Conceptual Climate Models Minitutorial Part II

Jim Walsh
Oberlin College

SIAM MPE Conference
Philadelphia, PA
September 30, 2016

Resources:

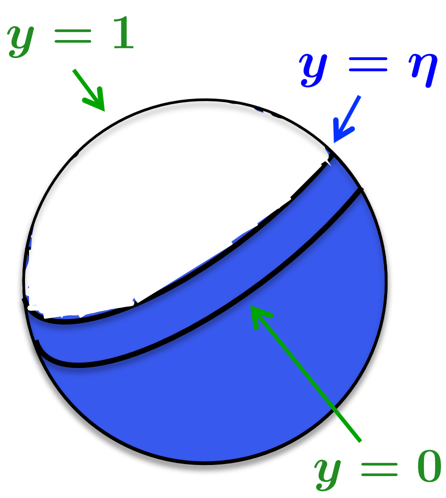
- https://mcrn.hubzero.org/resources/81#series. Videos of lectures and lecture slides from Introduction to the Mathematics of Climate, taught by Richard McGehee, School of Mathematics, University of Minnesota.

- https://mcrn.hubzero.org/resources/523/supportingdocs. Material developed for the MAA-NCS Summer Seminar Conceptual Climate Models, held in Minneapolis, July 2013. Contributors: A. Barry, R. McGehee, S. Oestreicher, J.A. Walsh, E. Widiasih.

- J.A. Walsh, Climate modeling in differential equations, *The UMAP Journal* **36** (4) (2015), 325-363.

Coupled zonal temperature – ice line model

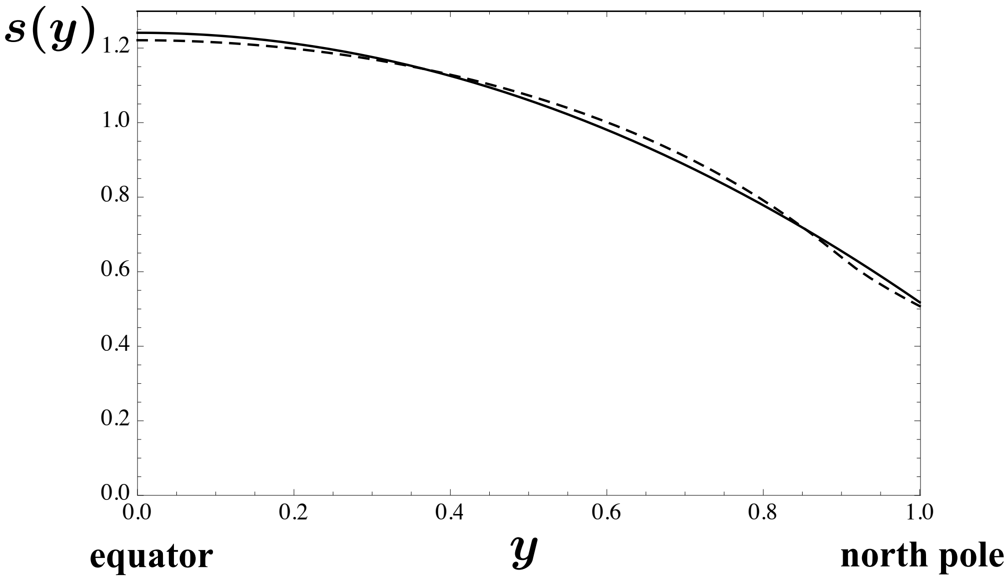
$$\begin{cases} R\dfrac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T}) \\ \dfrac{d\eta}{dt} = \epsilon(T(\eta,t)-T_c), \quad \epsilon > 0 \end{cases}$$

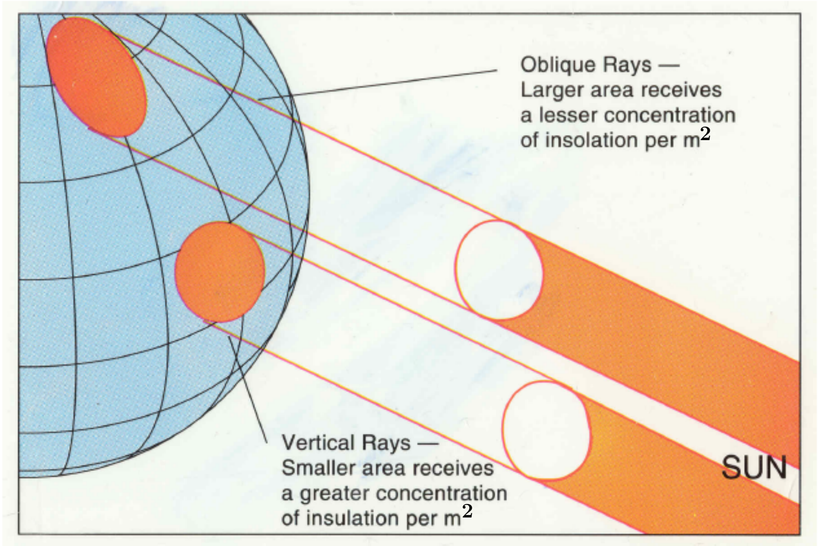


- An infinite-dimensional dynamical system
- Symmetry assumption ⟹ functions are even in y
- Consider an approximating system of ODEs

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T})$$

$$\int_0^1 s(y)\,dy = 1$$





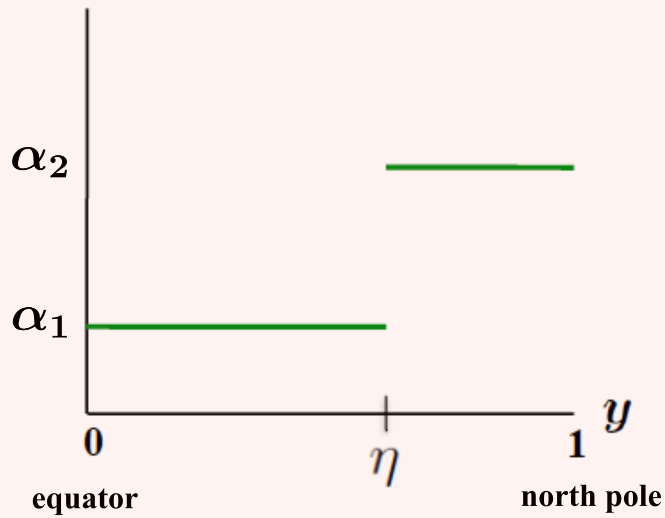
Dashed: actual (current) values

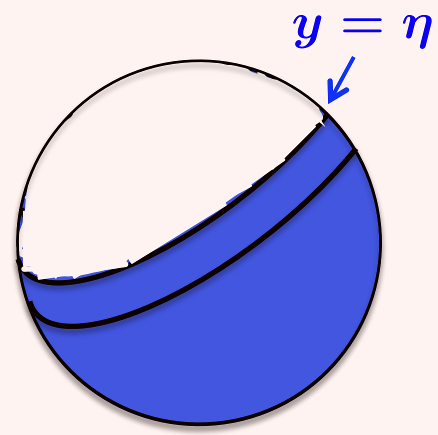
Solid: $s(y) = s_0 p_0(y) + s_2 p_2(y)$, $p_0(y) = 1$, $p_2(y) = \frac{1}{2}(3y^2 - 1)$

the first two even Legendre polys, $s_0 = 1$, $s_2 = -0.482$

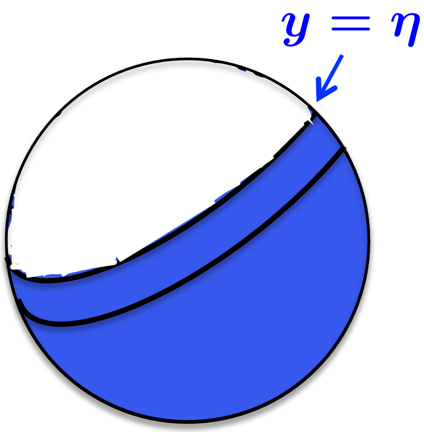
$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T})$$

$\alpha(y,\eta)$ $(albedo)$



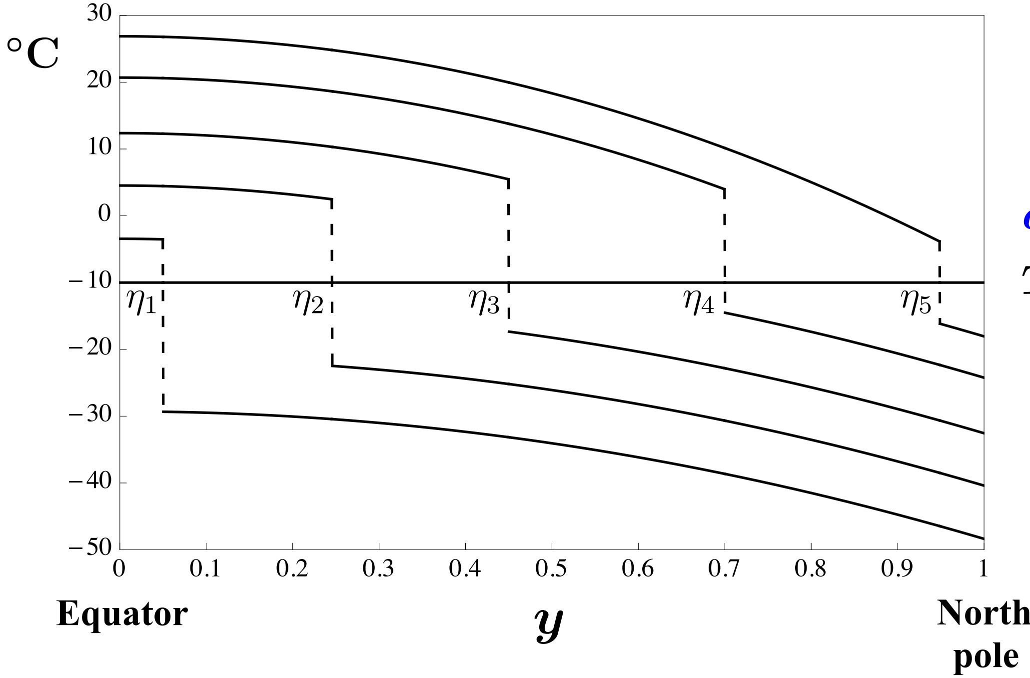


$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T})$$



Equilibrium temperature profiles $T^*_\eta(y)$

(piecewise quadratic)



critical temperature

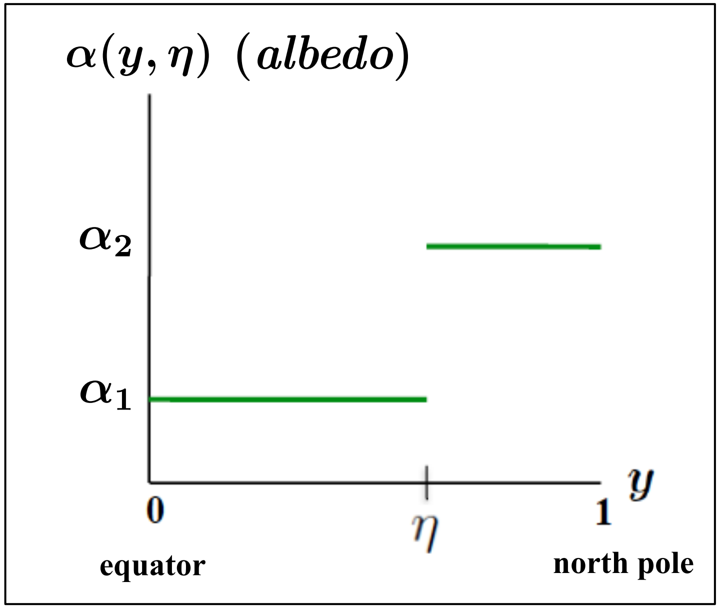
$T_c = -10°\mathrm{C}$

$$T(\eta,t) = \tfrac{1}{2}(\lim_{y\uparrow\eta} T(y,t) + \lim_{y\downarrow\eta} T(y,t))$$

Approximating system of ODEs

$$s(y) = s_0 p_0(y) + s_2 p_2(y)$$

$$\begin{cases} R\dfrac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T}) \\ \dfrac{d\eta}{dt} = \epsilon(T(\eta,t) - T_c), \quad \epsilon > 0 \end{cases}$$



$$T(y,t) = \begin{cases} U(y,t), & y < \eta \\ V(y,t), & y > \eta \end{cases}$$

$$U(y,t) = u_0(t)p_0(y) + u_2(t)p_2(y)$$

$$V(y,t) = v_0(t)p_0(y) + v_2(t)p_2(y)$$

$$T(\eta,t) = \tfrac{1}{2}(U(\eta,t) + V(\eta,t))$$

R. McGehee and E. Widiasih, A quadratic approximation to Budyko's ice albedo feedback model with ice line dynamics, *SIAM J. Appl. Dyn. Syst*. **13** (2014), 518-536.

$$p_0(y) = 1, \ \ p_2(y) = \frac{1}{2}(3y^2 - 1)$$

Approximating system of ODEs

$$R\frac{\partial T}{\partial t} = Qs(y)(1 - \alpha(y,\eta)) - (B + C)T - A + C\overline{T}$$

$$T(y,t) = \begin{cases} U(y,t), & y < \eta \\ V(y,t), & y > \eta \end{cases} \qquad \begin{aligned} U(y,t) &= u_0(t)p_0(y) + u_2(t)p_2(y) \\ V(y,t) &= v_0(t)p_0(y) + v_2(t)p_2(y) \end{aligned}$$

$$y < \eta$$

$$R\frac{\partial T}{\partial t} = R(\dot{u}_0 p_0(y) + \dot{u}_2 p_2(y))$$

$$Qs(y)(1 - \alpha(y,\eta)) = Q(s_0 p_0(y) + s_2 p_2(y))(1 - \alpha_1)$$

$$(B + C)T = (B + C)(u_0 p_0(y) + u_2 p_2(y))$$

$$-A + C\overline{T} = (-A + C\overline{T})p_0(y)$$

plug in, equate coefficients of $p_0(y)$ and $p_2(y)$, respectively...

Approximating system of ODEs

$$T(y,t) = \begin{cases} U(y,t), & y < \eta \\ V(y,t), & y > \eta \end{cases}$$

$$U(y,t) = u_0(t)p_0(y) + u_2(t)p_2(y)$$
$$V(y,t) = v_0(t)p_0(y) + v_2(t)p_2(y)$$

$$R\dot{u}_0 = Q(1-\alpha_1) - (B+C)u_0 - A + C\overline{T}$$
$$R\dot{u}_2 = Qs_2(1-\alpha_1) - (B+C)u_2$$

Repeat for $y > \eta$:

$$R\dot{v}_0 = Q(1-\alpha_2) - (B+C)v_0 - A + C\overline{T}$$
$$R\dot{v}_2 = Qs_2(1-\alpha_2) - (B+C)v_2$$

Include the ice line equation...

