) : numerical observability condition $= T > 1$

Shallow water model

$$
\begin{aligned}
\partial_t u - (f+\zeta)v + \partial_x B &= \frac{\tau_x}{\rho_0 h} - ru + \nu\Delta u \\
\partial_t v + (f+\zeta)u + \partial_y B &= \frac{\tau_y}{\rho_0 h} - rv + \nu\Delta v \\
\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0
\end{aligned}
$$

- $\zeta = \partial_x v - \partial_y u$ is the relative vorticity ;
- $B = g^* h + \frac{1}{2}(u^2+v^2)$ is the Bernoulli potential ;
- $g^* = 0.02\ m.s^{-2}$ is the reduced gravity ;
- $f = f_0 + \beta y$ is the Coriolis parameter (in the β-plane approximation), with $f_0 = 7.10^{-5}\ s^{-1}$ and $\beta = 2.10^{-11}\ m^{-1}.s^{-1}$;
- $\tau = (\tau_x, \tau_y)$ is the forcing term of the model (e.g. the wind stress), with a maximum amplitude of $\tau_0 = 0.05\ s^{-2}$;
- $\rho_0 = 10^3\ kg.m^{-3}$ is the water density ;
- $r = 9.10^{-8}\ s^{-1}$ is the friction coefficient.
- $\nu = 5\ m^2.s^{-1}$ is the viscosity (or dissipation) coefficient.

Shallow water model

2D shallow water model, state = height h and horizontal velocity (u, v)

Numerical parameters : (run example)

Domain : $L = 2000$ km $\times$ 2000 km ; Rigid boundary and no-slip BC ; Time step = 1800 s ; Assimilation period : 15 days ; Forecast period : $15 + 45$ days

Observations : of h only ($\sim$ satellite obs), every 5 gridpoints in each space direction, every 24 hours.

Background : true state one month before the beginning of the assimilation period + white gaussian noise ($\sim 10\%$)

Comparison BFN - 4DVAR : sea height h ; velocity :u and v.

[Auroux, Int J Numer Methods Fluids 2009]

Diffusion problem

Backward model and diffusion :

The main issue of the BFN is : how to handle diffusion processes in the backward equation ?

Let us consider only diffusion : heat equation (in 1D)

$$\partial_t u = \partial_{xx} u$$

The backward nudging model will be :

$$\partial_t \tilde{u} = \partial_{xx} \tilde{u} + K(\tilde{u} - u_{obs})$$

from time T to 0. By using a change of variable $t' = T - t$, we can rewrite the backward model as a forward one :

$$\partial_{t'} \tilde{u} = -\partial_{xx} \tilde{u} - K(\tilde{u} - u_{obs}),$$

and we can see that even if the nudging term stabilizes the model, the backward diffusion is a real issue (unbounded eigenvalues, except for discrete Laplacian).

Diffusion problem

Hopefully, in geophysical problems, diffusion is not a dominant term. The model has smoothing properties, and diffusion is small $\rightarrow$ diffusion processes are not highly unstable in backward mode, even if the model is clearly unstable without nudging.

Theoretically, there is a problem :

- **Viscous linear transport equation :** if the support of K is a strict sub-domain (i.e. some parts of the space domain are not observed), there does not exist a solution to the backward model, even in the distribution sense.

- **Viscous Burgers equation :** even if K is constant (in time and space $\Rightarrow$ full observations), the backward equation is ill-posed, as there is no stability (or continuity) with respect to the initial condition.

Without viscosity, one can prove the convergence of the BFN on these equations. [Auroux and Nodet, COCV 2012]

1. Nudging and observers

2. Back and Forth Nudging algorithm

⇒ 3. Diffusive BFN algorithm

Diffusive BFN

Diffusive free equations in the geophysical context :

In meteorology or oceanography, theoretical equations are usually diffusive free (e.g. Euler's equation for meteorological processes).

In a numerical framework, a diffusive term is added to the equations (or a diffusive scheme is used), in order to both stabilize the numerical integration of the equations, and take into consideration some subscale phenomena.

Example : weather forecast is done with Euler's equation (at least in Météo France...), which is diffusive free. Also, in quasi-geostrophic ocean models, people usually consider ∇^4 or ∇^6 for dissipation at the bottom, or for vertical mixing.

