Hybrid Large Deformation Diffeomorphic Metric Mapping

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LDDMM

The “large deformation diffeomorphic metric mapping” method is a family of algorithms designed for shape registration.

They provide a (local) representation of shape space in the diffeomorphism group.

They are routinely used in Computational Anatomy to study organ shape variation in relation to disease using medical images.

Notation and assumptions follow recent papers from S. Arguillère et al., and S. Arguillère’s dissertation.

Basic Principles of LDDMM

Denote by $Diff_0^p$ the space of diffeomorphisms ϕ in R^d who

- are C^p
- Are such that $\phi - id$ and its derivatives of order p or less tend to 0 at infinity
- $Diff_0^p - id$ is an open subset of $(C_0^p(\mathbb{R}^d, \mathbb{R}^d), \|\|_{p,\infty})$

Let V be a Hilbert space continuously included in $(C_0^p(\mathbb{R}^d, \mathbb{R}^d), \|\|_{p,\infty})$ for some $p \geq 1$.

Consider on $Diff_0^p$ the distribution
$\phi \mapsto V_\phi = V \circ \phi = \{v \circ \phi,\ v \in V\}$

with sub-Riemannian metric $\|v \circ \phi\|_\phi = \|v\|_V$.

Associated diffeomorphism subgroup

Denote by $Diff_V$ the group of attainable diffeomorphisms through finite energy paths $\phi(\cdot)$ such that $\dot{\phi}(t) \in V_{\phi(t)}$ and $\int_0^1 \|\dot{\phi}\|_{\phi}\, dt < \infty$

Basic example

Let $K:\mathbb{R}^d \times \mathbb{R}^d \to M_d(\mathbb{R}^d)$ be a positive kernel: $K(x,y) = K(y,x)^T$ and $\sum_{i,j=1}^{n} a_i^T K(x_i, x_j) a_j \geq 0$

for all $x_1, \ldots, x_n$, $a_1, \ldots, a_n \in \mathbb{R}^d$ (with equality only if $a_1 = \ldots = a_n = 0$.)

Take *V* as the associated RKHS

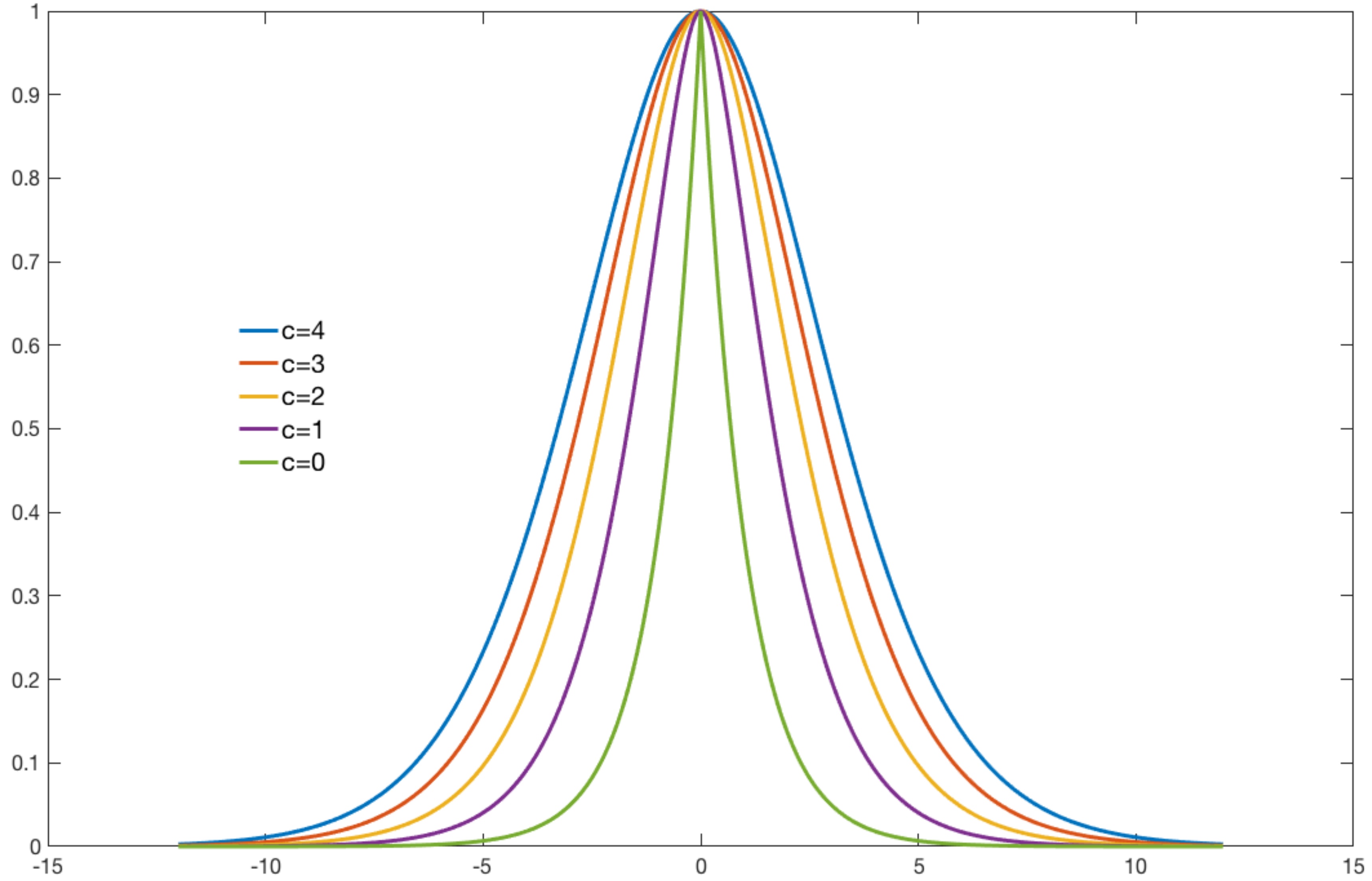
Let $V_\phi = V \circ \phi$.

Choosing V and its norm

Equivalent to choosing the positive kernel.

Gaussian kernel
$K(x,y)=e^{-|x-y|^2/2a^2}\ Id$

Laplacian, or Abel kernels:
$K(x,y)=P_c(|x-y|/a)e^{-|x-y|/2a}\ Id$
where P_c is a reverse Bessel polynomial of degree c.
Equivalent to Sobolev $H^{d+1/2+c}$ in odd dimension.



c=4
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-10
-5
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5
10
15

LDDMM Optimal Control Problem (version 1)

Minimize

$$\int_0^1 \|v(t)\|_V^2 \, dt + U(\phi(1))$$

subject to $\phi(0) = id$ and $\dot{\phi} = v \circ \phi$.

LDDMM Optimal Control Problem (version 2)

Assume that $Diff_0^p$ acts on a "shape space" $\mathfrak{M}$.

Minimize

$$\int_0^1 \|v(t)\|_V^2 \, dt + D(q(1), q_1)$$

subject to $q(0) = q_0$ and $q = v \cdot q$ (infinitesimal action).

Interpretation

LDDMM deforms the whole space in order to move the template to a position close to the target (up to invariance).

The deformation cost treats $\mathbb{R}^d$ as a homogeneous material or fluid.

In particular, this cost does not depend on the deformed objects.

This is good because…

The shape space geometry derives form a right-invariance Riemannian metric on Diff through a Riemannian submersion.

Geodesic equations are well known (EPDiff) and have important conservation laws.

Numerical procedures are well explored.

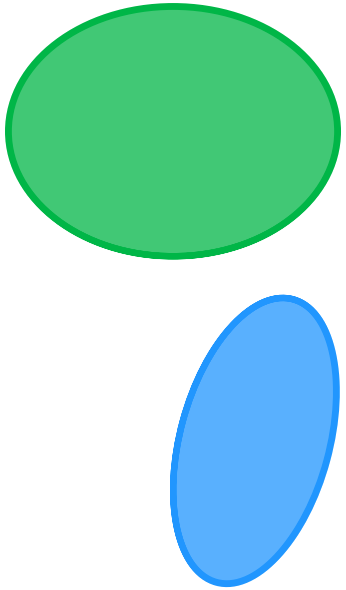
Dependency on shape can be brought in through the data attachment term.