Approximating system of ODEs

$$R\dot{u}_0 = Q(1-\alpha_1) - (B+C)u_0 - A + C\overline{T}$$

$$R\dot{u}_2 = Qs_2(1-\alpha_1) - (B+C)u_2$$

$$R\dot{v}_0 = Q(1-\alpha_2) - (B+C)v_0 - A + C\overline{T}$$

$$R\dot{v}_2 = Qs_2(1-\alpha_2) - (B+C)v_2$$

$$\dot{\eta} = \epsilon(T(\eta,t) - T_c)$$

- $\overline{T} = \int_0^\eta U(y,t)dy + \int_\eta^1 V(y,t)dy = \overline{T}(u_0,u_2,v_0,v_2,\eta)$
- $T(\eta,t) = \frac{1}{2}(U(\eta,t) + V(\eta,t)) = T(u_0,u_2,v_0,v_2,\eta)$

change variables $w = \frac{1}{2}(u_0 + v_0), \; z = u_0 - v_0$

Approximating system of ODEs

$\alpha_0 = \frac{1}{2}(\alpha_1 + \alpha_2)$

$$
\begin{aligned}
R\dot{w} &= Q(1-\alpha_0) - (B+C)w - A + C\overline{T} \\
R\dot{z} &= Q(\alpha_2 - \alpha_1) - (B+C)z \\
R\dot{u}_2 &= Qs_2(1-\alpha_1) - (B+C)u_2 \\
R\dot{v}_2 &= Qs_2(1-\alpha_2) - (B+C)v_2 \\
\dot{\eta} &= \epsilon(T(\eta,t) - T_c)
\end{aligned}
$$

- $\overline{T} = \overline{T}(w, z, u_2, v_2, \eta)$
- $T(\eta, t) = T(w, z, u_2, v_2, \eta)$

Assume z, u_2, v_2 are at equilibrium ➡

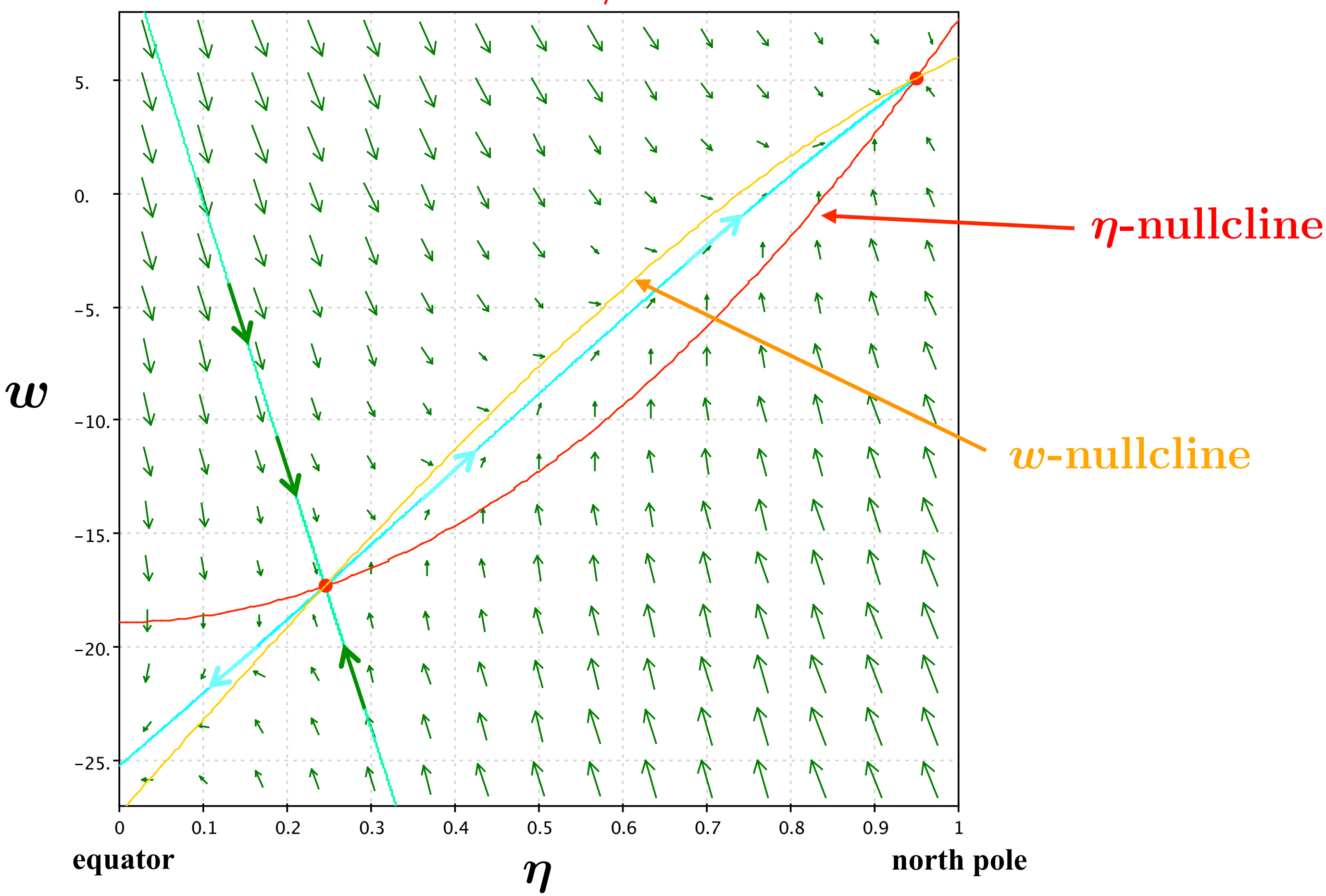
- $\overline{T} = \overline{T}(w, \eta)$
- $T(\eta, t) = T(w, \eta)$

$$\dot{w} = -\tau(w - F(\eta))$$
$$\dot{\eta} = \epsilon(w - G(\eta)),$$

$F(\eta) = \gamma(\eta) - A/B$ $\quad$ ($F(\eta)$ cubic in η; $\tau = B/R$)

$G(\eta)$ quadratic in η

$A = 202\ \mathrm{W/m^2}$

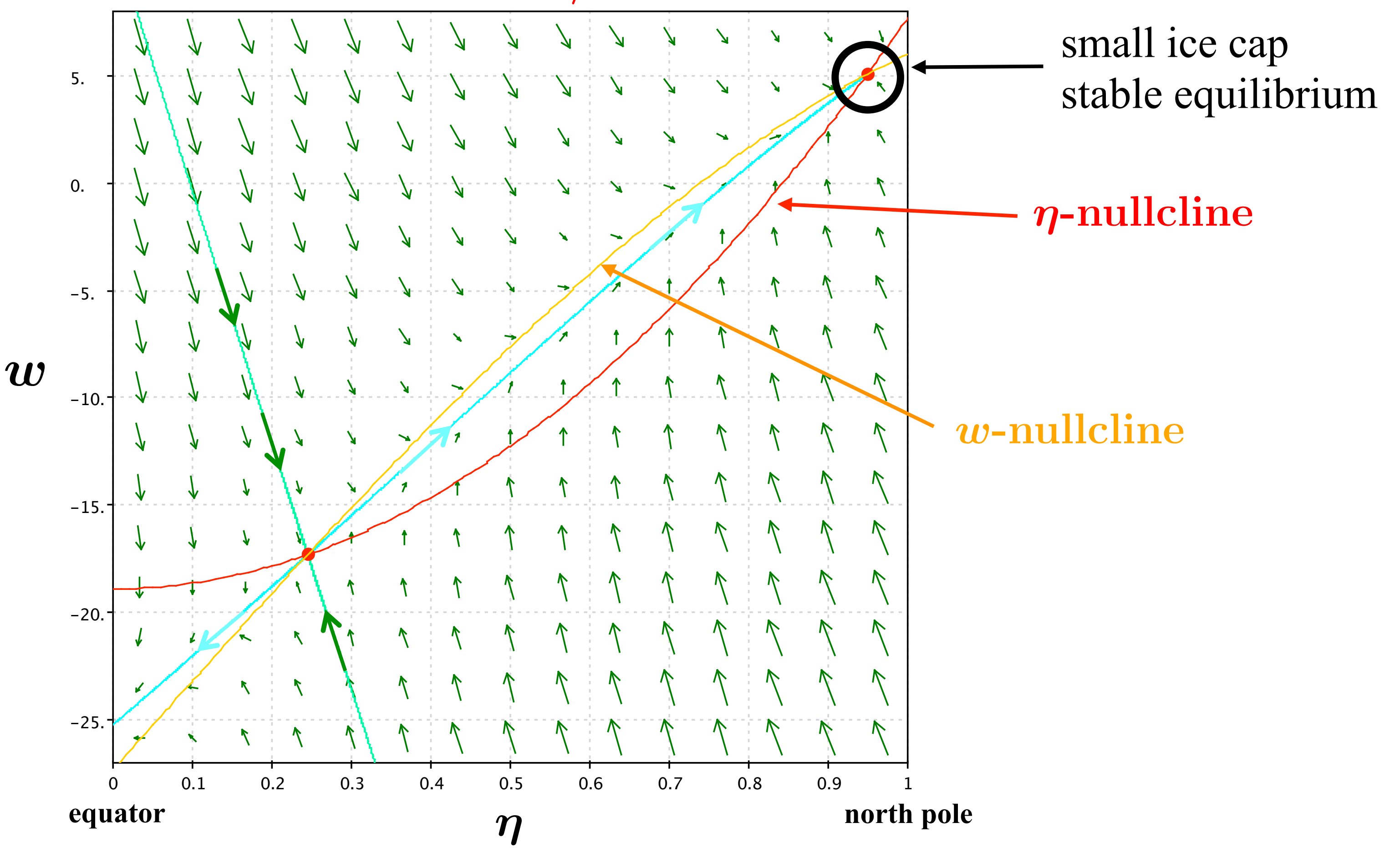


$$\dot{w} = -\tau(w - F(\eta))$$
$$\dot{\eta} = \epsilon(w - G(\eta)),$$

$F(\eta) = \gamma(\eta) - A/B$ $\quad$ ($F(\eta)$ cubic in η; $\tau = B/R$)

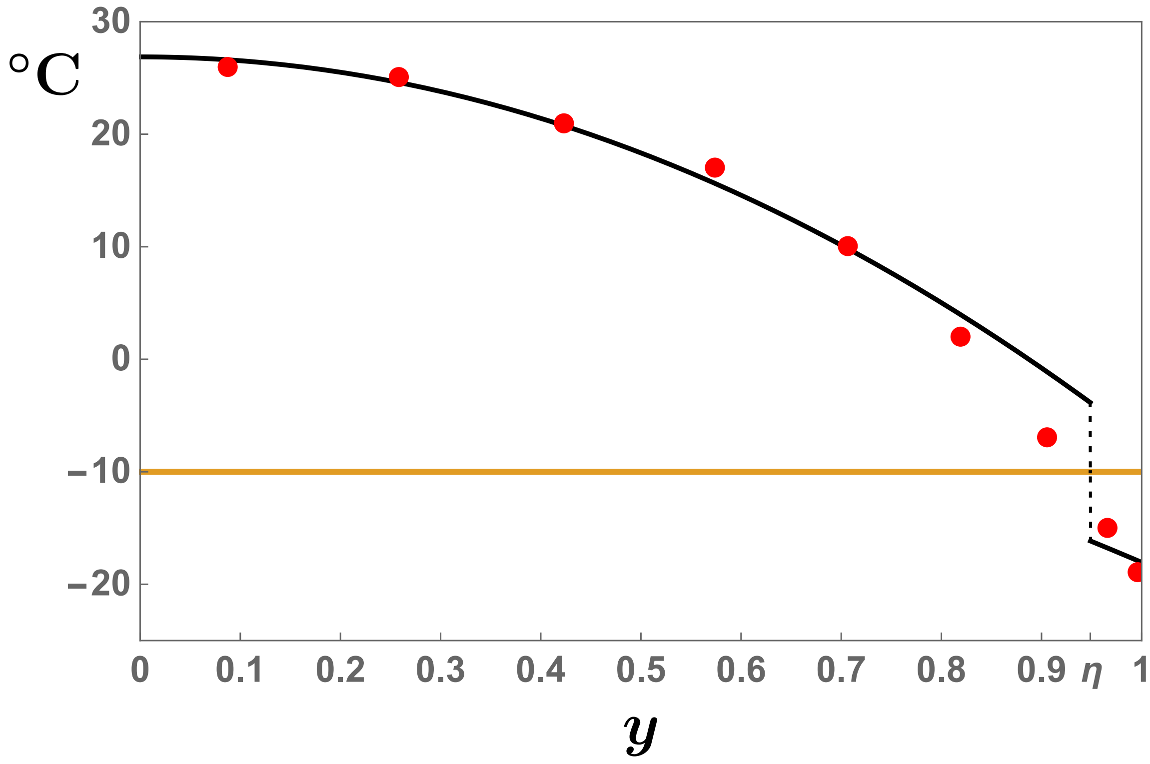
$G(\eta)$ quadratic in η

$A = 202\ \mathrm{W/m^2}$



Approximating system of ODEs

$$T(y,t) = \begin{cases} U(y,t), & y < \eta \\ V(y,t), & y > \eta \end{cases} \qquad \begin{aligned} U(y,t) &= u_0(t)p_0(y) + u_2(t)p_2(y) \\ V(y,t) &= v_0(t)p_0(y) + v_2(t)p_2(y) \end{aligned}$$