Diffusive BFN

Addition of a diffusion term :

$$\partial_t X = F(X) + \nu \Delta X, \quad 0 < t < T,$$

where F has no diffusive terms, ν is the diffusion coefficient, and we assume that the diffusion is a standard second-order Laplacian (could be a higher order operator).

We introduce the D-BFN algorithm in this framework, for $k \geq 1$:

$$\begin{cases} \partial_t X_k = F(X_k) + \nu \Delta X_k + K(\mathcal{Y} - H(X_k)), \\ X_k(0) = \tilde{X}_{k-1}(0), \quad 0 < t < T, \end{cases}$$

$$\begin{cases} \partial_t \tilde{X}_k = F(\tilde{X}_k) - \nu \Delta \tilde{X}_k - K'(\mathcal{Y} - H(\tilde{X}_k)), \\ \tilde{X}_k(T) = X_k(T), \quad T > t > 0. \end{cases}$$

Diffusive BFN

It is straightforward to see that the backward equation can be rewritten, using $t' = T - t$:

$$\partial_{t'} \tilde{X}_k = -F(\tilde{X}_k) + \nu \Delta \tilde{X}_k + K'(\mathcal{Y} - H(\tilde{X}_k)), \quad \tilde{X}_k(t' = 0) = X_k(T),$$

where $\tilde{X}$ is evaluated at time t'. As it is now forward in time, this equation can be compared with the forward nudging equation :

$$\partial_t X_k = F(X_k) + \nu \Delta X_k + K(\mathcal{Y} - H(X_k)), \quad X_k(0) = \tilde{X}_{k-1}(t' = T).$$

Then the backward equation can easily be solved, with an initial condition, and the same diffusion operator as in the forward equation. Only the physical model has an opposite sign.

The diffusion term both takes into account the subscale processes and stabilizes the numerical backward integrations, and the feedback term still controls the trajectory with the observations.

Linear transport equation

$$\partial_t u + a(x)\,\partial_x u = 0, \quad t \in [0,T],\ x \in \Omega, \quad u(t=0) = u_0 \in L^2(\Omega)$$

with periodic boundary conditions, and we assume that $a \in W^{1,\infty}(\Omega)$.

Numerically, for both stability and subscale modelling, the following equation would be solved :

$$\partial_t u + a(x)\,\partial_x u = \nu \partial_{xx} u, \quad t \in [0,T], x \in \Omega, \quad u(t=0) = u_0 \in L^2(\Omega),$$

where $\nu \geq 0$ is assumed to be constant.

Linear transport equation

Let us assume that the observations satisfy the physical model (without diffusion) :

$$\partial_t u_{obs} + a(x)\,\partial_x u_{obs} = 0, \quad t \in [0,T], x \in \Omega, \quad u_{obs}(t=0) = u^0_{obs} \in L^2(\Omega).$$

We assume in this idealized situation that the system is fully observed (and H is then the identity operator).

Then the D-BFN algorithm applied to this problem gives, for $k \geq 1$:

$$\begin{cases} \partial_t u_k + a(x)\,\partial_x u_k = \nu\partial_{xx}u_k + K(u_{obs,k} - u_k), \\ t \in [2(k-1)T, 2(k-1)T + T], x \in \Omega \\ u_k(2(k-1)T, x) = \tilde{u}_{k-1}(2(k-1)T, x) \end{cases}$$

$$\begin{cases} \partial_t \tilde{u}_k - a(x)\,\partial_x \tilde{u}_k = \nu\partial_{xx}\tilde{u}_k + K(\tilde{u}_{obs,k} - \tilde{u}_k), \\ t \in [2kT - T, 2kT], x \in \Omega \\ \tilde{u}_k(2kT - T, x) = u_k(2kT - T, x). \end{cases}$$