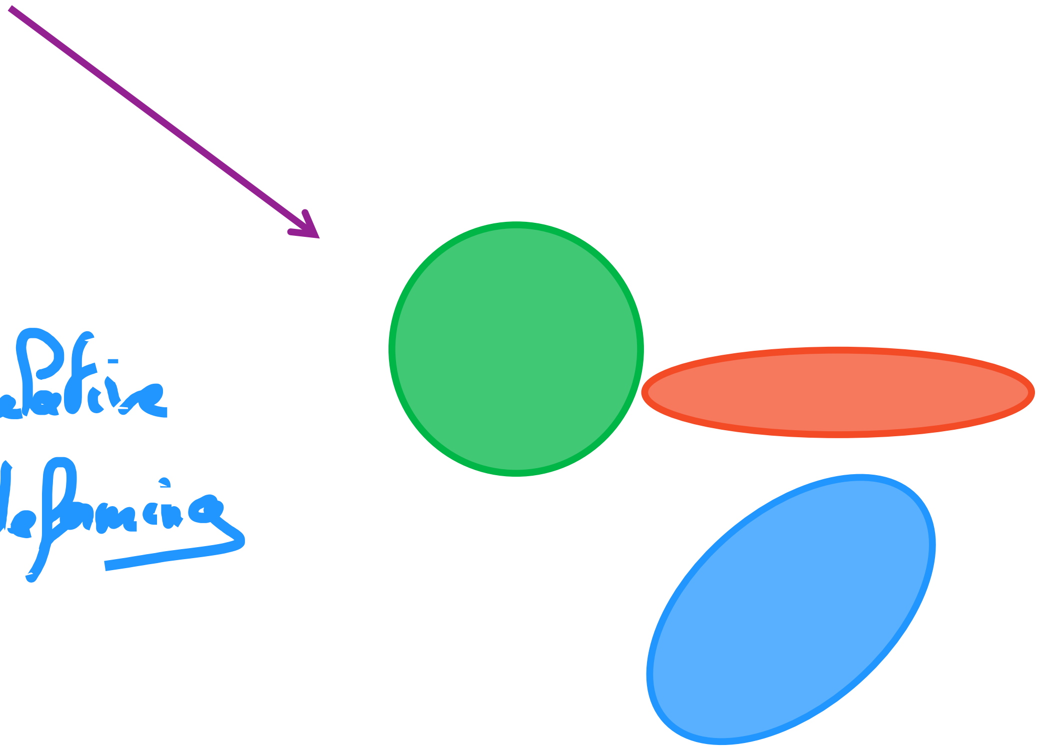
However…

Including object information to drive the deformation process can be beneficial in some important cases such as

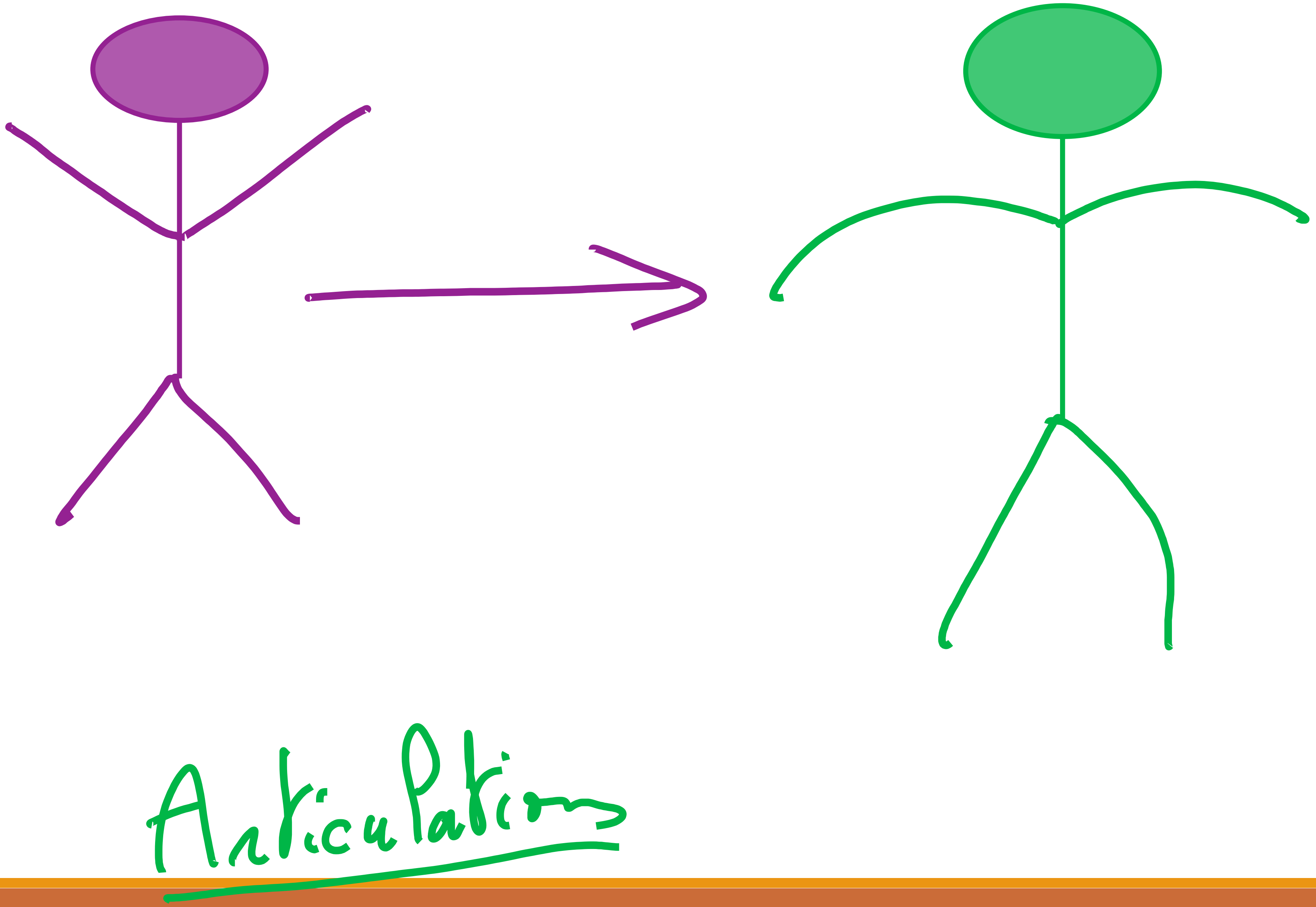
- Shape complexes (multi-shapes)
- Articulated shapes
- Near topological changes

Multi-Shape

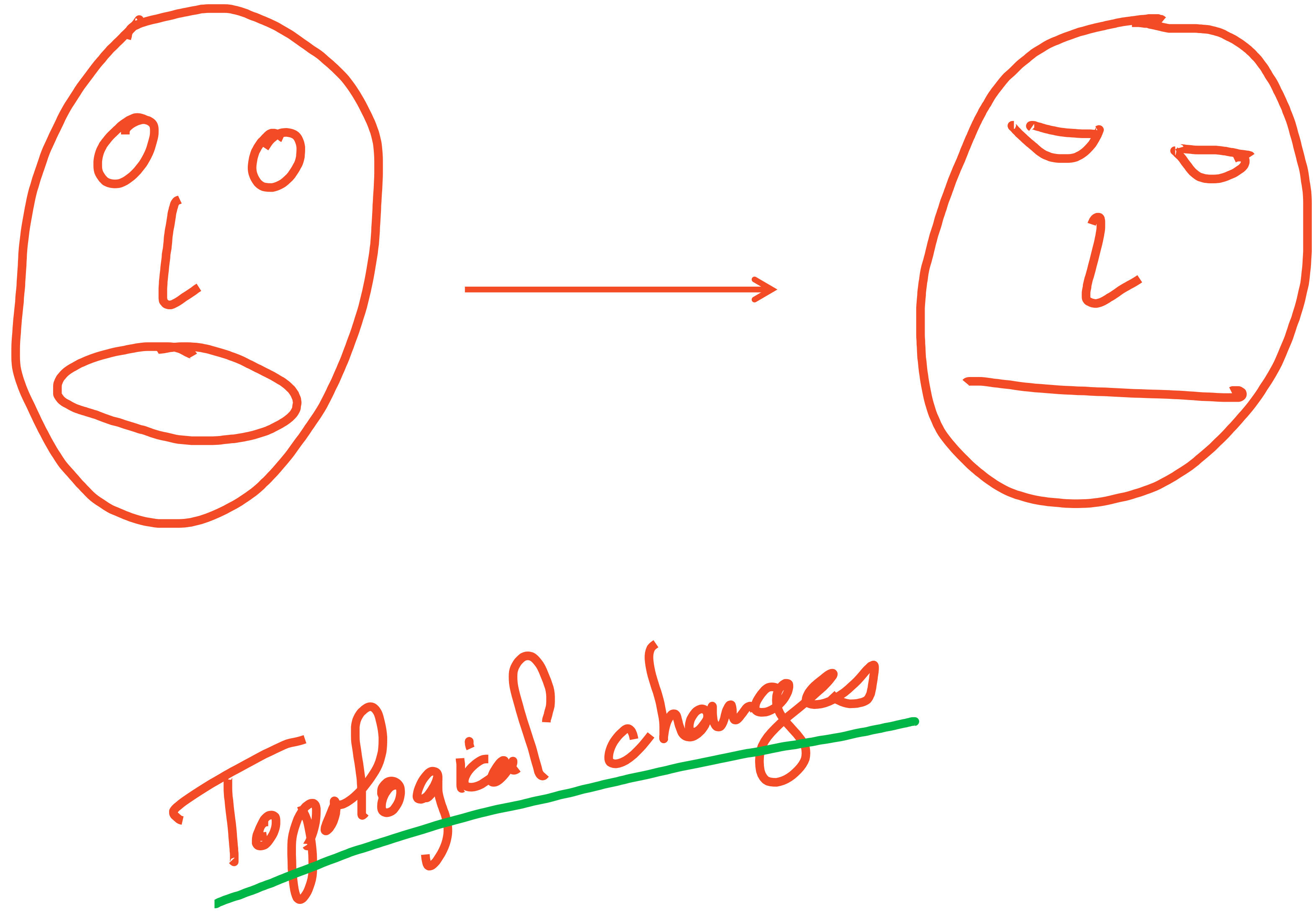




Shapes move easily relative to each other while deforming slightly.