Black: Equlibrium profile $T^*_\eta(y)$

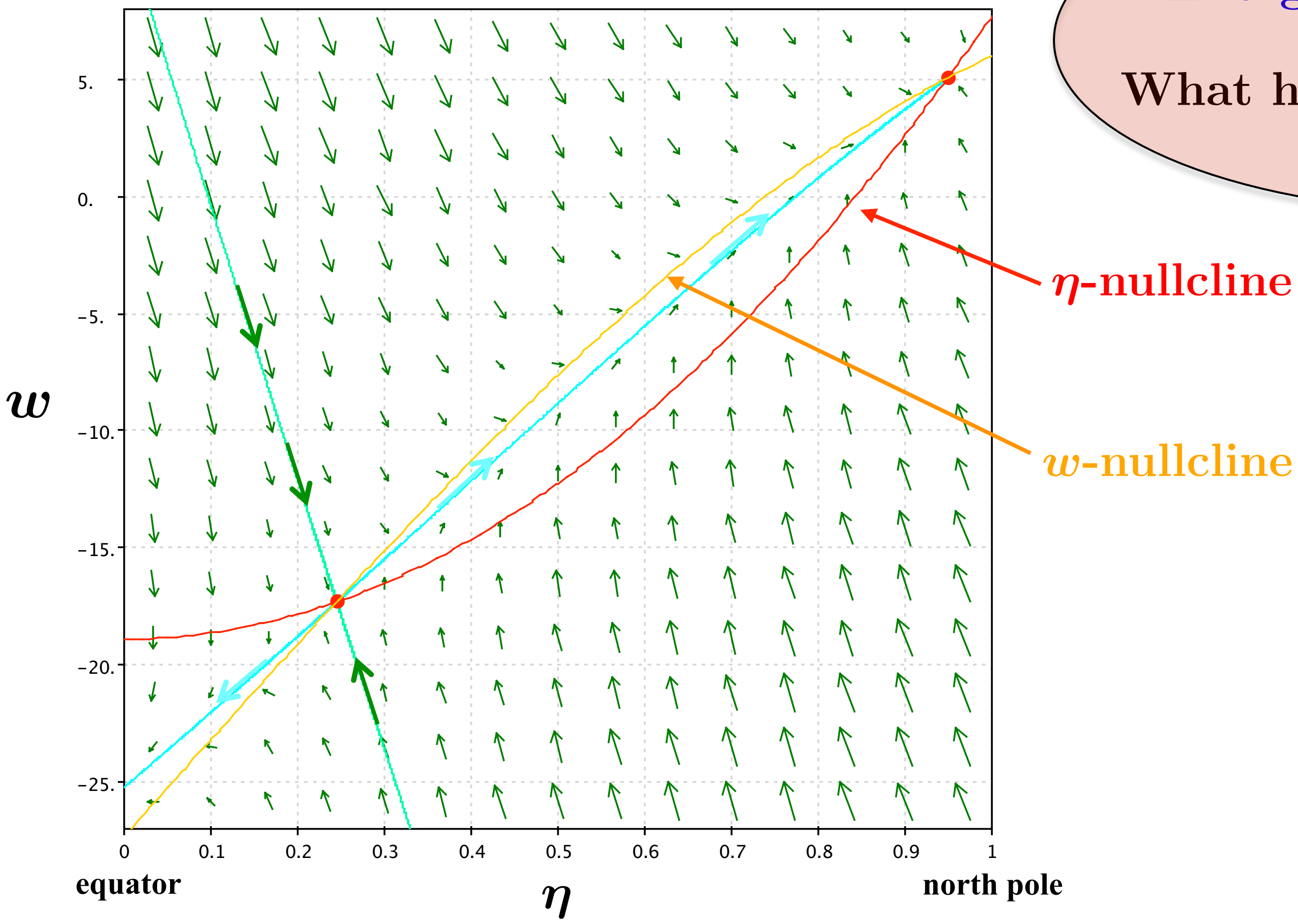
Red: Observations

$$\dot{w} = -\tau(w - F(\eta))$$
$$\dot{\eta} = \epsilon(w - G(\eta)),$$

$F(\eta) = \gamma(\eta) - A/B$ $\quad$ $(F(\eta)$ cubic in η; $\tau = B/R)$

$G(\eta)$ quadratic in η

$A = 202\ \mathrm{W/m^2}$



Energy$_{out} = A + BT(y,t)$

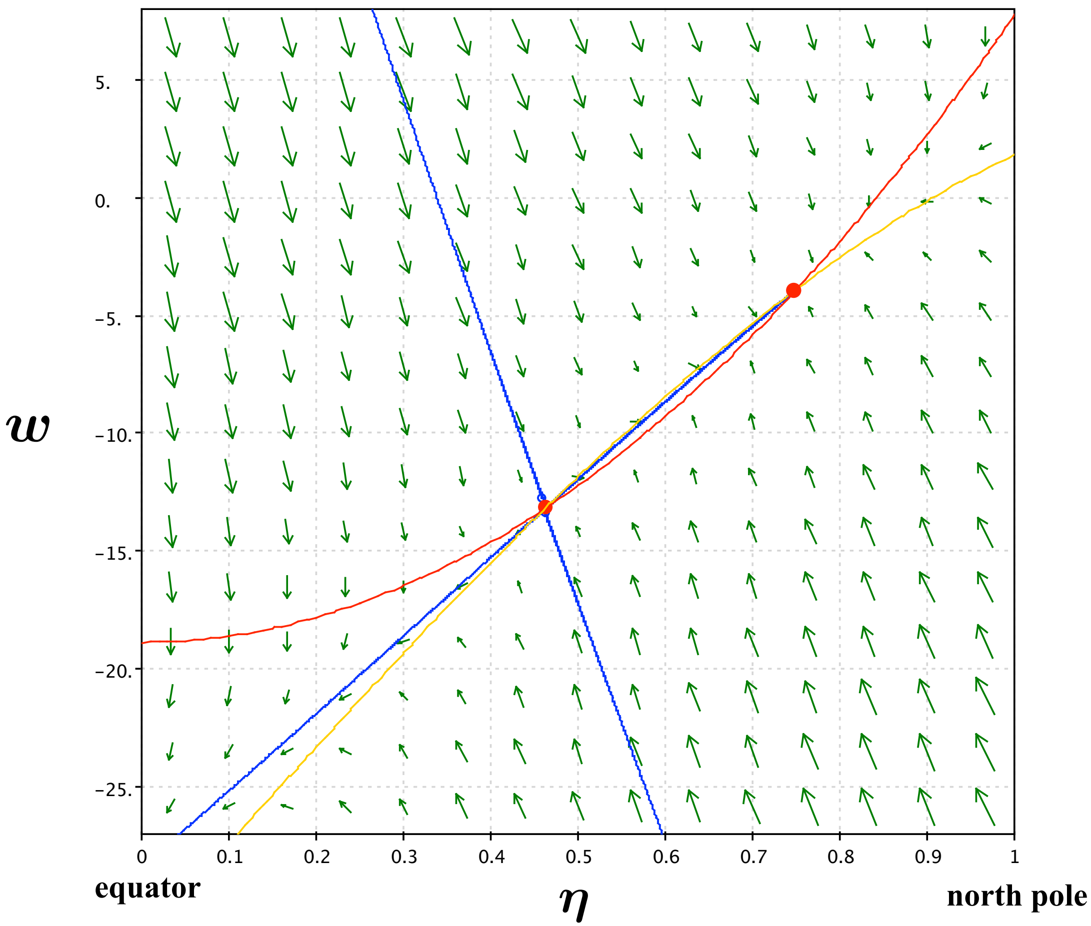
What happens if A increases?

$$\dot{w} = -\tau(w - F(\eta))$$
$$\dot{\eta} = \epsilon(w - G(\eta)),$$

$F(\eta) = \gamma(\eta) - A/B$ ($F(\eta)$ cubic in η; $\tau = B/R$)

$G(\eta)$ quadratic in η

$A = 210$ W/m^2

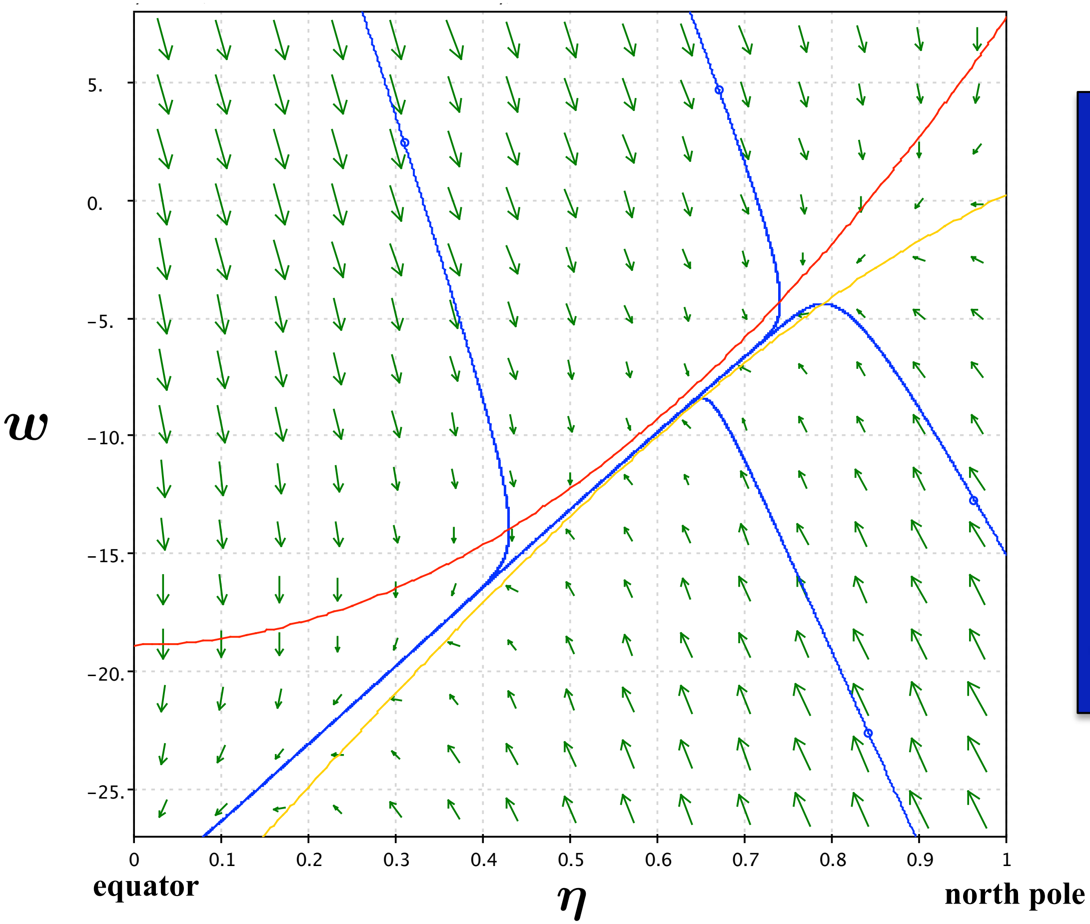


$$\boxed{\begin{aligned}\dot{w} &= -\tau(w - F(\eta)) \\ \dot{\eta} &= \epsilon(w - G(\eta)),\end{aligned}}$$

$F(\eta) = \gamma(\eta) - A/B$ $\quad$ ($F(\eta)$ cubic in η; $\tau = B/R$)

$G(\eta)$ quadratic in η

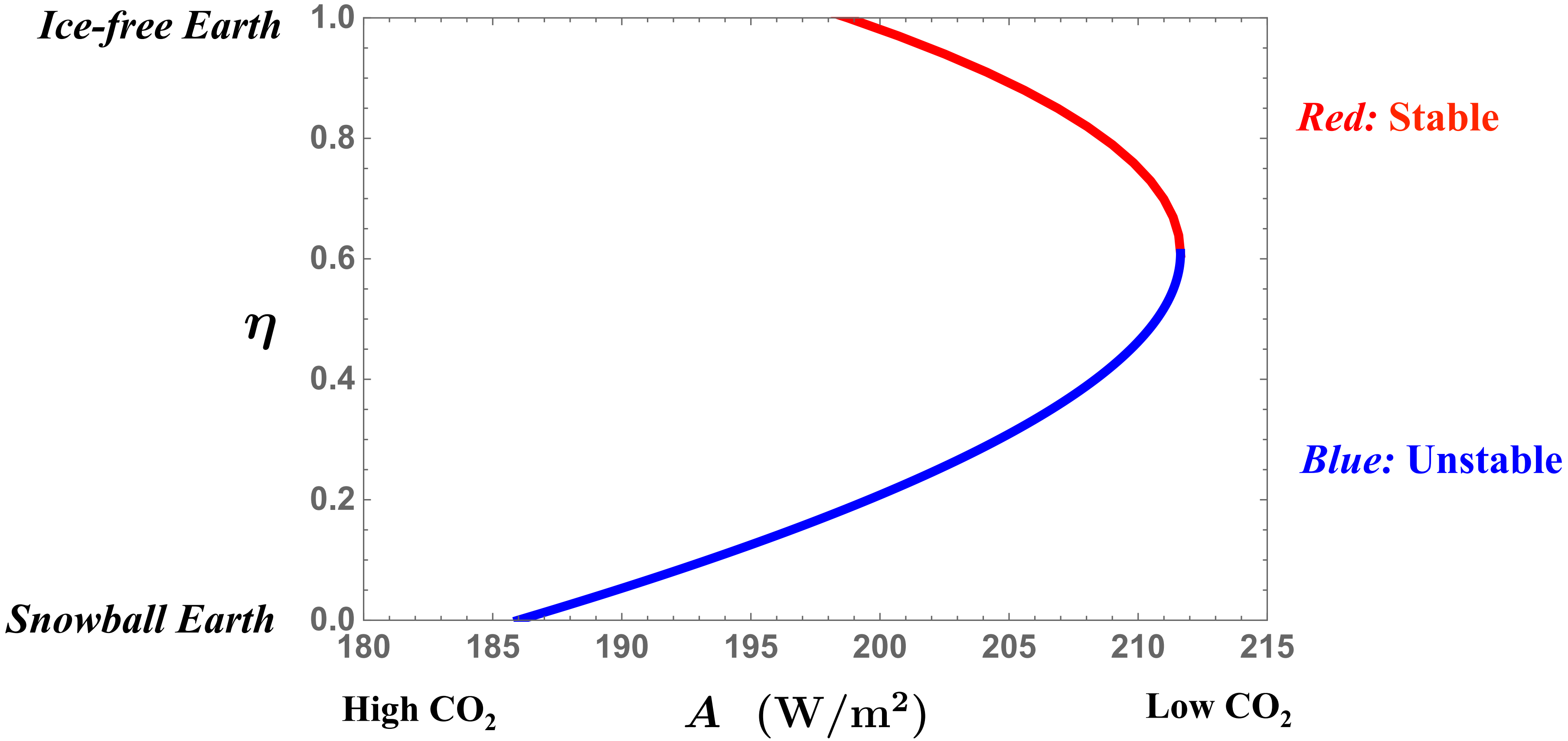
$A = 213\ \mathrm{W/m^2}$



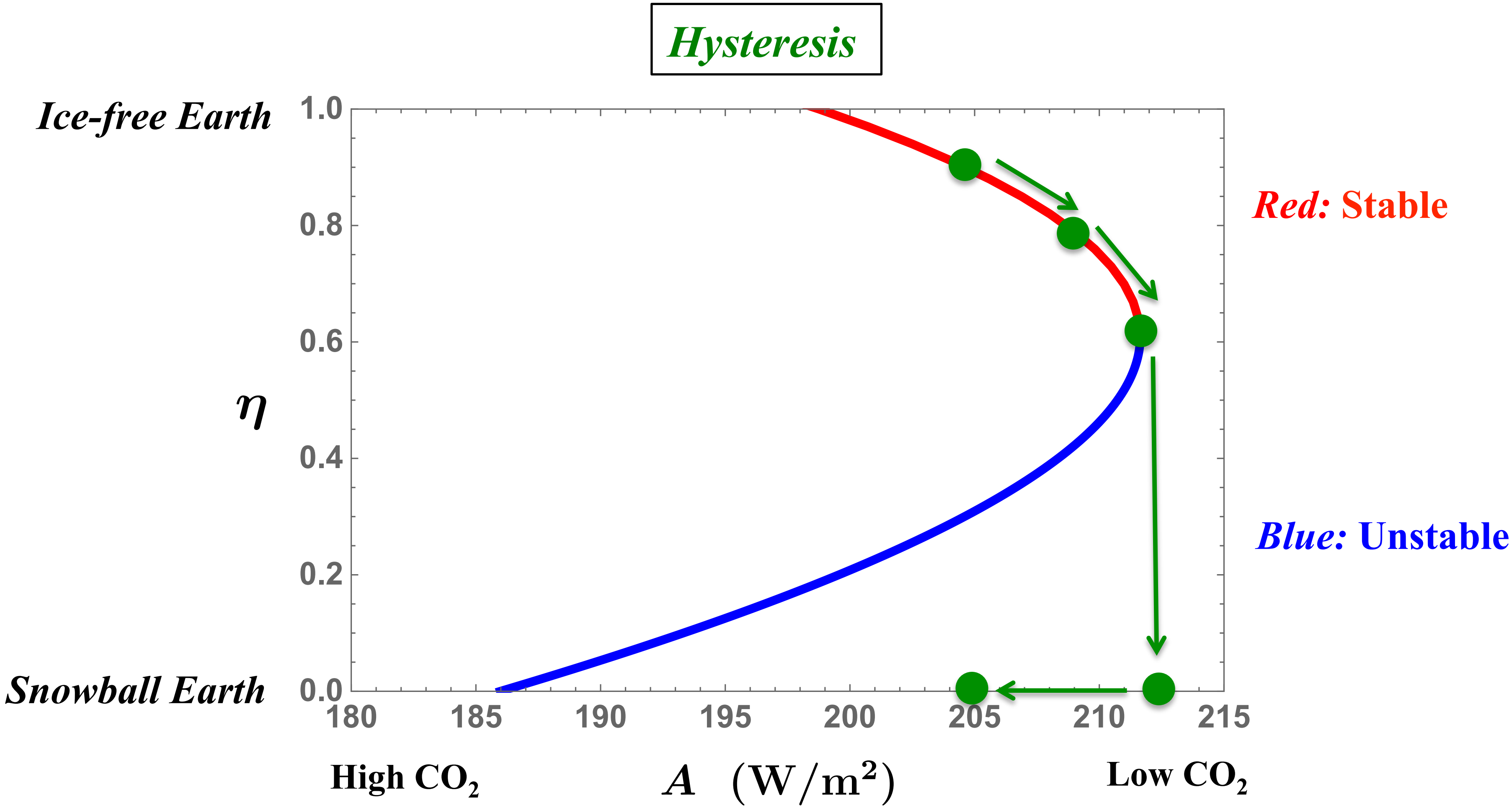
Snowball Earth!