Smoothing equation

At the limit $k \to \infty$, v_k and $\tilde{v}_k$ tend to $v_\infty(x)$ solution of

$$\nu \partial_{xx} v_\infty + K(u^0_{obs}(x) - v_\infty) = 0,$$

or equivalently

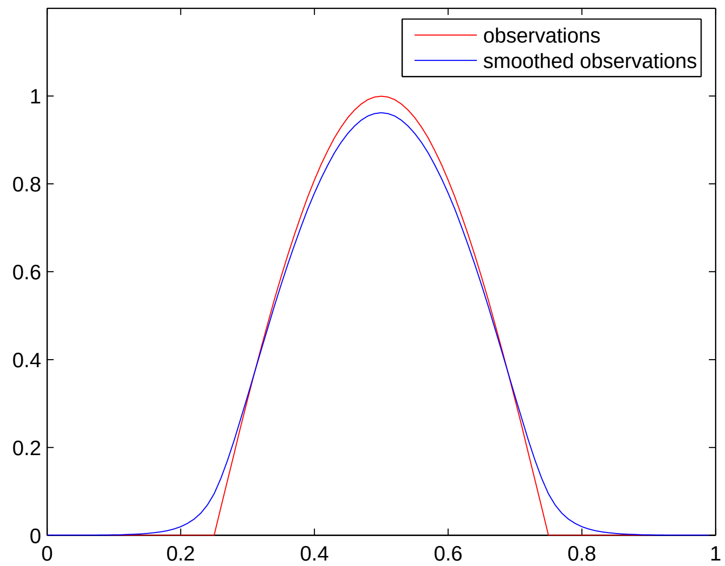
$$-\frac{\nu}{K} \partial_{xx} v_\infty + v_\infty = u^0_{obs}.$$

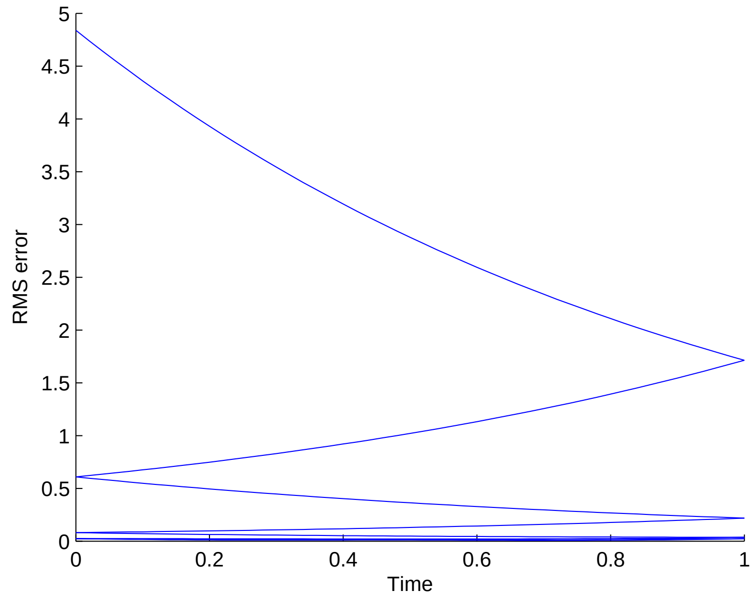
This equations is well known in signal or image processing, as being the standard linear diffusion restoration equation. In some sense, v_∞ is the result of a smoothing process on the observations u_{obs}, where the degree of smoothness is given by the ratio $\frac{\nu}{K}$.

Convergence result for constant advection equation.

[Auroux, Blum and Nodet, CRAS 2011]