Articulation



Topological changes

(Sub) Riemannian Submersion

Notation and Setting

Goal: “project” a sub-Riemannian structure on diffeomorphisms onto a shape space

Let V be a Hilbert space continuously embedded in $C_0^p(\mathbb{R}^d,\mathbb{R}^d)$ for $p\geq 1$.

Associate to each $\phi\in Diff_0^p$ a Hilbert norm on V denoted $\|\,\|_{V,\phi}$ such that

$$\|v\|_{V,\phi} \geq c_\phi \|v\|_V$$

for some c_ϕ.

Denote $V_\phi=\{v\circ\phi,\ v\in V\}$, $\|v\circ\phi\|_\phi=\|v\|_{V,\phi}$.

Notation and Setting (cont.)

$Diff_0$: group of attainable diffeomorphisms, endpoints of paths $\phi(\cdot)$ such that $\int_0^1 \|\phi(t)\|_{\phi(t)}^2 \, dt < \infty$.

$\mathfrak{M}$: shape space with $Diff_0$ acting on $\mathfrak{M}$.
$(\phi, q) \mapsto \phi \cdot q = \pi_q(\phi)$
$(v, q) \mapsto v \cdot q = \xi_q v = d\pi_q(id) v$
(action and infinitesimal action).

Assume that $\mathfrak{M}$ is open in Q, a Banach space.

Isometry Hypothesis

Fix $q_0 \in \mathfrak{M}$: the template.

Let $\mathfrak{M}_0 = \{\pi_{q_0}(\phi), \phi \in Diff_0\}$.

For $\phi \in Diff_0$, define $H_\phi = Null(\xi_q)^{\perp_{V,\phi}} \subset V$, with $q = \pi_{q_0}(\phi)$.

$v \in H_\phi \Leftrightarrow (\xi_q w = 0 \Rightarrow \langle v, w\rangle_{V,\phi} = 0)$

Isometry Hypothesis (cont.)

If $\pi_{q_0}(\phi)=\pi_{q_0}(\psi)=q$, the condition

$\xi_q(I(v))=\xi_q v$

uniquely defines an isomorphism $I: H_\phi \to H_\psi$.

($I(v)$ is the orthogonal projection of 0 on the space $\{w : \xi_q w = \xi_q v\}$ for the $\langle , \rangle_{V,\phi}$ dot product).

Assumption: I is an isometry between H_ϕ and H_ψ.

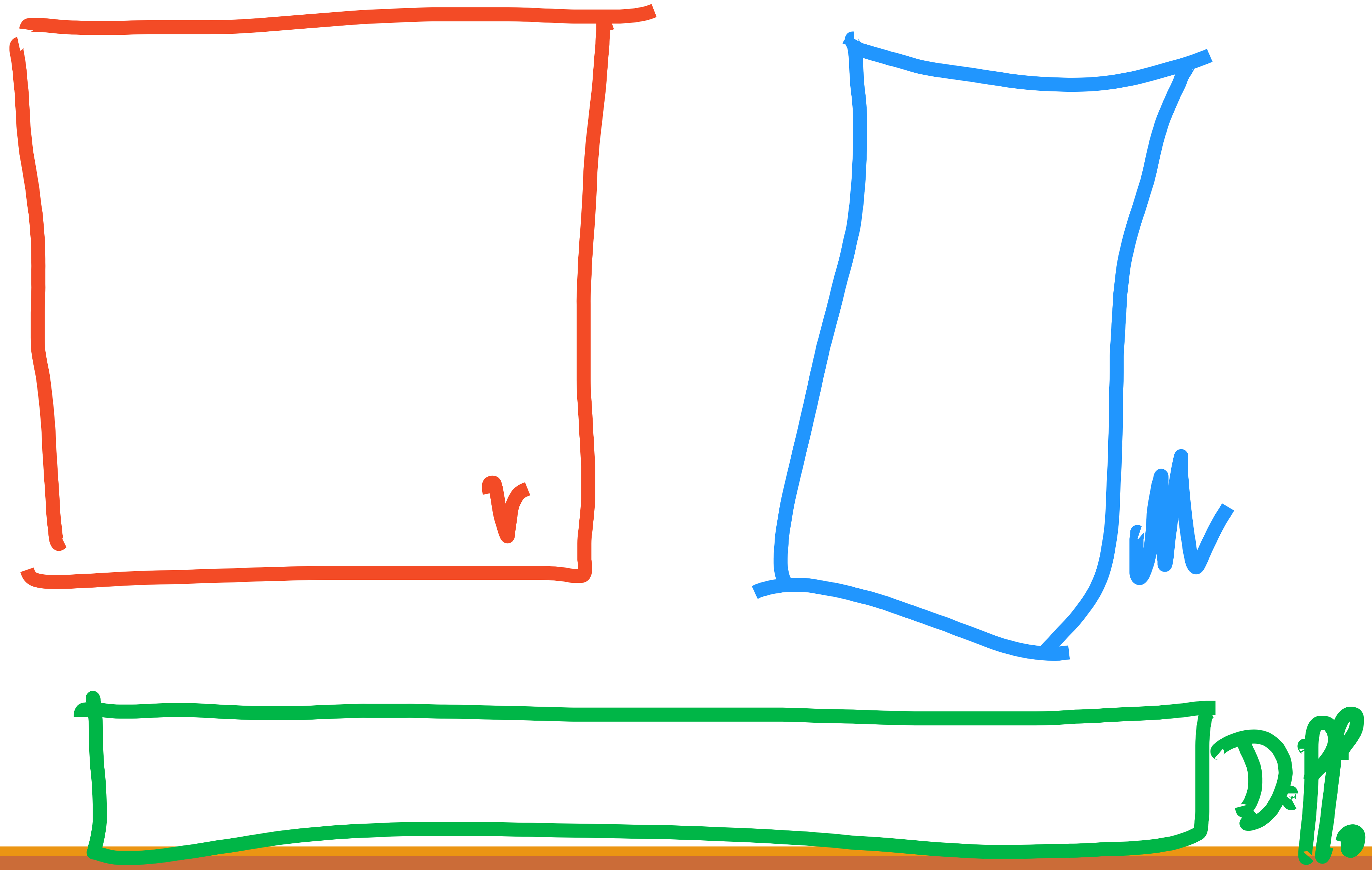
Shape space distribution and metric

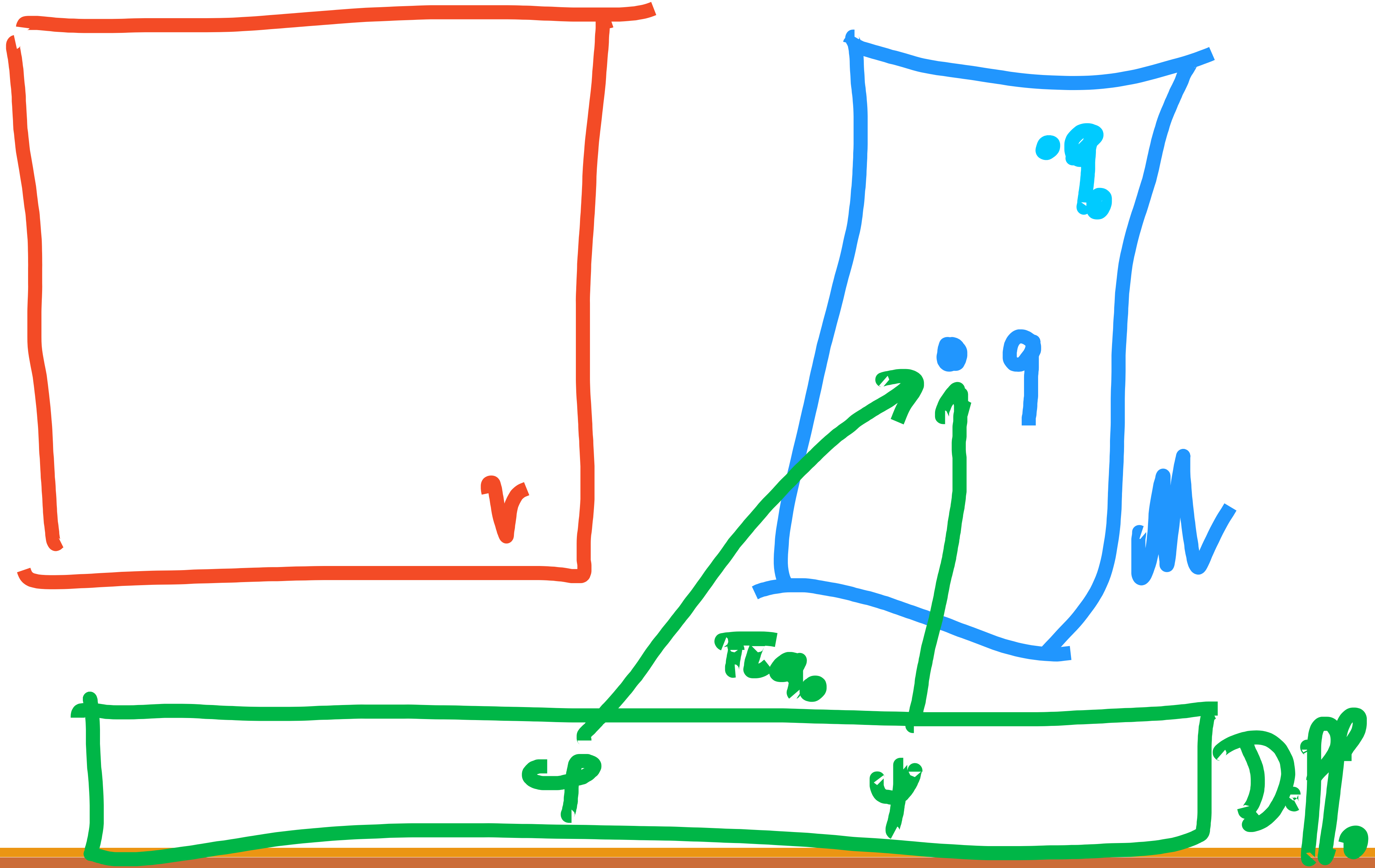
Define $\mathcal{H}_q = \xi_q H_\phi = \{\xi_q v,\ v \in H_\phi\}$ for $\pi_{q_0}(\phi) = q$.