Approximating system of ODEs

Bifurcation Diagram



Approximating system of ODEs

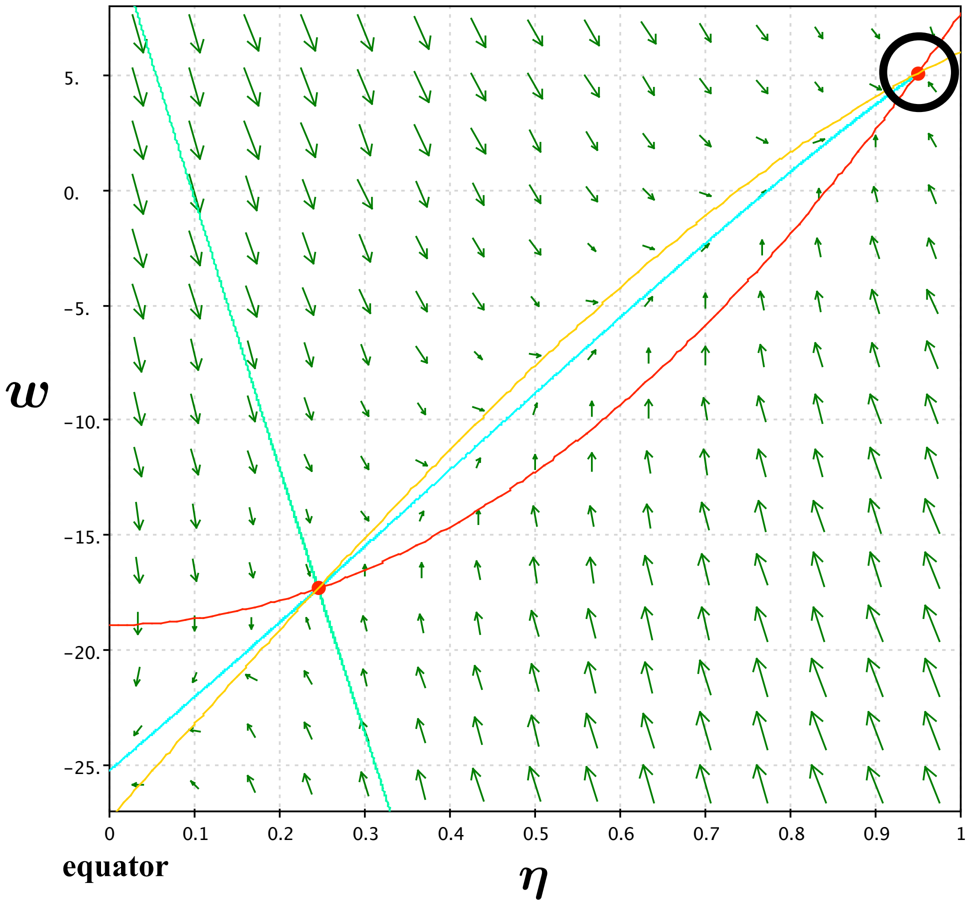


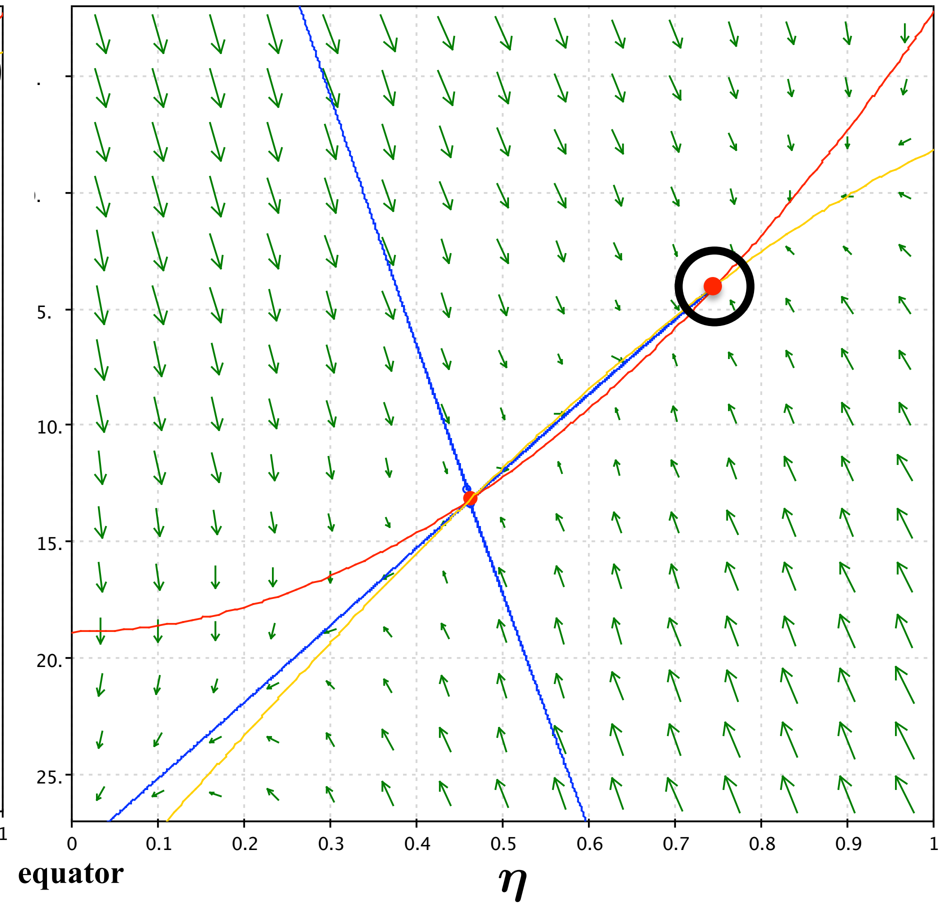
$$\dot{w} = -\tau(w - F(\eta))$$
$$\dot{\eta} = \epsilon(w - G(\eta)),$$

Saturday, October 1 MS13

Mathematics and Conceptual Climate Modeling

9:30-9:55 Conceptual Models: Understanding Past Climate Through Mathematics, Esther Widiasih





Meridional heat transport

$$R\frac{\partial T}{\partial t} = E_{\text{in}} - E_{\text{out}} - E_{\text{transport}}$$

$$= Qs(y)(1-\alpha_\eta(y)) - (A + BT(y,t)) - C\left(T(y,t) - \overbrace{\int_0^1 T(y,t)dy}^{\overline{T}}\right)$$

- $E_{\text{transport}} = C(T - \overline{T})$ (*relaxation to the mean*)
- $E_{\text{transport}} = D\nabla^2 T = D\frac{\partial}{\partial y}(1-y^2)\frac{\partial T}{\partial y}$ (*diffusion process*)

Models: Heat flux resulting from horizontal redistribution by the circulations of the oceans and atmosphere

Thermohaline Circulation

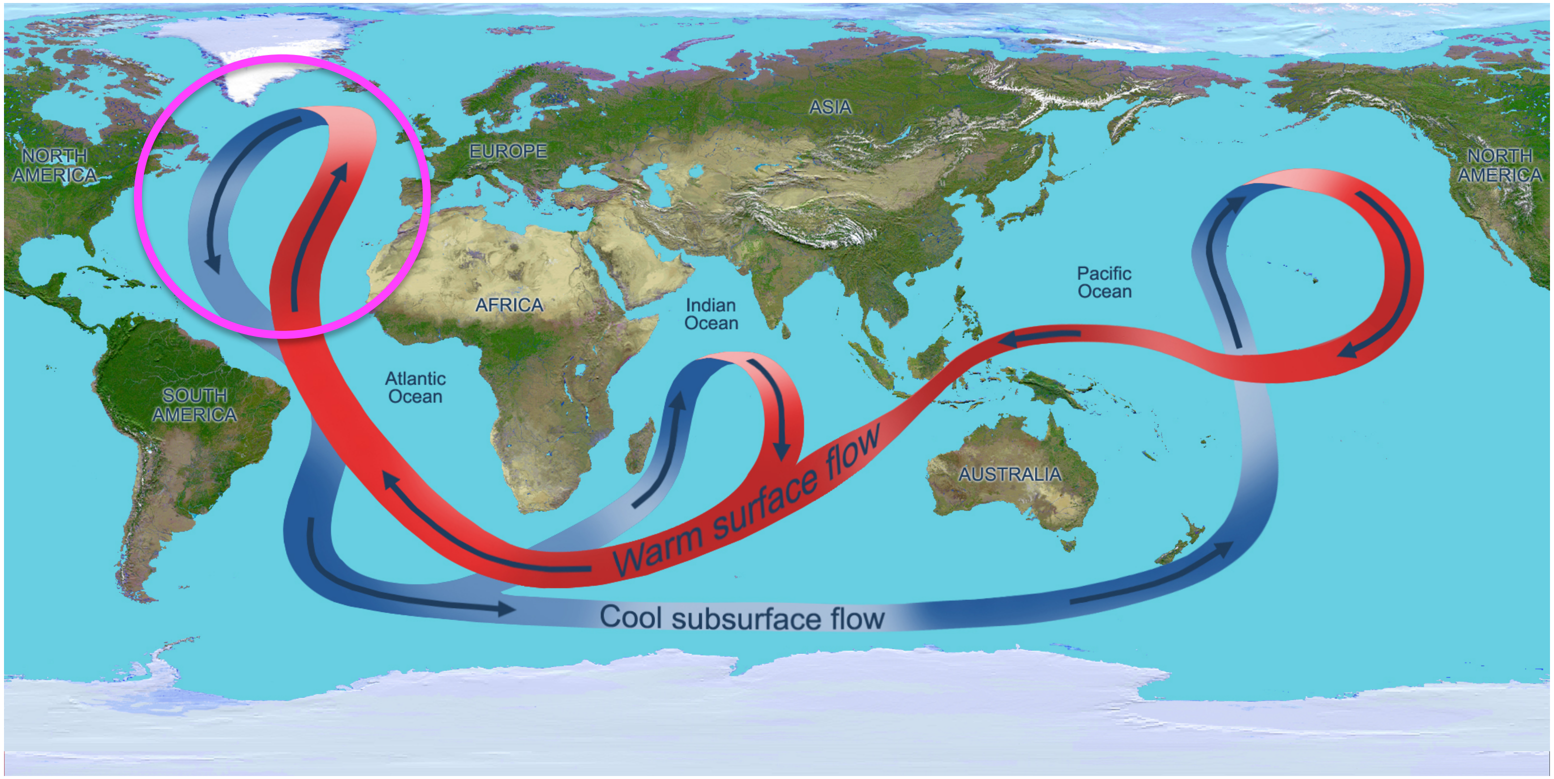


Image Credit: *NASA*

thermo: heat
haline: salt

The rate of circulation is a function of the temperature and salinity and can change over time.

Atlantic Meridional Overturning Circulation

Part of the
thermohaline circulation

illustration by jg. Source for Earth's topology: NASA/JPL-Caltech

An ocean box model

Resources:

- H. Stommel, Thermohaline convection with two stable regimes of flow, *Tellus* XIII (2) (1961)

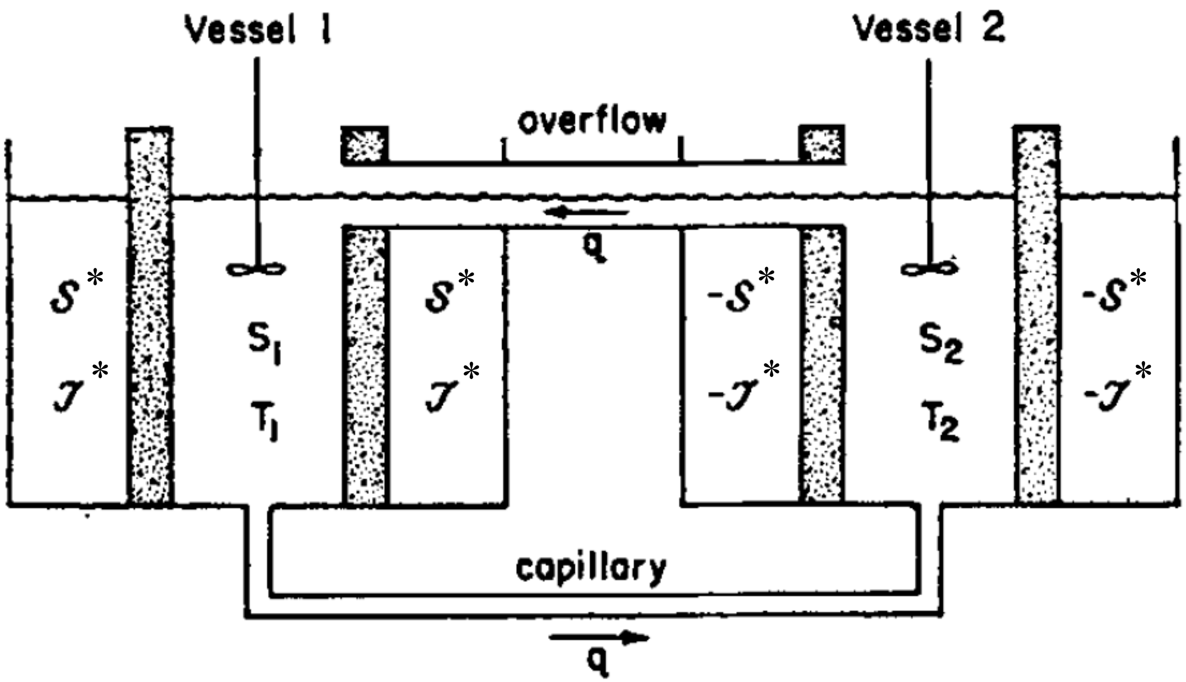
- https://mcrn.hubzero.org/resources/81#series. Videos of lectures and lecture slides from Introduction to the Mathematics of Climate, taught by Richard McGehee, School of Mathematics, University of Minnesota.