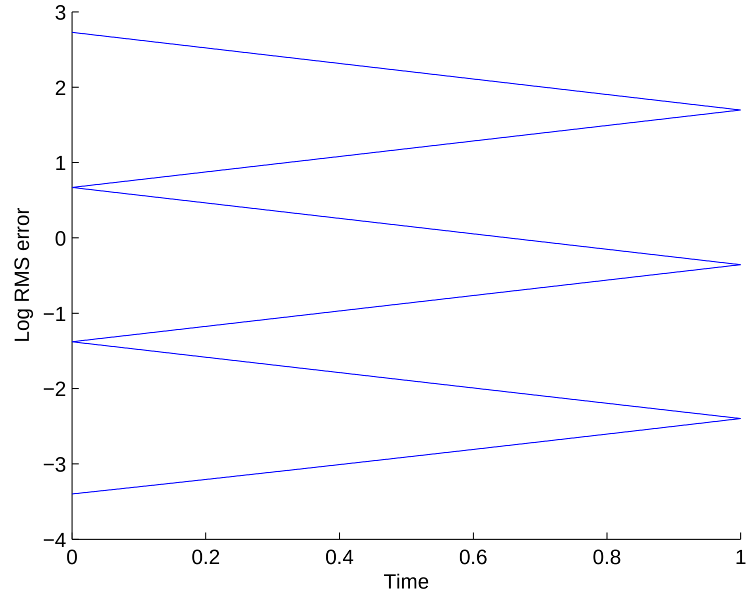
Numerical experiments





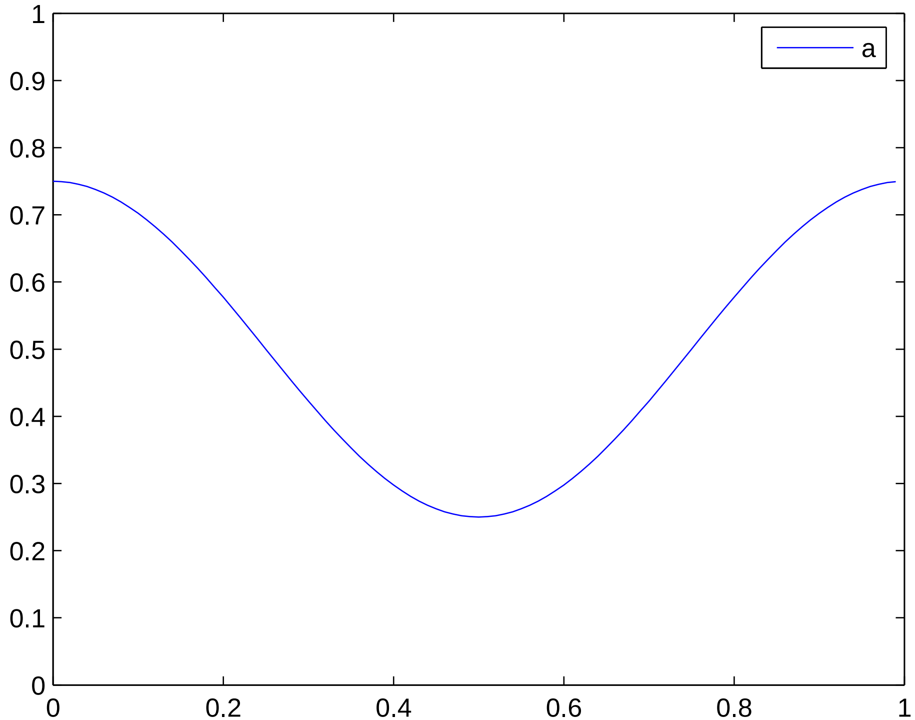
Initial condition of the observation and corresponding smoothed solution ; RMS difference between the BFN iterates and the smoothed observations ; same in semi-log scale.

Movie



Numerical experiments

Linear transport equation with non-constant transport :



Movie

Full primitive ocean model

Primitive equations : Navier-Stokes equations (velocity-pressure), coupled with two active tracers (temperature and salinity).

Momentum balance :

$$\frac{\partial U_h}{\partial t} = -\left[(\nabla \wedge U) \wedge U + \frac{1}{2}\nabla(|U|^2)\right]_h - f.z \wedge U_h - \frac{1}{\rho_0}\nabla_h p + D^U + F^U$$

Incompressibility equation : $\nabla.U = 0$

Hydrostatic equilibrium : $\dfrac{\partial p}{\partial z} = -\rho g$

Heat and salt conservation equations :

$$\frac{\partial T}{\partial t} = -\nabla.(TU) + D^T + F^T \quad \text{(+ same for S)}$$