On this set, let
$\|\xi_q v\|_q = \|v\|_{V,\phi}$: $v \in H_\phi$

Independent of $\phi \in \pi_{q_0}^{-1}(q)$ by assumption.

This provides a sub-Riemannian metric on $\mathfrak{M}$.


v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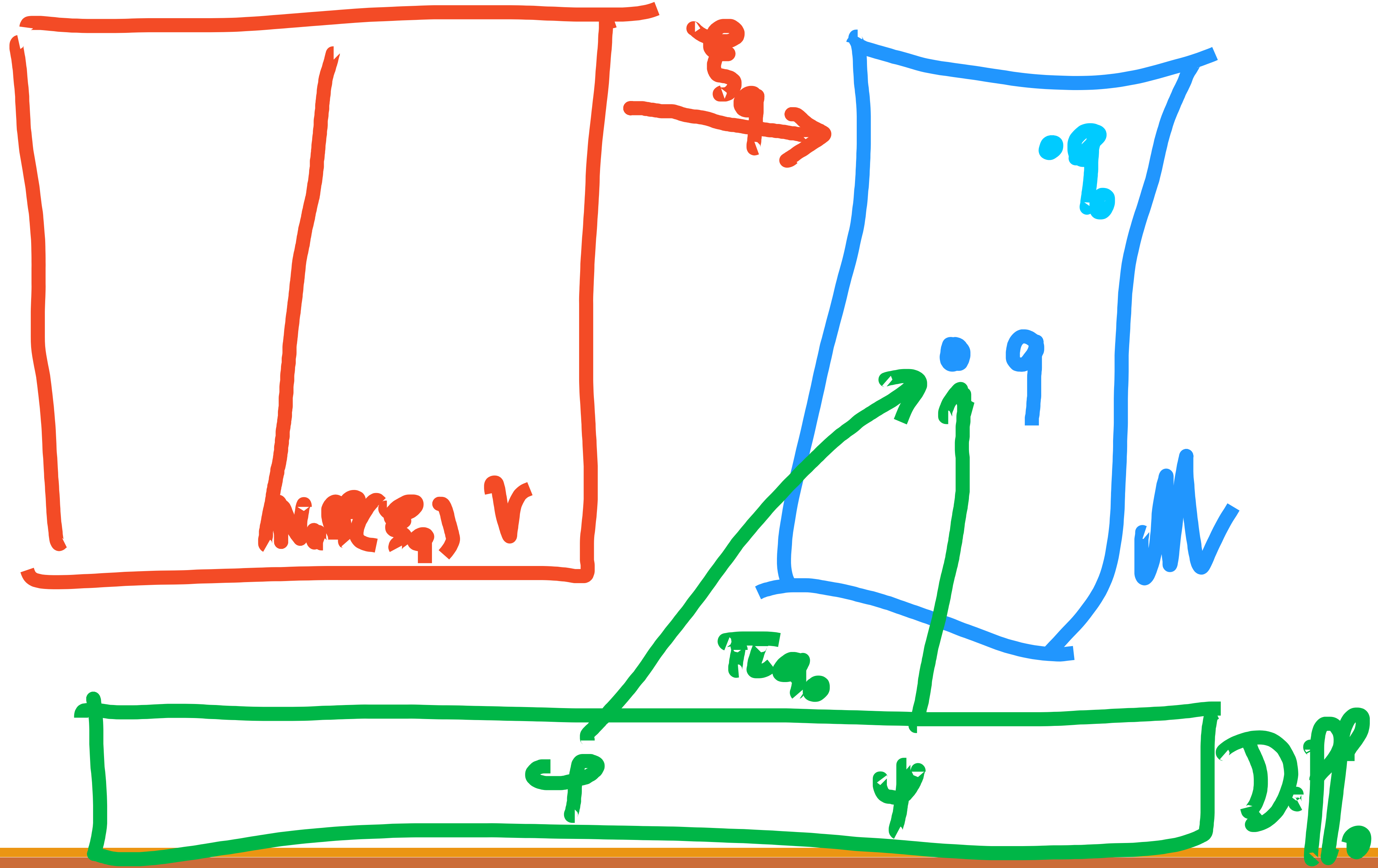