- H. Kaper and H. Engler, *Mathematics and Climate*, SIAM (2013)

Stommel's ocean box model

	bath 1	bath 2
temperature	$\mathcal{T}^*$	$-\mathcal{T}^*$
salinity	$\mathcal{S}^*$	$-\mathcal{S}^*$

Low latitudes **High latitudes**



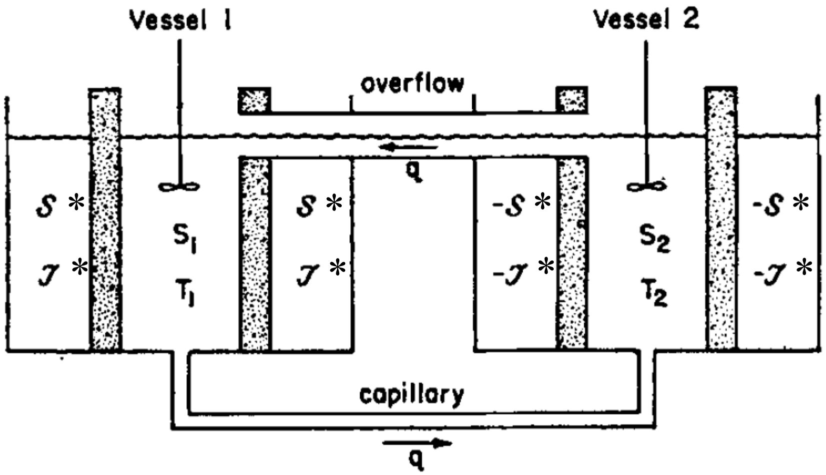
Capillary flow: assumed to be proportional to density differences

Density: Decreases as temperature (T) increases; Inreases as salinity (S) increases

Stommel's ocean box model

Low latitudes

High latitudes



Temperature

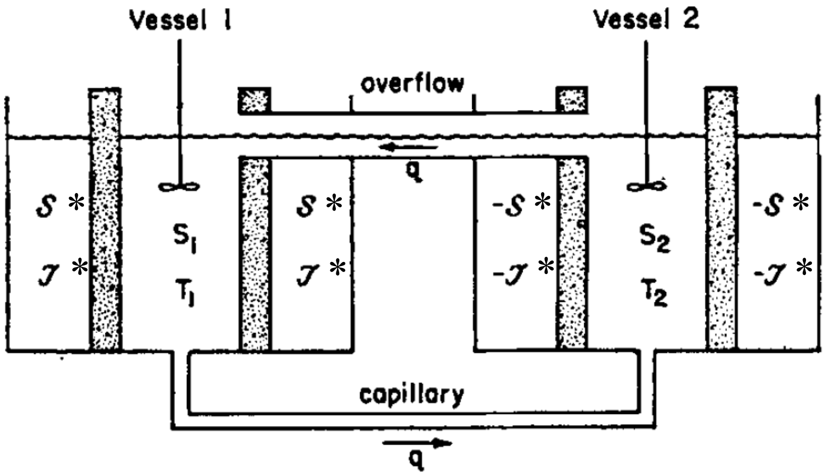
$$\dot{T}_1 = c(\mathcal{T}^* - T_1) - |q|(T_1 - T_2)$$

$$\dot{T}_2 = c(-\mathcal{T}^* - T_2) + |q|(T_1 - T_2)$$

$$\frac{d}{dt}(T_1 + T_2) = -c(T_1 + T_2) \;\Rightarrow\; T_1 + T_2 \to 0.$$

Set $T_1 = -T_2 = T$

Stommel's ocean box model



Temperature

$$\dot{T}_1 = c(\mathcal{T}^* - T_1) - |q|(T_1 - T_2)$$
$$\dot{T}_2 = c(-\mathcal{T}^* - T_2) + |q|(T_1 - T_2)$$

Salinity

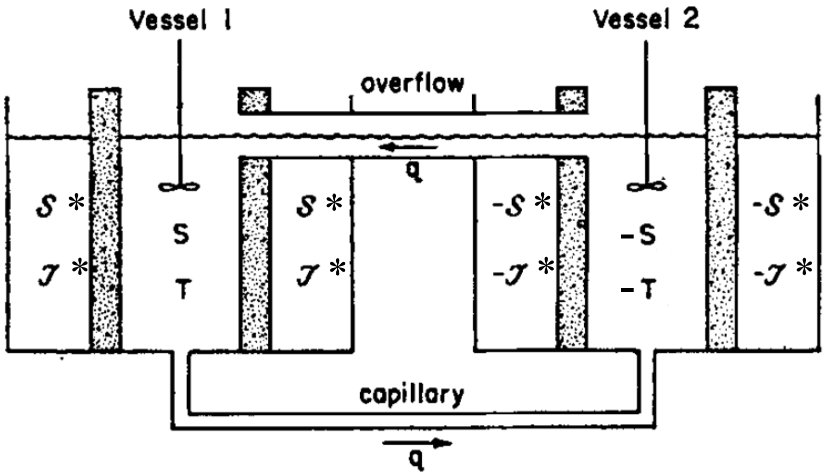
$$\dot{S}_1 = d(\mathcal{S}^* - S_1) - |q|(S_1 - S_2)$$
$$\dot{S}_2 = d(-\mathcal{S}^* - S_2) + |q|(S_1 - S_2)$$

$$\frac{d}{dt}(T_1 + T_2) = -c(T_1 + T_2) \;\Rightarrow\; T_1 + T_2 \to 0.$$

Set $T_1 = -T_2 = T$

Similarly, set $S_1 = -S_2 = S$

Stommel's ocean box model



$$\dot{T} = c(\mathcal{T}^* - T) - 2|q|T$$
$$\dot{S} = d(\mathcal{S}^* - S) - 2|q|S$$

Flow rate q: proportional to density differences

$$kq = \rho_1 - \rho_2$$

Density $\rho = \rho(T,S) = \rho_0(1 - \alpha T + \beta S), \quad \alpha, \beta > 0$

[ρ_0 reference density]

Flow rate: $kq = \rho_1 - \rho_2 = 2\rho_0(-\alpha T + \beta S)$

Stommel's ocean box model

$$\dot{T} = c(\mathcal{T}^* - T) - 2|q|T$$
$$\dot{S} = d(\mathcal{S}^* - S) - 2|q|S$$

$$kq = 2\rho_0(-\alpha T + \beta S)$$

Nondimensionalize … and more!

flow rate

$$y = \frac{T}{\mathcal{T}^*}, \; x = \frac{S}{\mathcal{S}^*}, \; \tau = ct, \; \delta = \frac{d}{c}, \; f = 2\frac{q}{c}$$

$$\frac{dx}{d\tau} = \delta(1 - x) - |f|x$$
$$\frac{dy}{d\tau} = 1 - y - |f|y$$

Stommel's ocean box model

$$y = \frac{T}{\mathcal{T}^*},\ x = \frac{S}{\mathcal{S}^*},\ \tau = ct,\ \delta = \frac{d}{c},\ f = 2\frac{q}{c}$$

$$\frac{dx}{d\tau} = \delta(1-x) - |f|x$$

$$\frac{dy}{d\tau} = 1 - y - |f|y$$

$$kq = 2\rho_0(-\alpha T + \beta S)$$

Flow rate: $kq = 2\rho_0\alpha\mathcal{T}^*(-y + Rx) = k\frac{cf}{2}$

$$\left(R = \frac{\beta\mathcal{S}^*}{\alpha\mathcal{T}^*}\right)$$

$$f = \frac{1}{\lambda}(-y + Rx), \quad \lambda = \frac{ck}{4\rho_0\alpha\mathcal{T}^*}$$

Stommel's ocean box model

$$y = \frac{T}{\mathcal{T}^*}, \quad x = \frac{S}{\mathcal{S}^*}$$

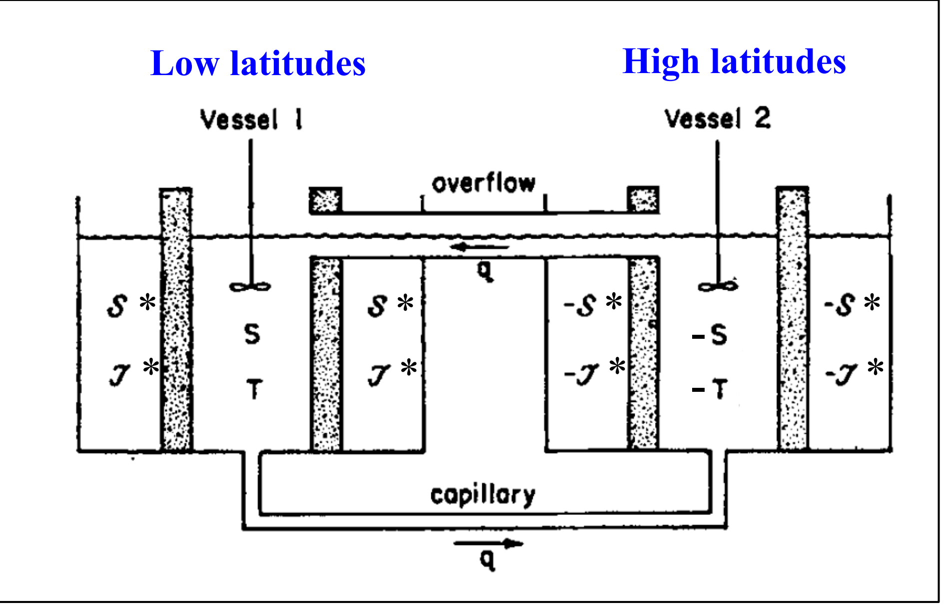
$$\frac{dx}{d\tau} = \delta(1 - x) - |f|x$$
$$\frac{dy}{d\tau} = 1 - y - |f|y$$

$$f = \frac{1}{\lambda}(-y + Rx), \quad \lambda = \frac{ck}{4\rho_0 \alpha \mathcal{T}^*}$$

At equilibrium: $$x^* = \frac{\delta}{\delta + |f|}, \quad y^* = \frac{1}{1 + |f|}$$

$$\lambda f = -y^* + Rx^* = -\frac{1}{1 + |f|} + R\frac{\delta}{\delta + |f|} = \phi(f, R, \delta)$$

Solutions of $\lambda f = \phi(f, R, \delta)$ yield equilibria (x^*, y^*)



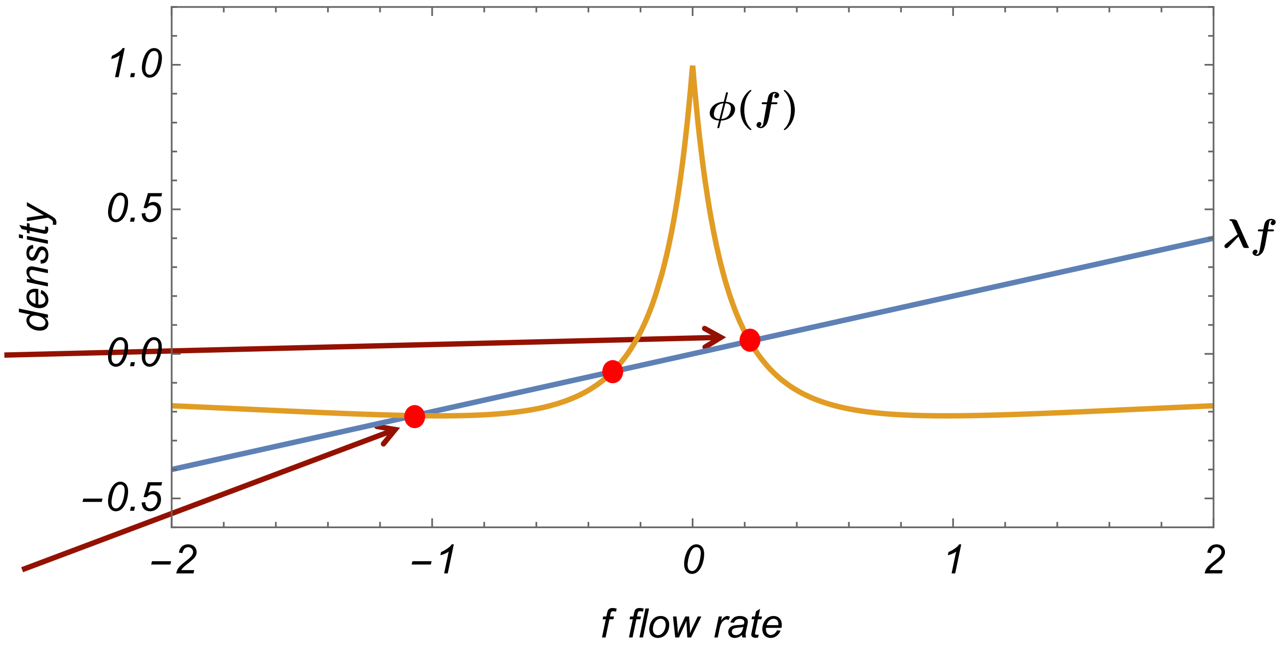
$$y = \frac{T}{\mathcal{T}^*},\ x = \frac{S}{\mathcal{S}^*}$$

$$\lambda f = \phi(f; R, \delta) = -\frac{1}{1+|f|} + \frac{R\delta}{\delta + |f|}$$
$$= -y^* + Rx^*$$