Equation of state : $\rho = \rho(T, S, p)$

Full primitive ocean model

Free surface formulation : the height of the sea surface η is given by

$$\frac{\partial \eta}{\partial t} = -div_h((H+\eta)\bar{U_h}) + [P - E]$$

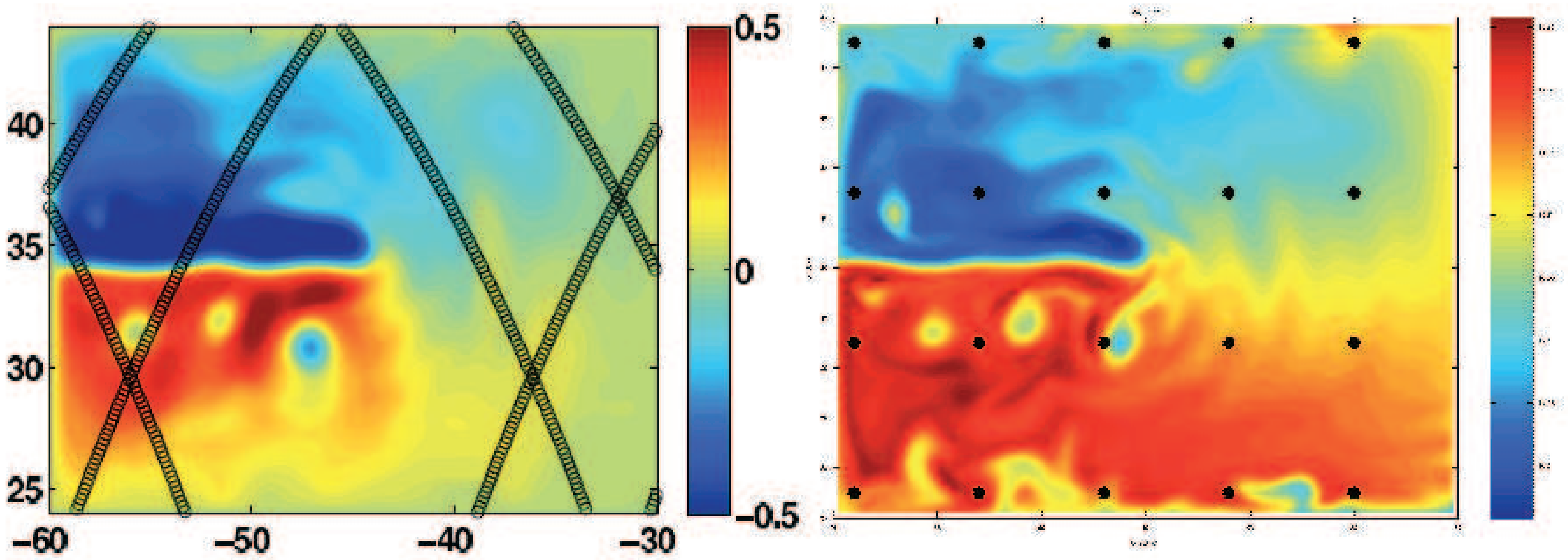
The surface pressure is given by : $p_s = \rho g \eta.$

This boundary condition is then used for integrating the hydrostatic equilibrium and calculating the pressure.

Numerical experiments : double gyre circulation confined between closed boundaries (similar to the shallow water model). The circulation is forced by a sinusoidal (with latitude) zonal wind.