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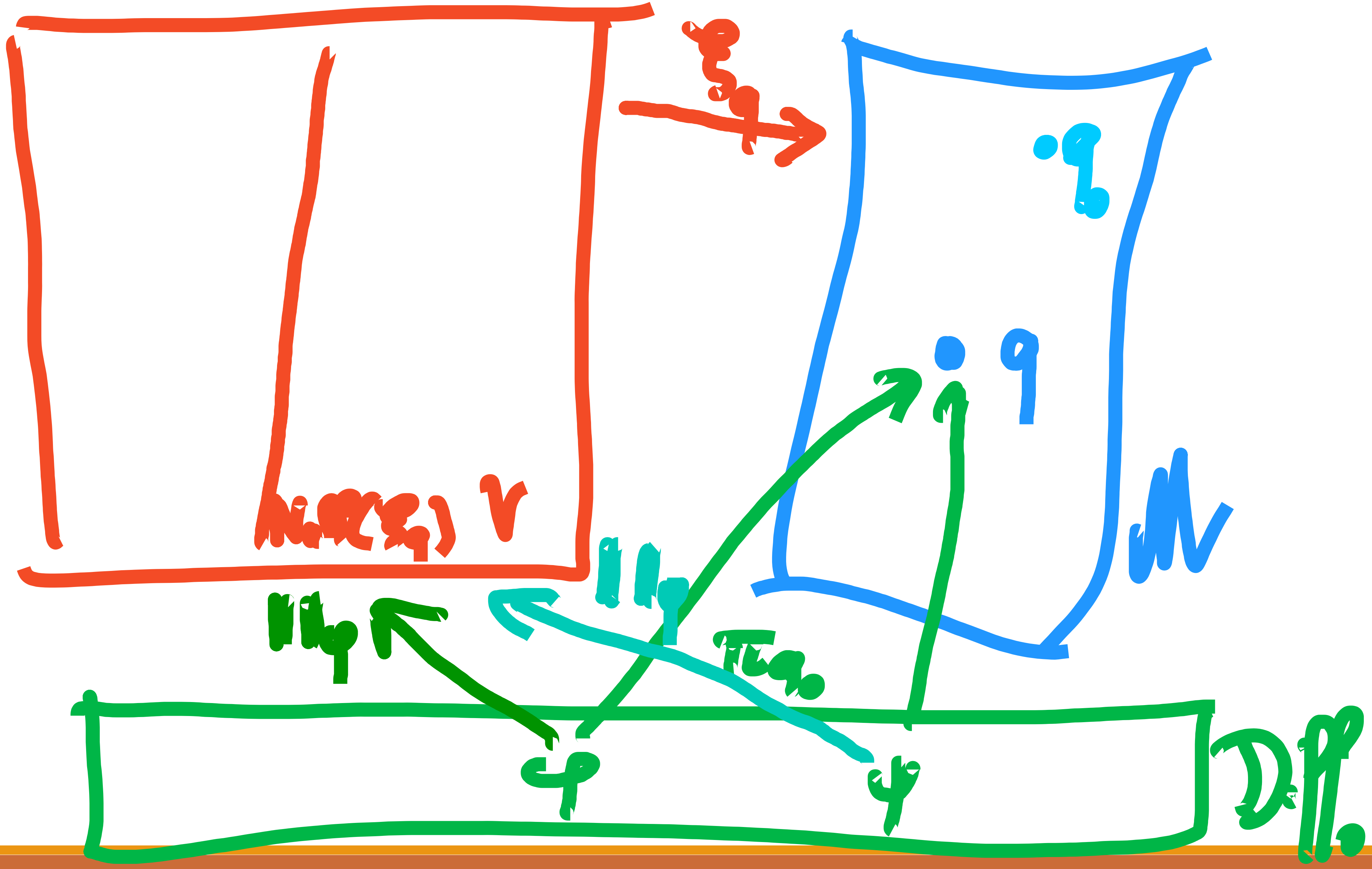
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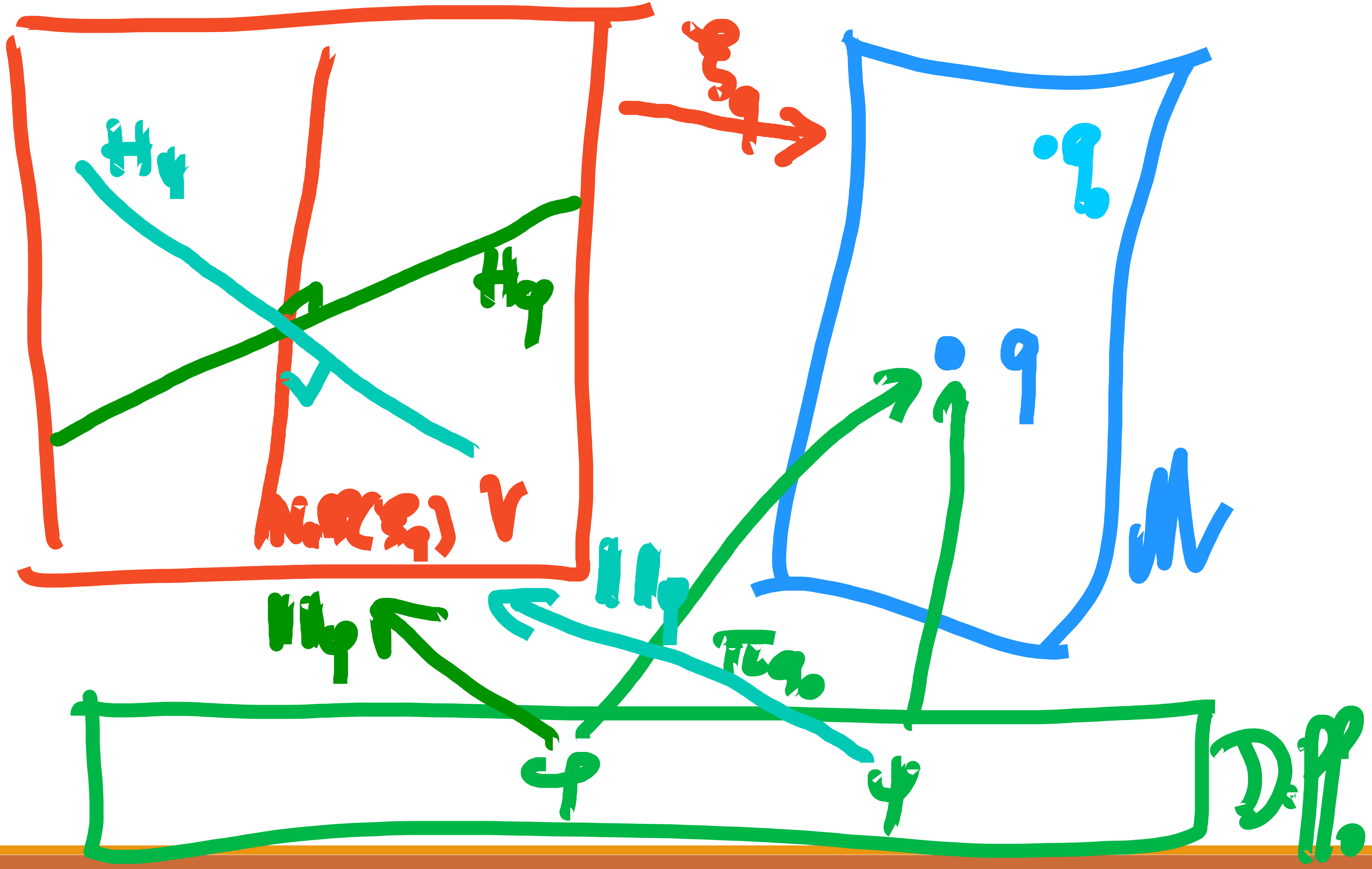
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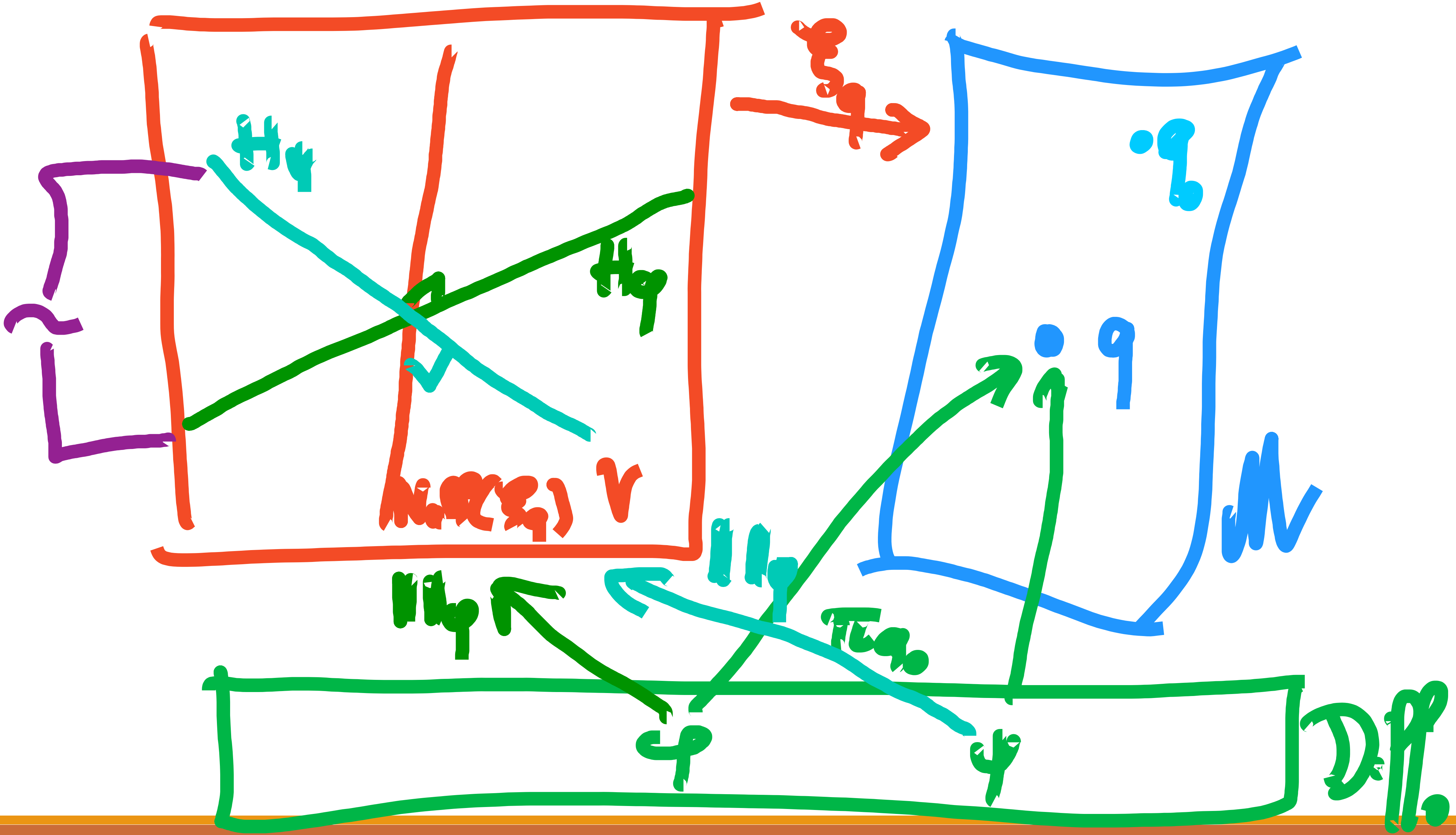
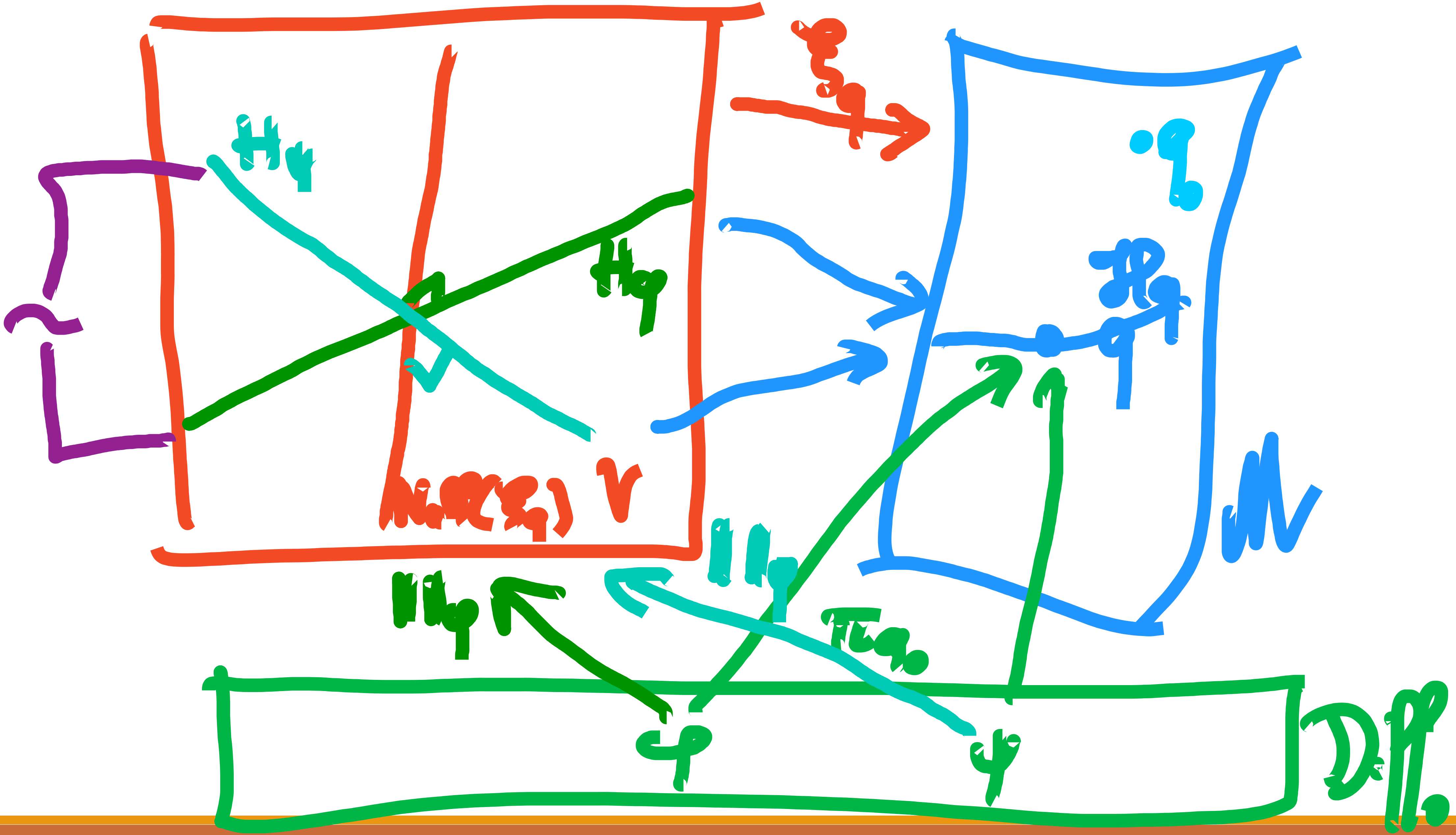