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$

salinity dominates
capillary flow: warm to cold

temperature dominates
capillary flow: cold to warm



$$\frac{dx}{d\tau} = \delta(1-x) - |f|x$$

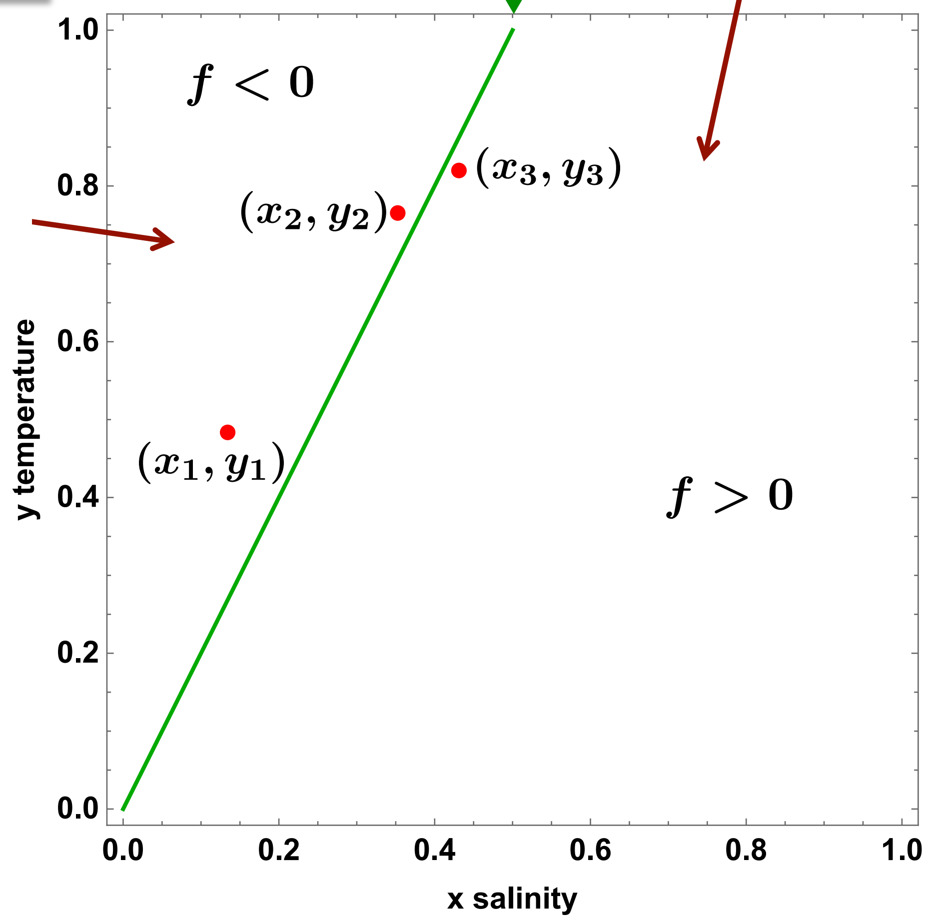
$$\frac{dy}{d\tau} = 1 - y - |f|y$$

$$y = \frac{T}{\mathcal{T}^*}, \quad x = \frac{S}{\mathcal{S}^*}$$

$f = 0$

salinity dominates
capillary flow: warm to cold

temperature dominates
capillary flow: cold to warm



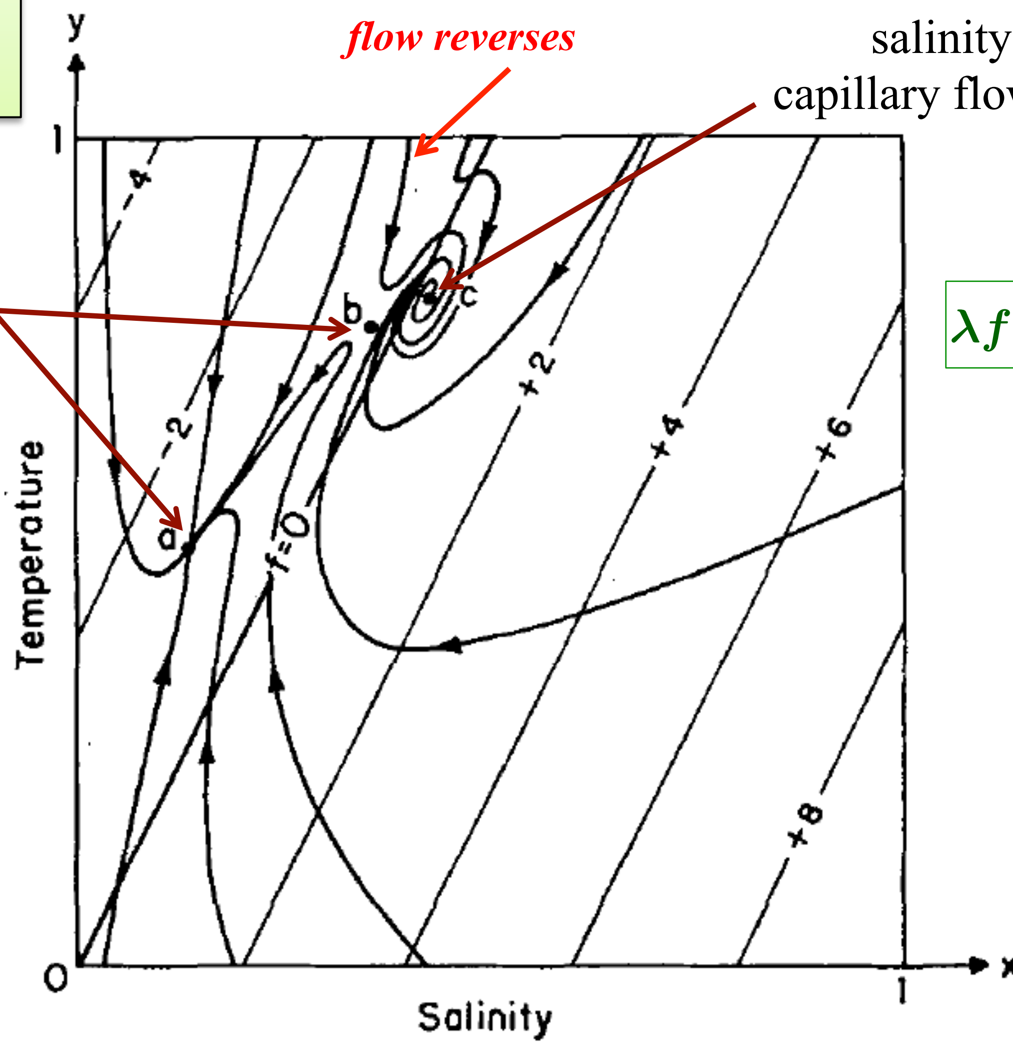
$$\lambda f = -y + Rx$$

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$

$$\frac{dx}{d\tau} = \delta(1-x) - |f|x$$

$$\frac{dy}{d\tau} = 1 - y - |f|y$$

$$y = \frac{T}{\mathcal{T}^*}, \quad x = \frac{S}{\mathcal{S}^*}$$



flow reverses

salinity dominates
capillary flow: warm to cold

temperature dominates
capillary flow: cold to warm

$$\lambda f = -y + Rx$$

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$

Stommel, *Tellus* (1961)

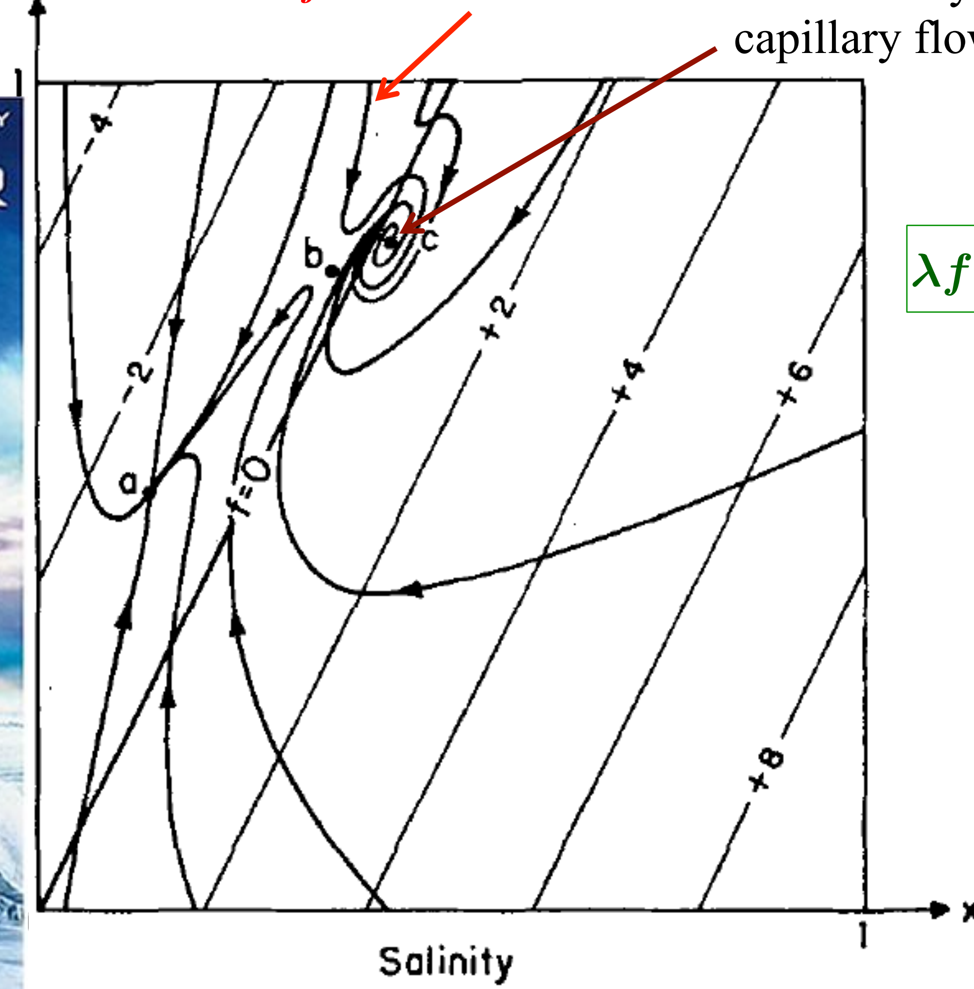
$$\frac{dx}{d\tau} = \delta(1 - x) - |f|x$$

$$\frac{dy}{d\tau} = 1 - y - |f|y$$

$$y = \frac{T}{\mathcal{T}^*}, \; x = \frac{S}{\mathcal{S}^*}$$

flow reverses

salinity dominates
capillary flow: warm to cold



$$\lambda f = -y + Rx$$

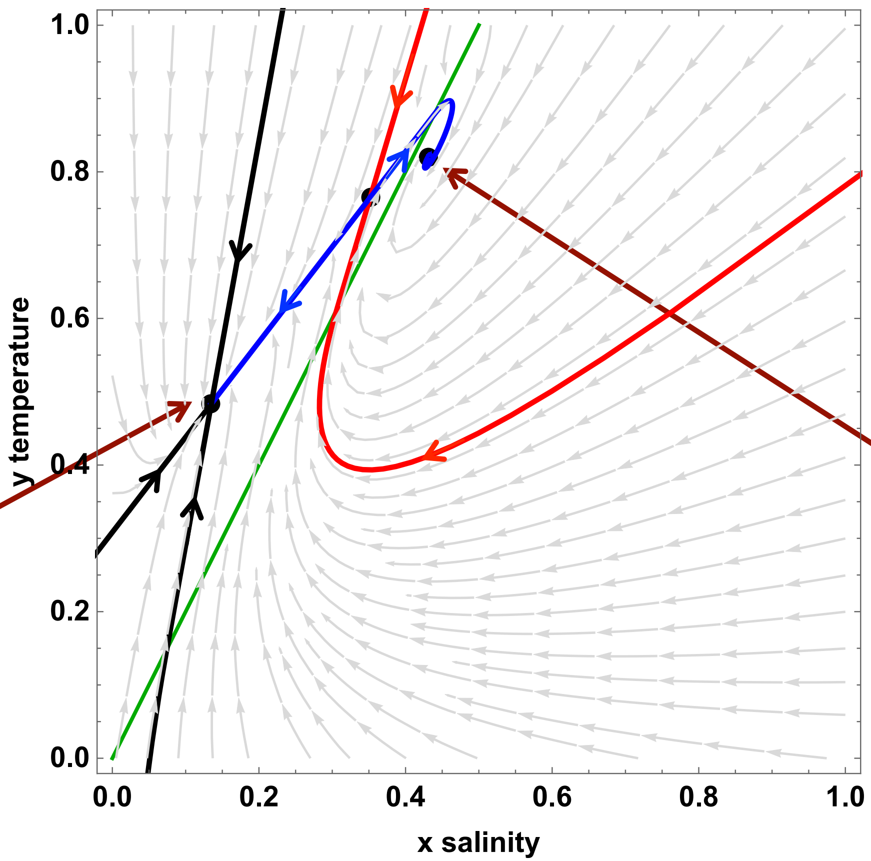
$$\frac{dx}{d\tau} = \delta(1-x) - |f|x$$

$$\frac{dy}{d\tau} = 1 - y - |f|y$$

$$y = \frac{T}{\mathcal{T}^*}, \quad x = \frac{S}{\mathcal{S}^*}$$

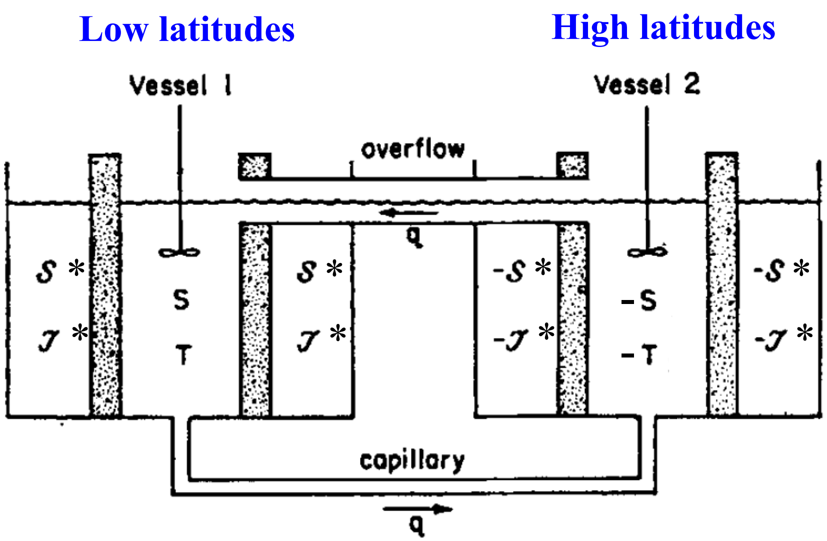
$$\lambda f = -y + Rx$$

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$



Gulf Stream flowing north

Gulf Stream flowing south



$$\frac{dx}{d\tau} = \delta(1-x) - |f|x$$
$$\frac{dy}{d\tau} = 1 - y - |f|y$$

$$\lambda f = -y + Rx$$

$$y = \frac{T}{\mathcal{T}^*},\ x = \frac{S}{\mathcal{S}^*},\ \tau = ct,\ \delta = \frac{d}{c},\ f = 2\frac{q}{c},\ \lambda = \frac{ck}{4\rho_0 \alpha \mathcal{T}^*}$$

Flow rate: $kq = \rho_1 - \rho_2$

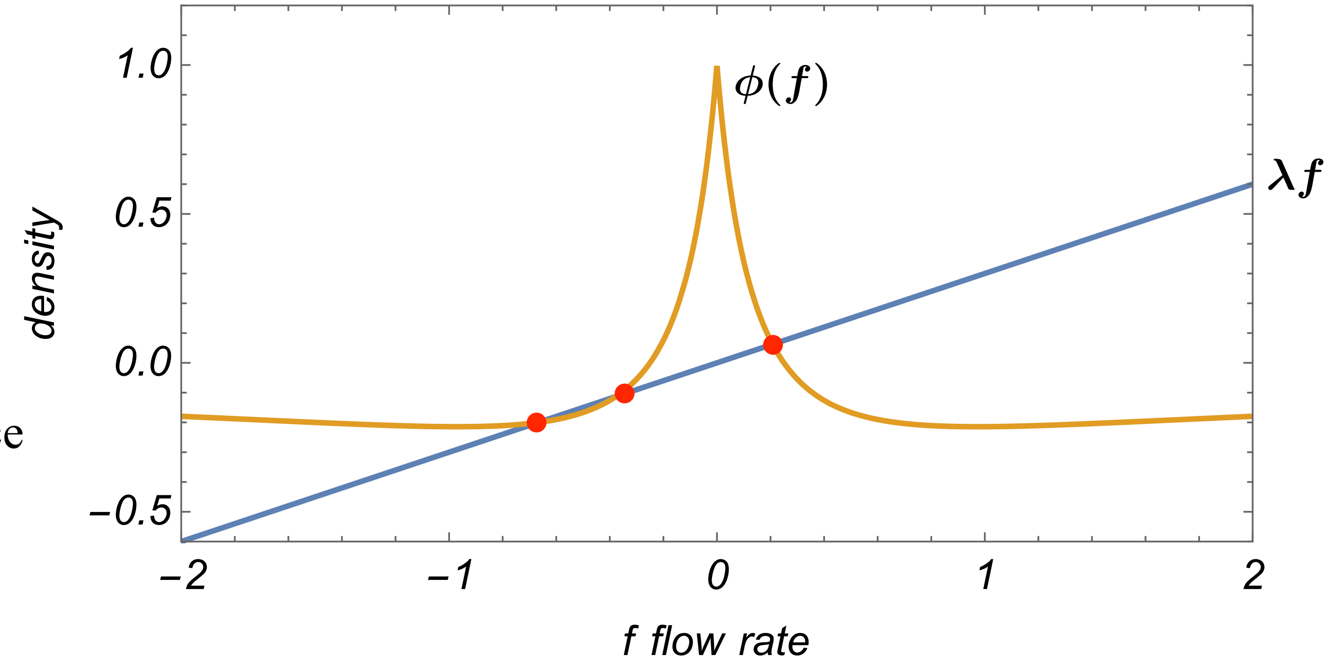
Increase k $\Rightarrow$ increase λ : Increase the resistance in the capillary

Stommel's model: Bifurcations

$$\lambda f = \phi(f; R, \delta) = -\frac{1}{1+|f|} + \frac{R\delta}{\delta + |f|} = -y^* + Rx^*$$