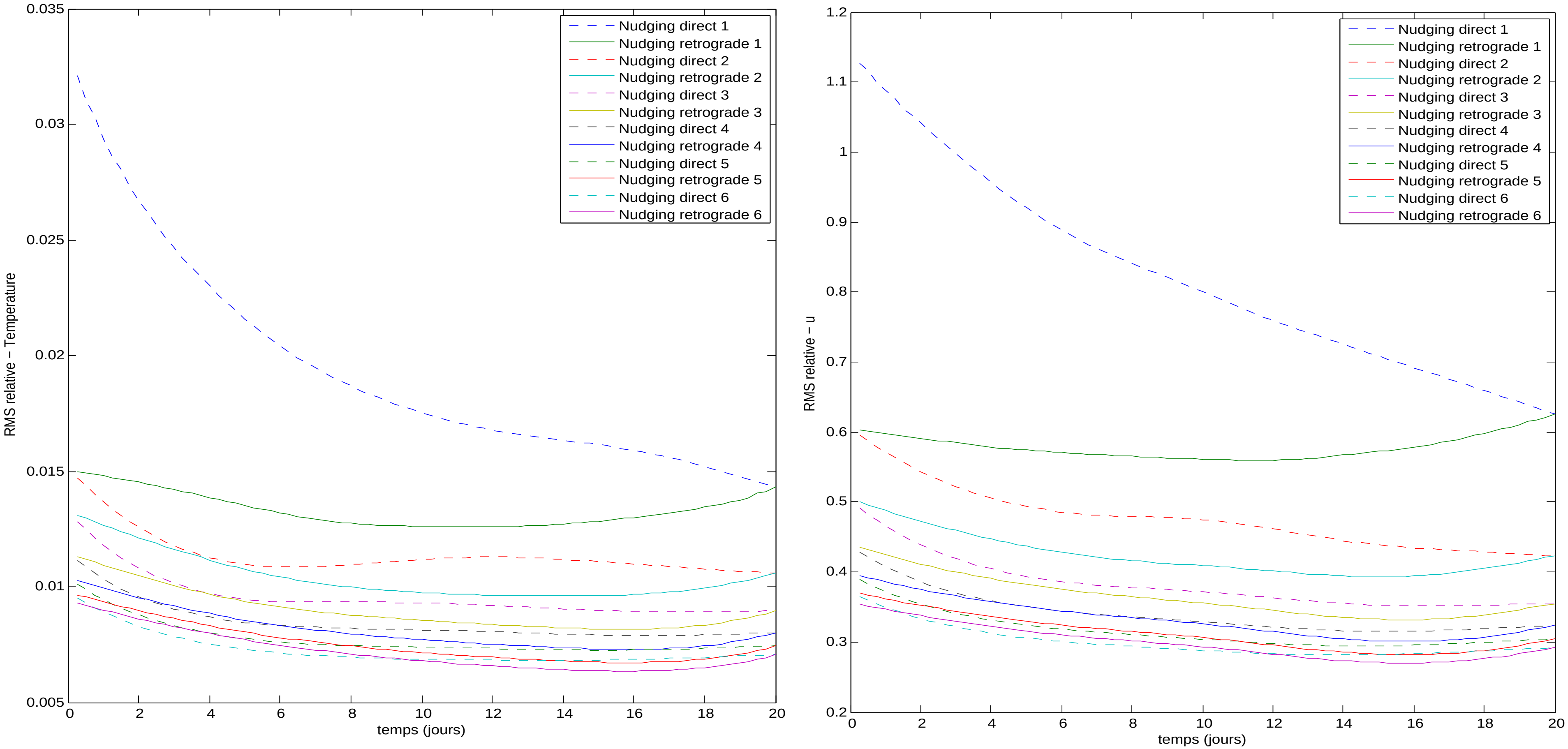
Twin experiments : observations are extracted from a reference run, according to networks of realistic density : SSH is observed similarly to TOPEX/POSEIDON, and temperature is observed on a regular grid that mimics the ARGO network density.