Hq
Hq
V
M
Diff.


V
M
Diff.

Special case: LDDMM

Take $\|v\|_{V,\phi} = \|v\|_V$ so that $H_\phi = H_\psi$ and $I = id$.

$\|\xi_q v\|_V = \|v\|_V$ for $v \in H_\phi$, $\phi \cdot q_0 = q$.

Slightly less trivial…

Assume that $\|\cdot\|_{V,\phi} = \|\cdot\|_{V,\psi}$ when $\phi \cdot q_0 = \psi \cdot q_0$.

Then again $H_\phi = H_\psi$ and $I = id$.

All examples today fall in this category.

Running Construction

Let $(q,h) \mapsto G_q(h,h)$ be a pseudo-Riemannian metric on $\mathfrak{M}$.

Let
$\|v\|_q^2 = \|v\|_{v,\phi}^2 = \lambda \|v\|_V^2 + G_q(\xi_q v, \xi_q v)$
with $q = \pi_{q_0}(\phi)$.

Hybrid LDDMM problem

Minimize

$$\int_0^1 \|v(t)\|_V^2 \, dt + D(q(1), q_1)$$

subject to $q(0) = q_0$ and $\dot{q} = \xi_q v$.

Two interpretations

1. Enrich the LDDMM norm with shape-dependent (geometric) information.
2. Modify the shape space pseudo norm for force geodesics to evolve diffeomorphically.

Important note

It is easy to apply the construction to products of shape spaces.

Replace $\mathfrak{M}$ by $\mathfrak{M}^n$ with the product pseudo-Riemannian metric.

Use action $\phi \cdot (q_1, \ldots, q_n) = (\phi \cdot q_1, \ldots, \phi \cdot q_n)$.

Application to spaces of curves

A lot of pseudo-Riemannian metrics have been described and studied in the literature, notably by Peter Michor's group in Vienna, or by Srivastava, Klassen, Mumford, Shah, etc.

One works with parametrized curves, or embeddings.

The shape spaces of interest are curves modulo parametrization. Invariance is achieved by selecting a parametrization-invariant cost function.

Scientific Articles

[146] Martin Bauer, Martins Bruveris, Philipp Harms, Peter W. Michor: **Soliton solutions for the elastic metric on spaces of curves.** 22 pages. arxiv:1702.04344 (pdf).

[145] Martins Bruveris, Peter W. Michor: **Geometry of the Fisher-Rao metric on the space of smooth densities.** 13 pages. arxiv:1607.04550 (pdf).

[144] Martins Bruveris, Peter W. Michor, Adam Parusinski, Armin Rainer: **Moser's theorem on manifolds with corners.** 9 pages. arxiv:1604.07787 (pdf).

[143] Martin Bauer, Peter W. Michor, Olaf Müller: **Riemannian geometry of the space of volume preserving immersions.** *Differential Geometry and its Applications* **49** (December 2016), 23â€“42. doi:10.1016/j.difgeo.2016.07.002. arxiv:1603.05916. (pdf).

[142] Peter W. Michor: **Manifolds of mappings and shapes.** In the book: *The legacy of Bernhard Riemann after one hundred and fifty years*. Editors: Lizhen Ji, Frans Oort, Shing-Tung Yau. Series: Advanced Lectures of Mathematics 35, pp. 459â€“486. Higher Education Press and International Press Beijingâ€“Boston 2016. arXiv:1505.02359. (pdf).

[141] Martin Bauer, Martins Bruveris, Peter W. Michor: **Why use Sobolev metrics on the space of curves.** IN: *Riemannian Computing in Computer Vision*. Ed.: Pavan K. Turaga, Anuj Srivastava. Pages 233-255. Springer-Verlag, 2016. ISBN 978-3-319-22956-0. arXiv:1502.03229. (pdf).

[140] Martin Bauer, Martins Bruveris, Peter W. Michor: **Uniqueness of the Fisher-Rao metric on the space of smooth densities.** Bulletin of the London Mathematical Society. **48**, 3 (2016), 499-506. doi:10.1112/blms/bdw020. arXiv:1411.5577. (pdf). Erratum

[139] Andreas Kriegl, Peter W. Michor, Armin Rainer: **The exponential law for spaces of test functions and diffeomorphism groups.** Indagationes Mathematicae **27**, 1 (2016), 225â€“265. doi:10.1016/j.indag.2015.10.006. arXiv:1411.0483. (pdf).

[138] Andreas Kriegl, Thomas Hotz, Peter W. Michor: **Frölicher spaces as a setting for tree spaces and stratified spaces.** *Oberwolfach Report* **44/2014**, 24-29. doi:10.4171/OWR/2014/44. (pdf).

[137] Andreas Kriegl, Peter W. Michor, Armin Rainer: **An exotic zoo of diffeomorphism groups on $\mathbb{R}^n$.** *Ann. Glob. Anal. Geom.* **47**, 2 (2015), 179-222. doi:10.1007/s10455-014-9442-0. arXiv:1404.7033. (pdf).

[136] Martins Bruveris, Peter W. Michor, David Mumford: **Geodesic Completeness for Sobolev Metrics on the Space of Immersed Plane Curves.** *Forum of Mathematics, Sigma* **2**, e19, 38 pages, 2014. doi:10.1017/fms.2014.19. arXiv:1312.4995. (pdf).

[135] Martin Bauer, Martins Bruveris, Peter W. Michor: **R-transforms for Sobolev H^2-metrics on spaces of plane curves.** *Geometry, Imaging and Computing* **1**,1, 1-56, 2014. doi:10.4310/GIC.2014.v1.n1.a1. arXiv:1311.3526. (pdf).

[134] Martin Bauer, Martins Bruveris, Peter W. Michor: **Overview of the Geometries of Shape Spaces and Diffeomorphism Groups.** *Journal of Mathematical Imaging and Vision*, **50**, 1-2, 60-97, 2014. doi:10.1007/s10851-013-0490-z. arXiv:1305.1150. (pdf).

[133] Giuseppe Marmo, Peter W. Michor, Yuri Neretin: **The Lagrangian Radon Transform and the Weil representation.** *Journal of Fourier Analysis and Applications* **20**, 2 (2014), 321-361. doi:10.1007/s00041-013-9315-0. arXiv:1212.4610. (pdf).

[132] Martin Bauer, Martins Bruveris, Peter W. Michor: **Geodesic distance for right invariant Sobolev metrics of fractional order on the diffeomorphism group. II.** *Ann. Glob. Anal. Geom.* **44**, 4 (2013), 361-368. doi:10.1007/s10455-013-9370-4. arXiv:1211.7254. (pdf).

[131] Peter W. Michor and David Mumford: **A zoo of diffeomorphism groups on $\mathbb{R}^n$.** *Ann. Glob. Anal. Geom.* **44**, 4 (2013), 529-540. doi:10.1007/s10455-013-9380-2. arXiv:1211.5704. (pdf).

Maximum Principle: Assumptions

$\mathfrak{M} = \{C^r \text{ embeddings from } S^1 \text{ (or } [0,1]) \text{ to } \mathbb{R}^2\}$, $Q = C^r(S^1, \mathbb{R}^2)$.

$V \subset C_0^p(\mathbb{R}^2, \mathbb{R}^2)$, $p \geq r$.

$G_q(h,h) \leq c_q \|h\|_{r,\infty}$.

$q \mapsto G\_q(h,h)$ is C^1.

$q \mapsto D(q, q_1)$ is C^1.

Maximum Principle

These assumptions ensure that Pontryagin's maximum principle is true: let $H_v(p,q) = p\, v \circ q - \frac{1}{2} \|v\|_q^2$.

Then, along optimal solutions, there exists $p : [0,1] \to Q^*$ such that

$$\begin{cases} q = \partial_p H_v \\ p = -\partial_q H_v \\ v = \operatorname{argmax}_w H_w(p,q) \end{cases}$$

Reduction

The PMP implies that $v = K \xi_q^* \alpha$ for some $\alpha \in Q^*$.