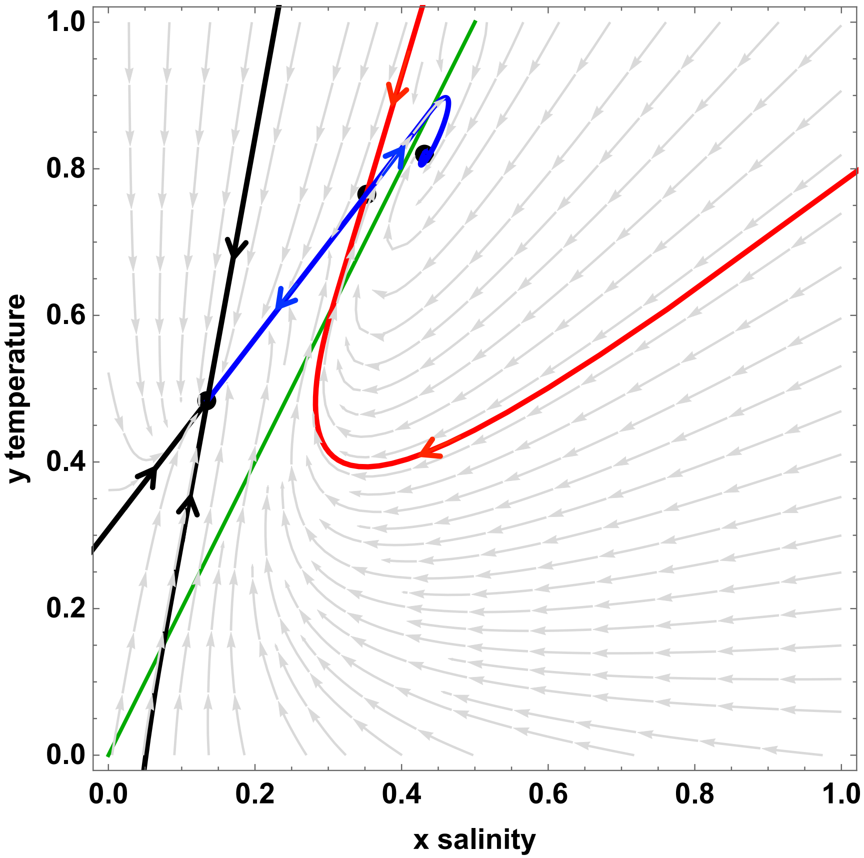
$\delta = 1/6$
$R = 2$
$\lambda = 0.3$

increased resistance from 0.2 to 0.3

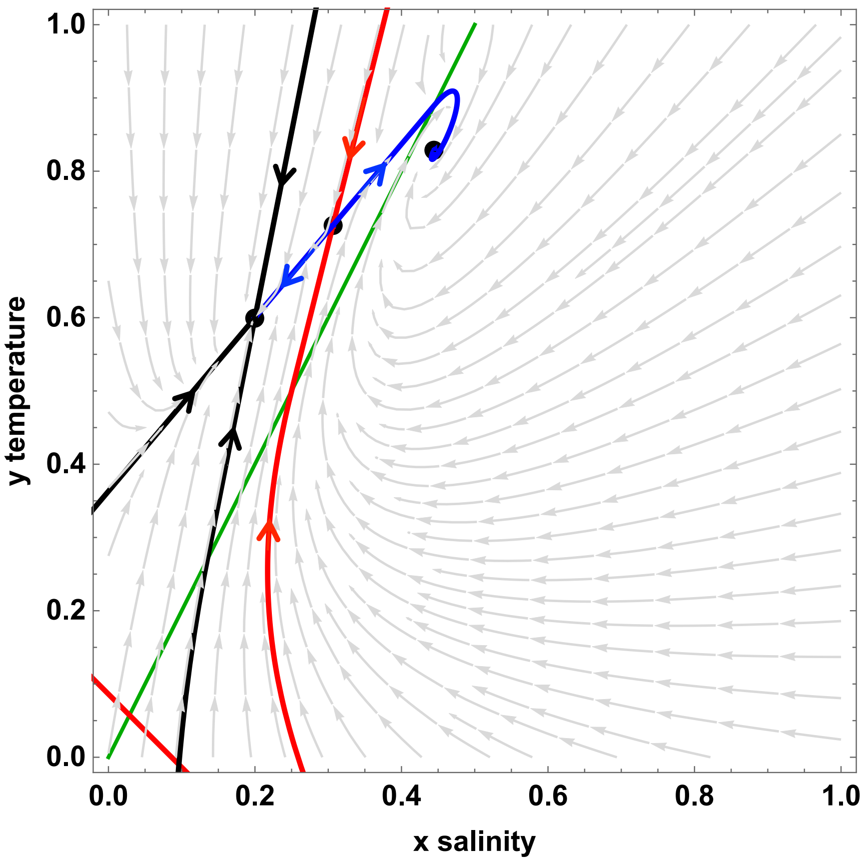


Stommel's model: Bifurcations

$\delta = 1/6$
$R = 2$
$\lambda = 1/5$



$\delta = 1/6$
$R = 2$
$\lambda = 0.3$

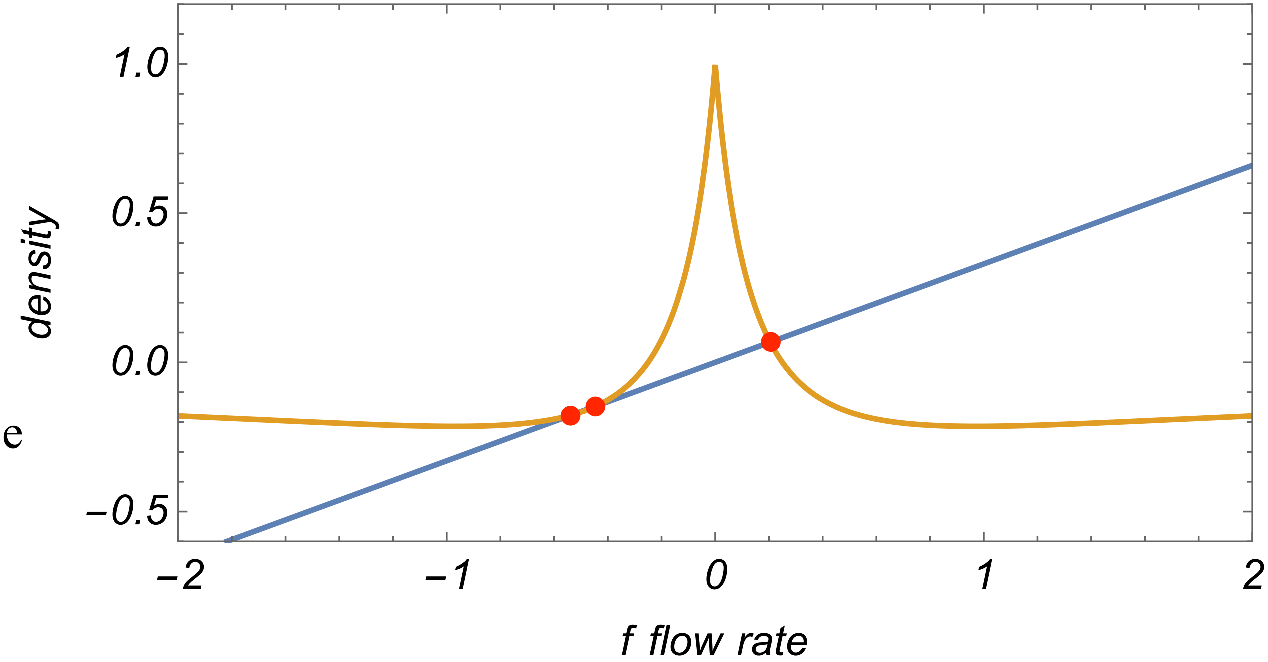


Stommel's model: Bifurcations

$$\lambda f = \phi(f; R, \delta) = -\frac{1}{1+|f|} + \frac{R\delta}{\delta+|f|} = -y^* + Rx^*$$

$\delta = 1/6$
$R = 2$
$\lambda = 0.33$

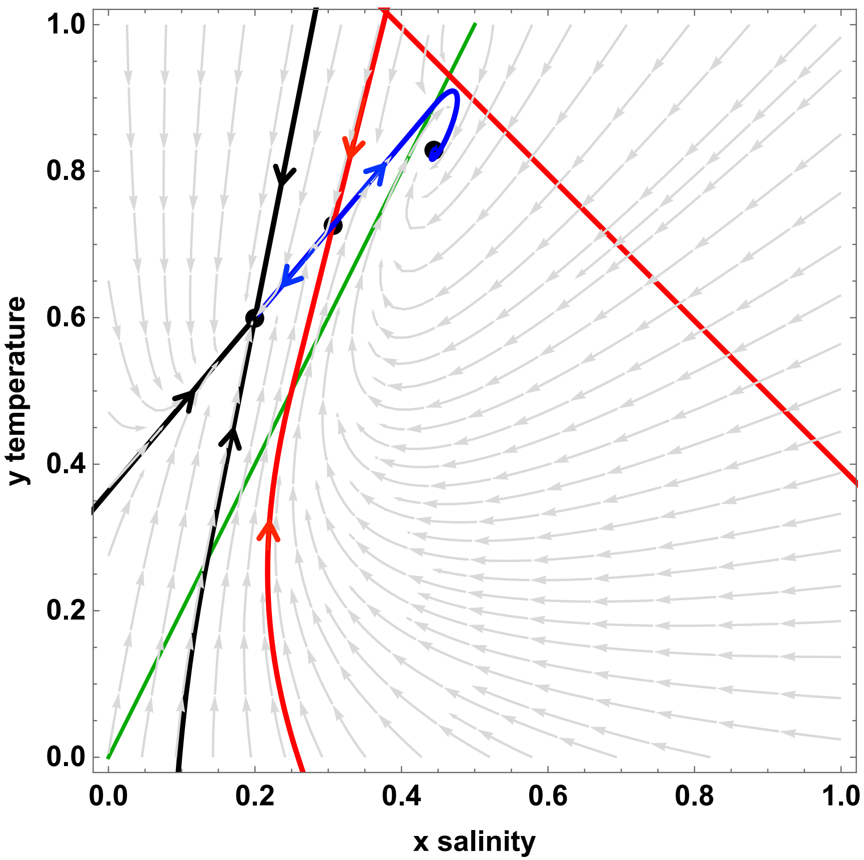
increased resistance from 0.3 to 0.33



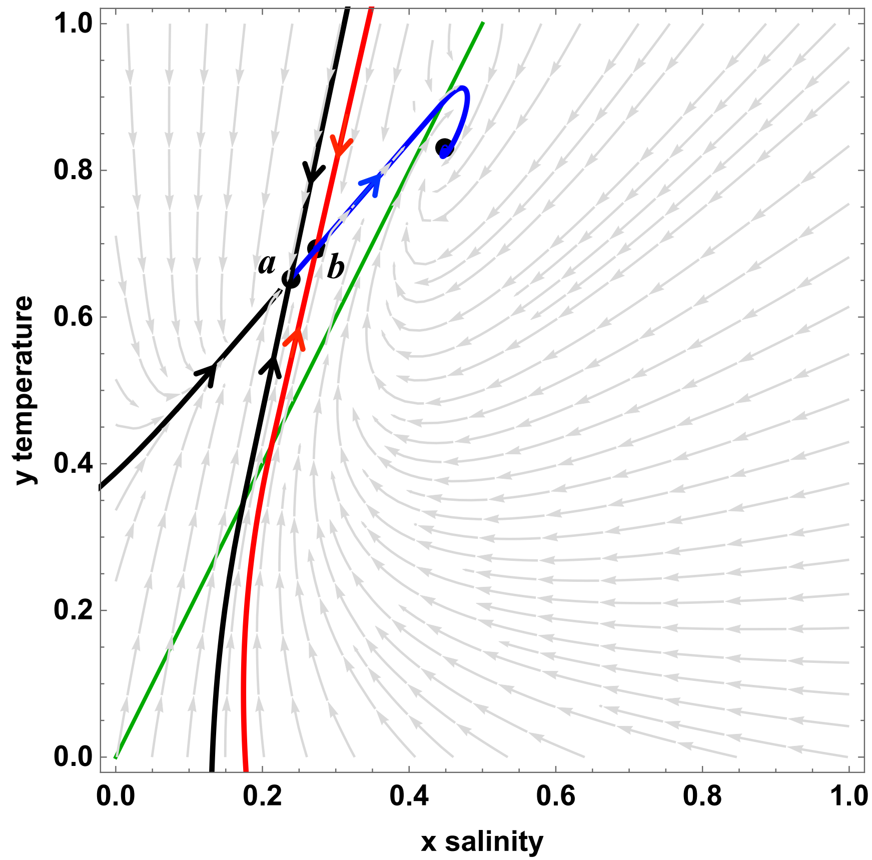
Stommel's model: Bifurcations

The stable node *a* and the saddle *b* begin to merge

$\delta = 1/6$
$R = 2$
$\lambda = 0.3$



$\delta = 1/6$
$R = 2$
$\lambda = 0.33$

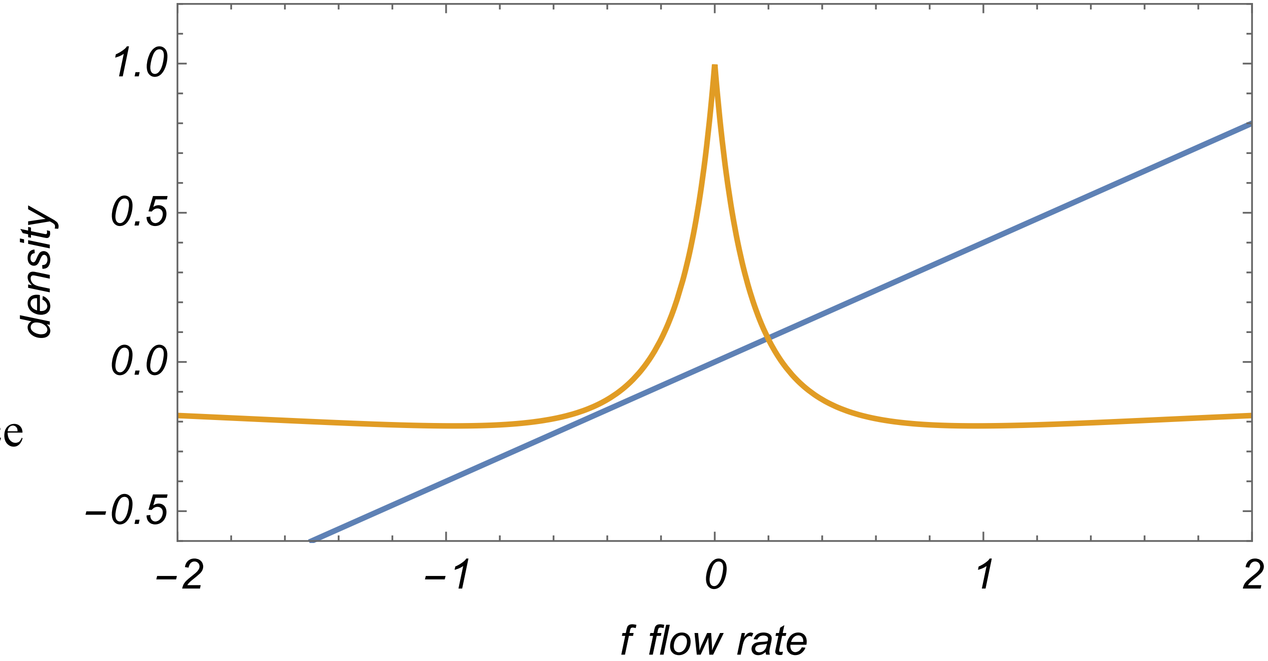


Stommel's model: Bifurcations

$$\lambda f = \phi(f; R, \delta) = -\frac{1}{1+|f|} + \frac{R\delta}{\delta + |f|} = -y^* + Rx^*$$