Full primitive ocean model



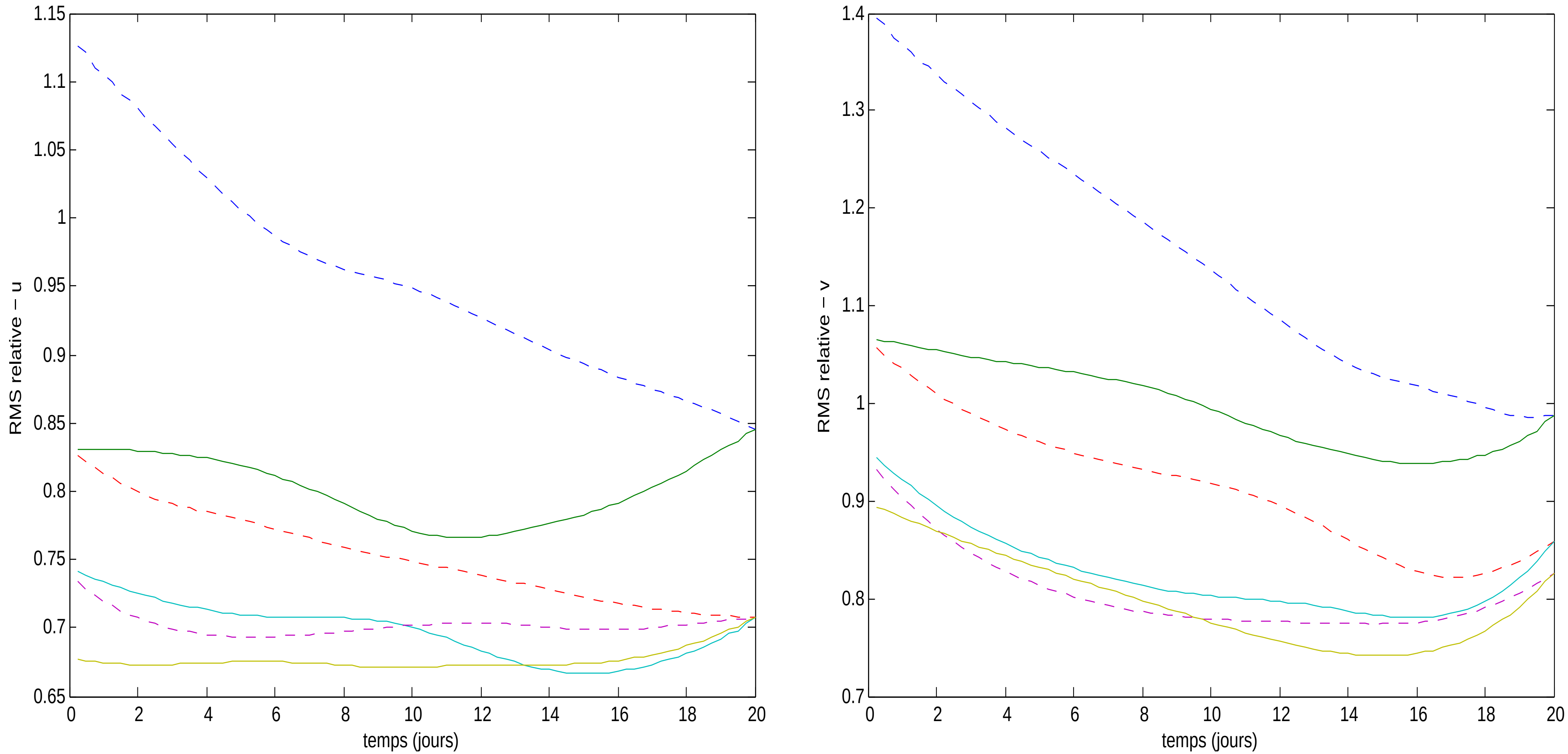
Example of observation network used in the assimilation : along-track altimetric observations (Topex-Poseidon) of the SSH every 10 days ; vertical profiles of temperature (ARGO float network) every 18 days.