Use α to reparametrize the problem.

Minimize
$\frac{1}{2} \int_0^1 ||\alpha(t)||_{q(t)}^2 \, dt + d(q(1), q_1)$
with $q = K_q \alpha$, where

$K_q = \xi_q K \xi_q^*$ and $||\alpha||_q^2 = \lambda \alpha K_q \alpha + G_q (K_q \alpha, K_q \alpha)$.

H^1 norms in experiments

Let h be a vector field along q.

H^1 norm:
$G_q(h,h)=\int_0^{l(q)} |\partial_s h|^2 \, ds$
where $l(q)=length(q)$.

Rescaled H^1 :
$G_q(h,h)=1/l(q) \int_0^{l(q)} |\partial_s h|^2 \, ds$

H^1 norms in experiments

Rotation corrected H^1 :

$$G_q(h,h)=\int_0^{l(q)} |\partial_s h|^2 \, ds - \frac{1}{l(q)} \left(\int_0^1 \partial_s h^T N_q \, ds\right)^2$$

Rotation and scale corrected rescaled H^1 :

$$G_q(h,h)=\frac{1}{l(q)} \int_0^{l(q)} |\partial_s h|^2 \, ds - \left(\frac{1}{l(q)} \int_0^1 \partial_s h^T N_q \, ds\right)^2 - \left(\frac{1}{l(q)} \int_0^1 \partial_s h^T T_q \, ds\right)^2$$

where T_q is the unit tangent to q and N_q the unit normal.

Cost function

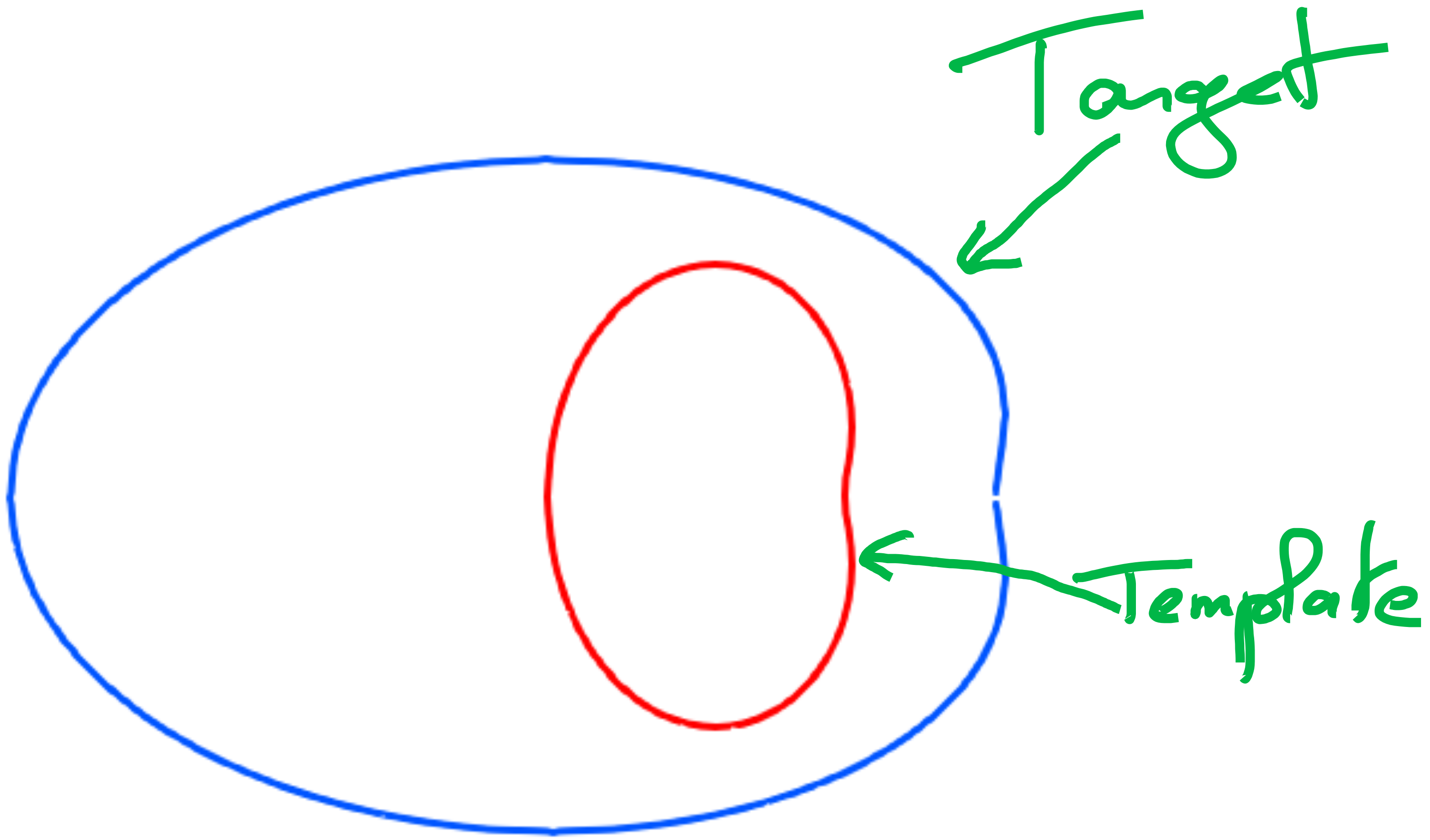
We used a version of the varifold norm introduced by Trouvé and Charon:

$$D(q,q_1) = \|q\|_{\chi}^{2} - 2\langle q,q_1\rangle_{\chi} + \|q_1\|_{\chi}^{2}$$