$\delta = 1/6$
$R = 2$
$\lambda = 0.4$

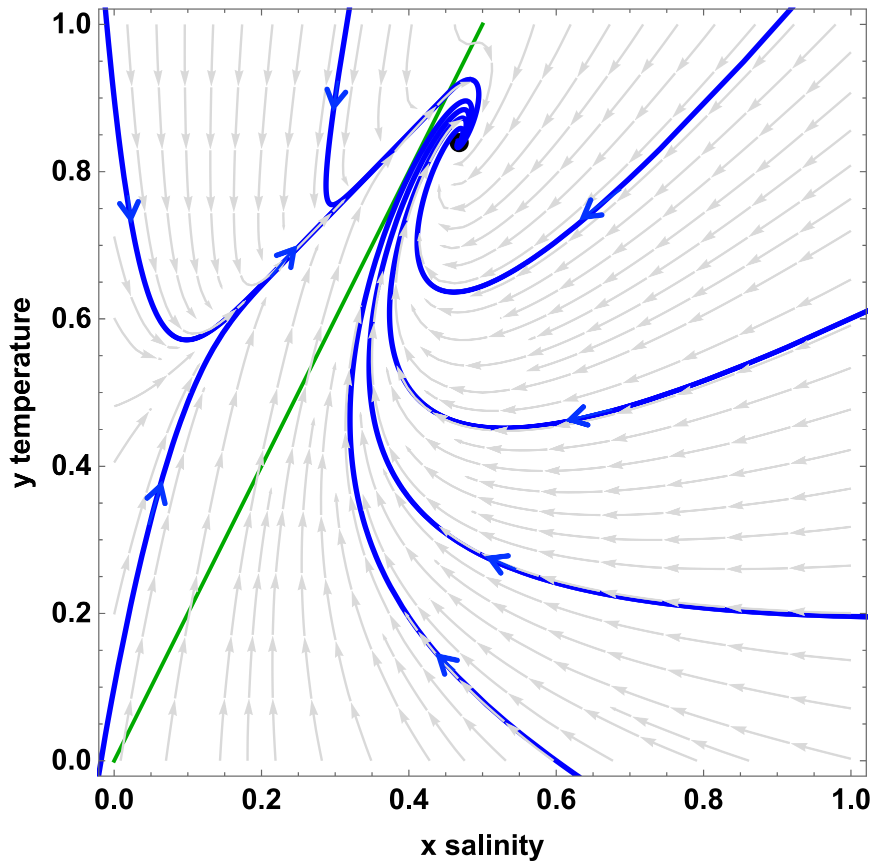
increased resistance from 0.33 to 0.4



Stommel's model: Bifurcations

$\delta = 1/6$
$R = 2$
$\lambda = 0.4$

Increase the flow resistance sufficiently and the stable node and saddle disappear. The Gulf Stream will eventually reverse.



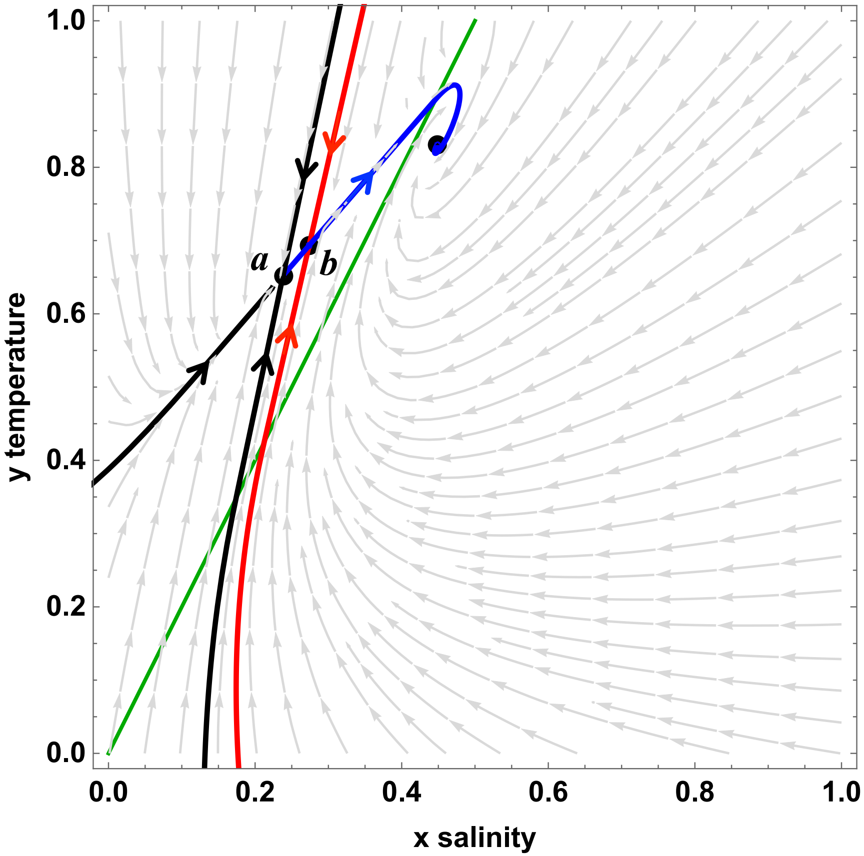
Now decrease λ $\Rightarrow$ *decrease* the resistance in the capillary

Stommel's model: Hysteresis

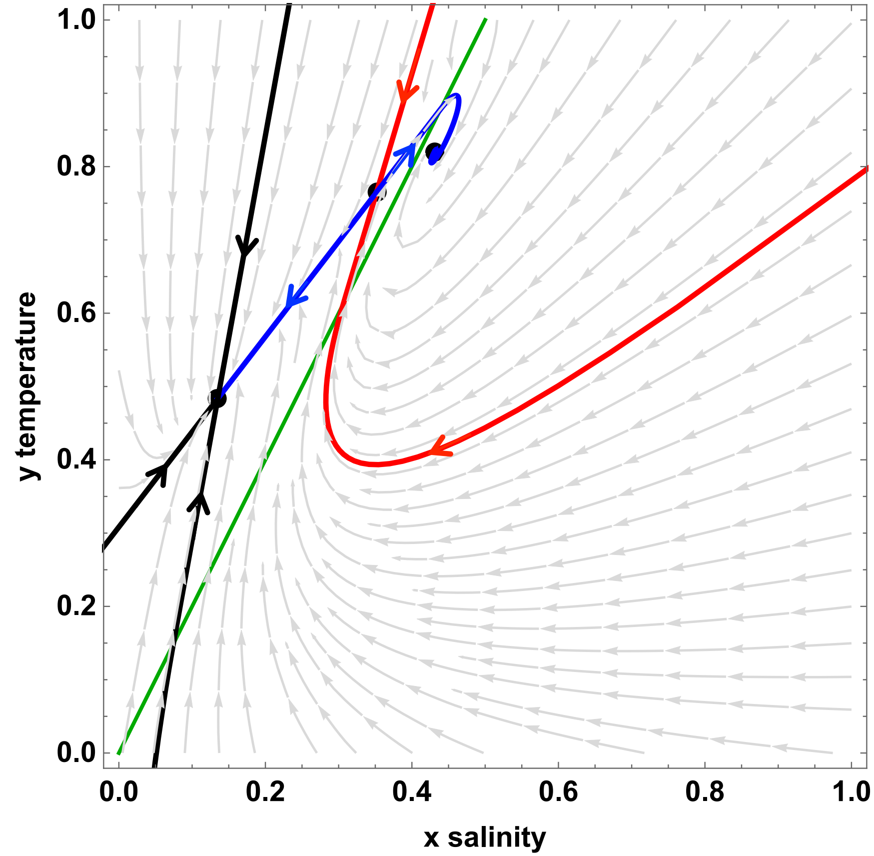
The stable node ***a*** and the saddle ***b*** have remerged, but it is difficult to get to ***a***. The Gulf Stream is still reversed.

We are back to our original parameters, but the Gulf Stream is still reversed.

$\lambda = 0.33$



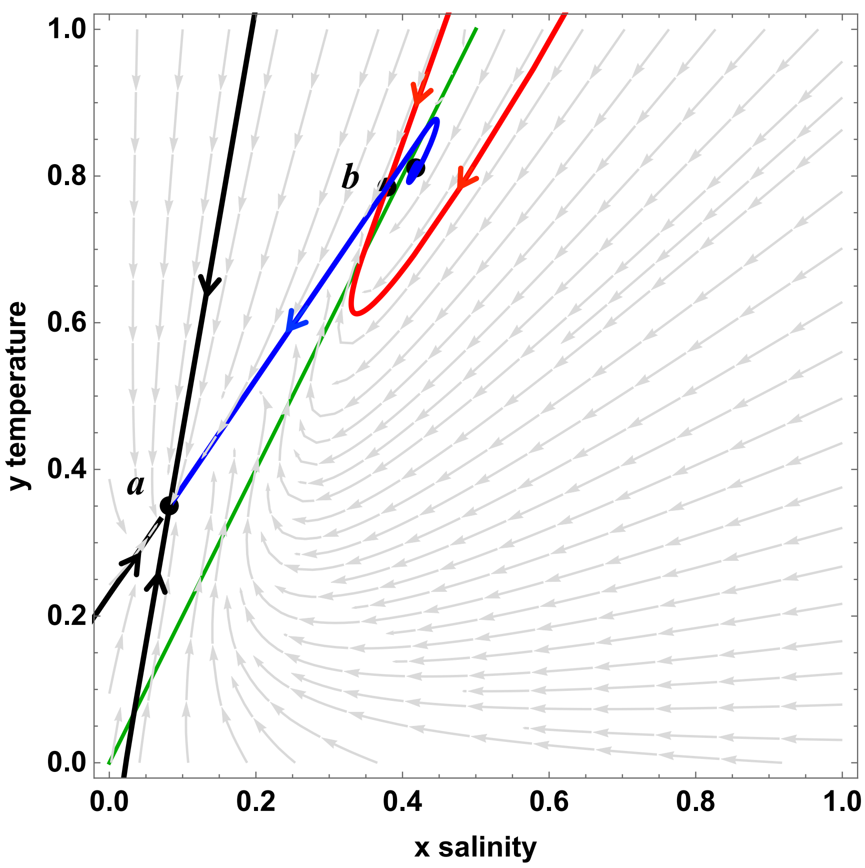
$\lambda = 0.2$



Stommel's model: Hysteresis

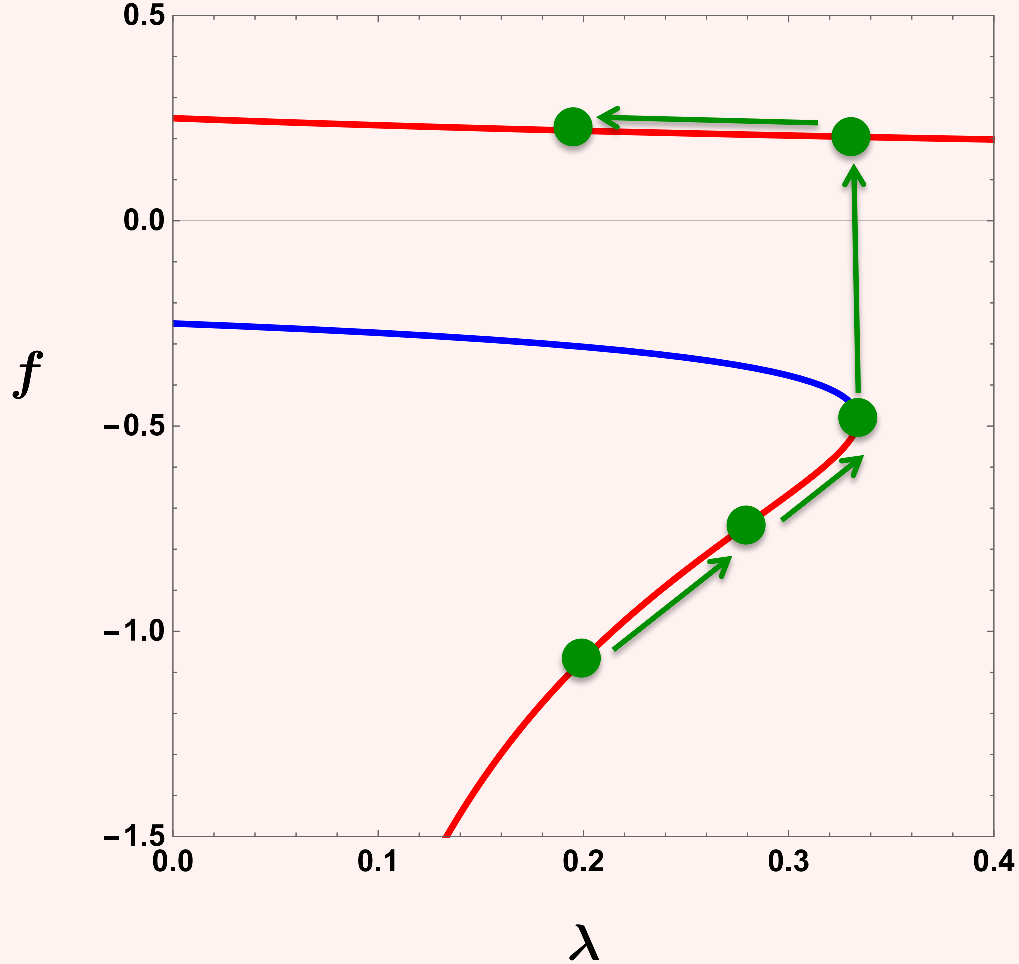
$\delta = 1/6$
$R = 2$
$\lambda = 0.1$

The flow resistance is below the original value. Point ***a*** is the dominant attractor. Perhaps the Gulf Stream will find a way to return to normal.



Stommel's model: Hysteresis

Bifurcation Diagram



Red: Stable

Blue: Unstable

Atlantic Meridional Overturning Circulation

L. G. Henry, J. F. McManus, W. B. Curry, N. L. Roberts, A. M. Piotrowski, L. D. Keigwin, North Atlantic ocean circulation and abrupt climate change during the last glaciation, *Science*, June 30, 2016.

"The last ice age wasn't one long big chill. Dozens of times temperatures abruptly rose or fell, causing all manner of ecological change.

Now, scientists have implicated the culprit behind those seesaws—changes to a conveyor belt of ocean currents known as the Atlantic Meridional Overturning Circulation (AMOC).

These currents, which today drive the Gulf Stream, bring warm surface waters north and send cold, deeper waters south. ***But they weakened suddenly and drastically, nearly to the point of stopping, just before several periods of abrupt climate change****, researchers report today in Science."*

Eric Hand, http://www.sciencemag.org/news/2016/06/crippled-atlantic-conveyor-triggered-ice-age-climate-change

Advertisement

Thank you for your attention!

Saturday, October 1

MS13
Mathematics and Conceptual Climate Modeling

9:30 AM - 11:30 AM Room: Concerto B - 3rd Floor

9:30-9:55 Conceptual Models: Understanding Past Climate Through Mathematics ***Esther Widiasih***

10:00-10:25 Peatlands, Agriculture, and the Carbon Budget: A Conceptual Model for 15kyr Bp to the Present, ***Alice Nadeau***

10:30-10:55 Palaeoclimate Dynamics Modelled with Delay Equations, ***Courtney Quinn***

11:00-11:25 Improved Validation of Conceptual Climate Models Using Data Analysis Techniques, ***Charles D. Camp***