Numerical results



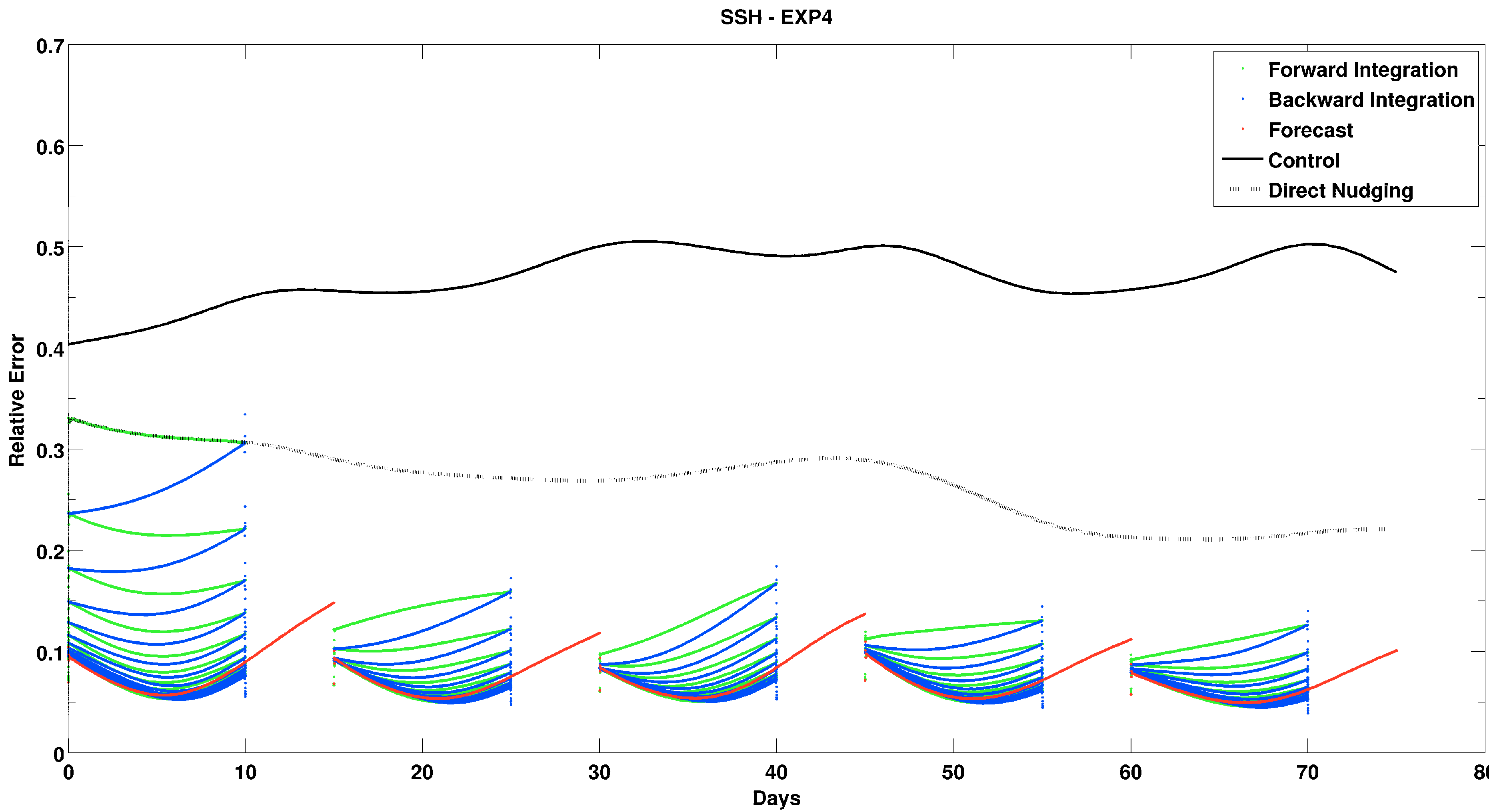
Relative RMS error of the temperature (left) and longitudinal velocity (right), 6 iterations of BFN (nudging terms in the temperature and SSH equations only), with full and unnoisy SSH observations every day. [Ruggiero, PhD thesis 2014]

Numerical results



Relative RMS error of the longitudinal and transversal velocities, 3 iterations of BFN (nudging terms in the temperature and SSH equations only), with "realistic" SSH observations (T/P track + 15% noise).

Numerical results



Evolution of the errors during the Back and Forth iterations and during the forecast phase. In black : evolution of the error for the control and direct nudging experiments.

Conclusions

Back and Forth Nudging algorithm :

- Easy implementation (no linearization, no adjoint state, no minimization process)
- Very efficient in the first iterations (faster convergence)
- Lower computational and memory costs than other DA methods
- Stabilization of the backward model
- Excellent preconditioner for 4D-VAR (or Kalman filters)

Diffusive BFN algorithm :

- Converges even faster, with smaller backward nudging coefficients
- Still produces very precise forecasts

Perspectives

Extension to more (but not too) complex Back and Forth Observers :

- Observers for N-d compresible Navier-Stokes [Apte et al 2017] : reconstruction of velocity from density or density from velocity, + arbitrary choice of the error decay rate

- Use of physical considerations [Ruggiero 2014] : e.g. geostrophic equilibrium (Coriolis force $\simeq$ pressure gradient) to correct non observed variables

Extension to parameter estimation :

- Add an equation for the parameter (e.g. $\frac{d\alpha}{dt} = 0$), observe the physical variables, and try to build an observer that corrects all variables (including the parameter)

- Use of observers in a similar way as Kalman filtering for parameter estimation ($\rightsquigarrow$ Fourier decompositions, energy estimates, Lyapunov theory, ...) [undergoing work @Nice]

Thank you for your attention !