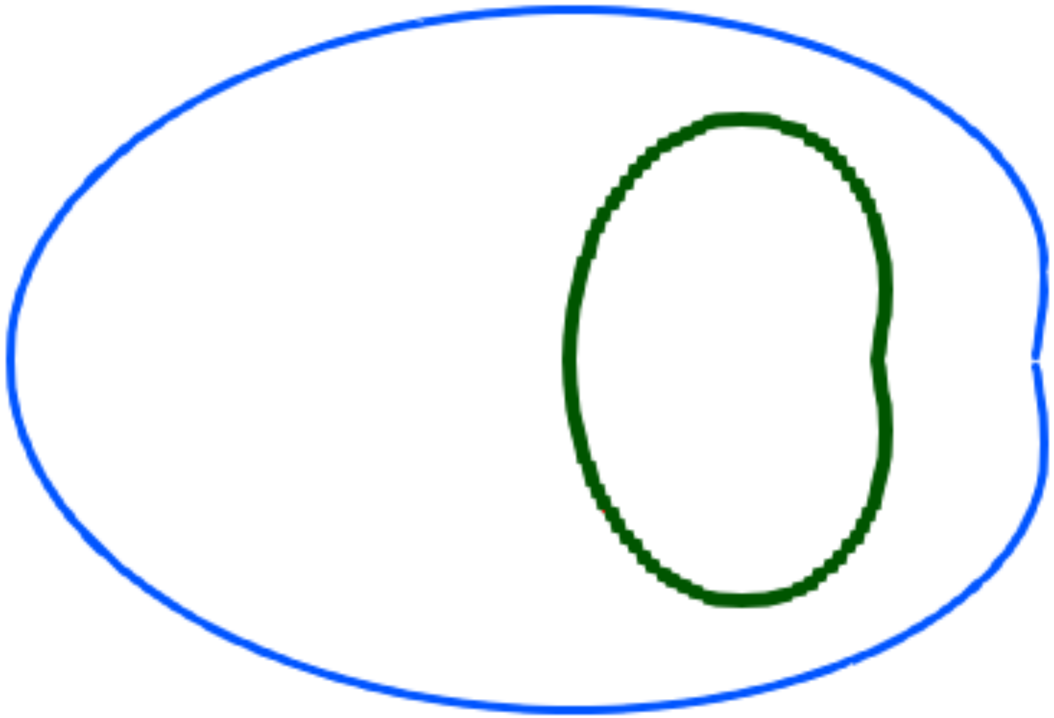
with

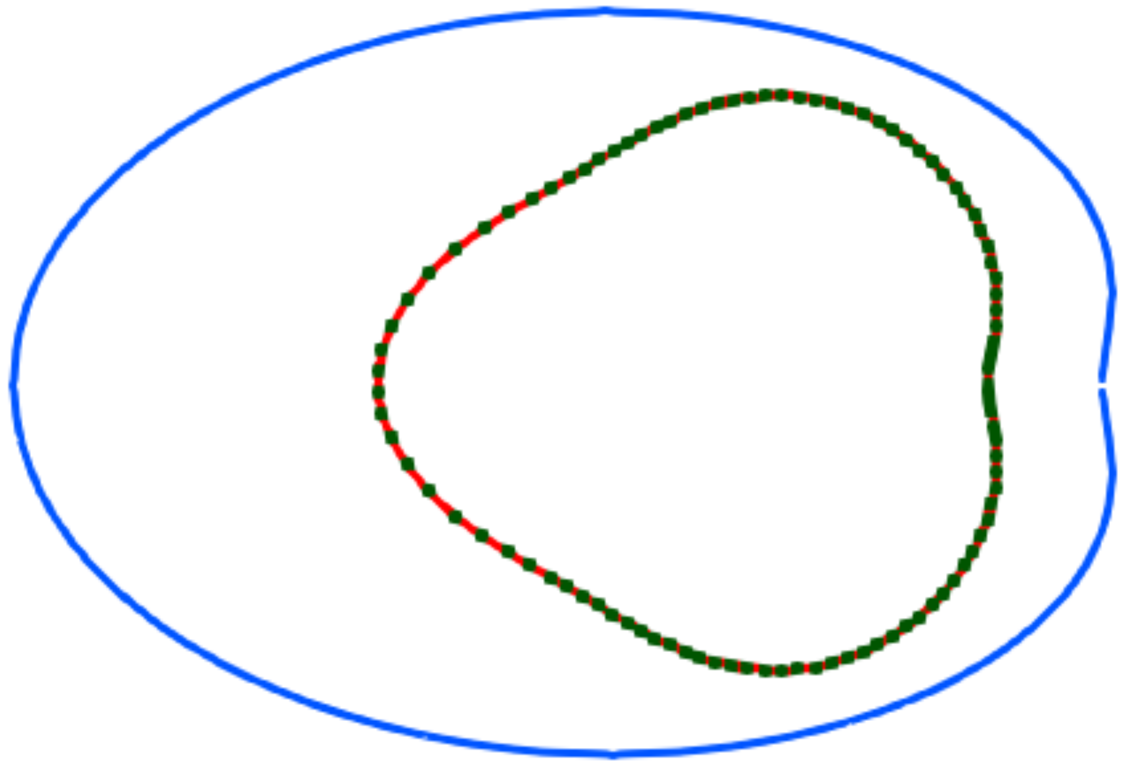
$$\langle q,q_1\rangle_{\chi} = \int_{S^1}\int_{S^1} \chi(q(u),q(u_1))\left(1+c\left(N_q(u)^T N_{q_1}(u_1)\right)^2\right)\times |q'(u)|\,|q_1'(u_1)|\,du_1\,du$$

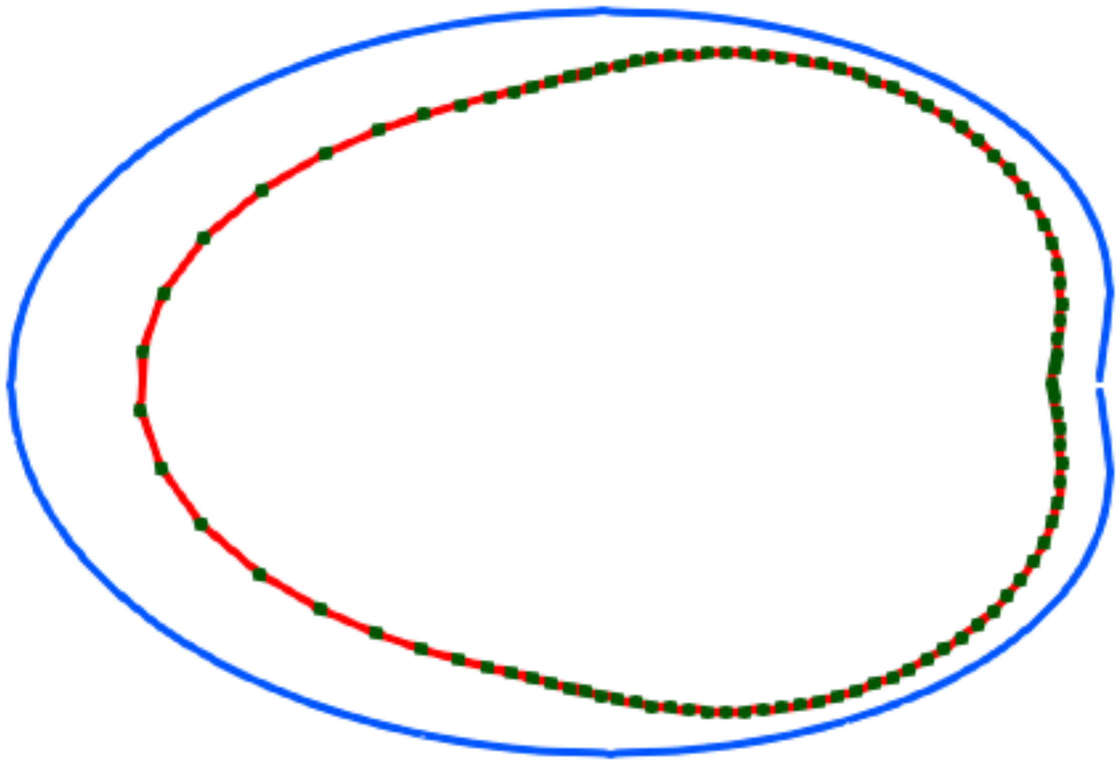
EXAMPLES

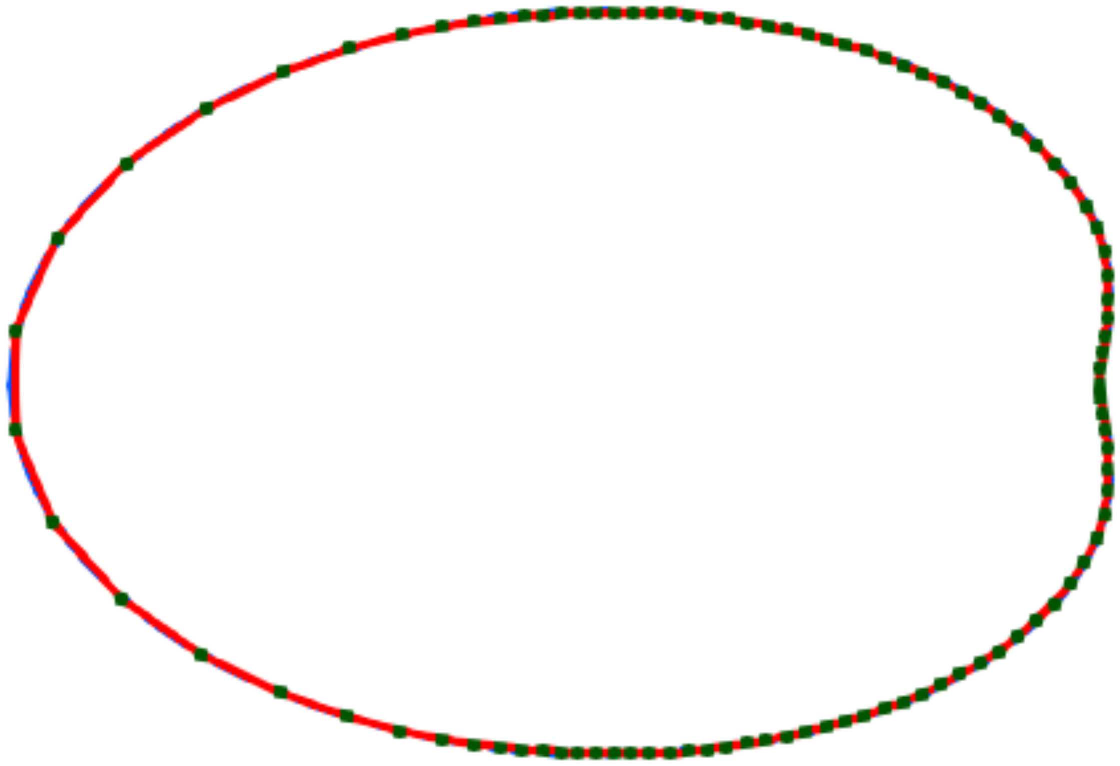

Target
Template

LDDMM

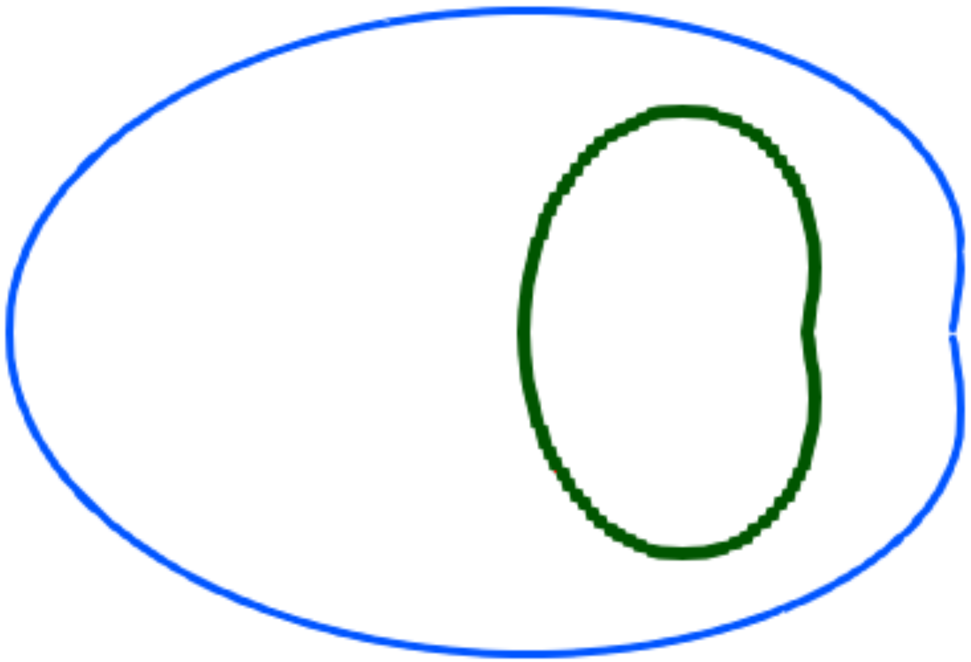


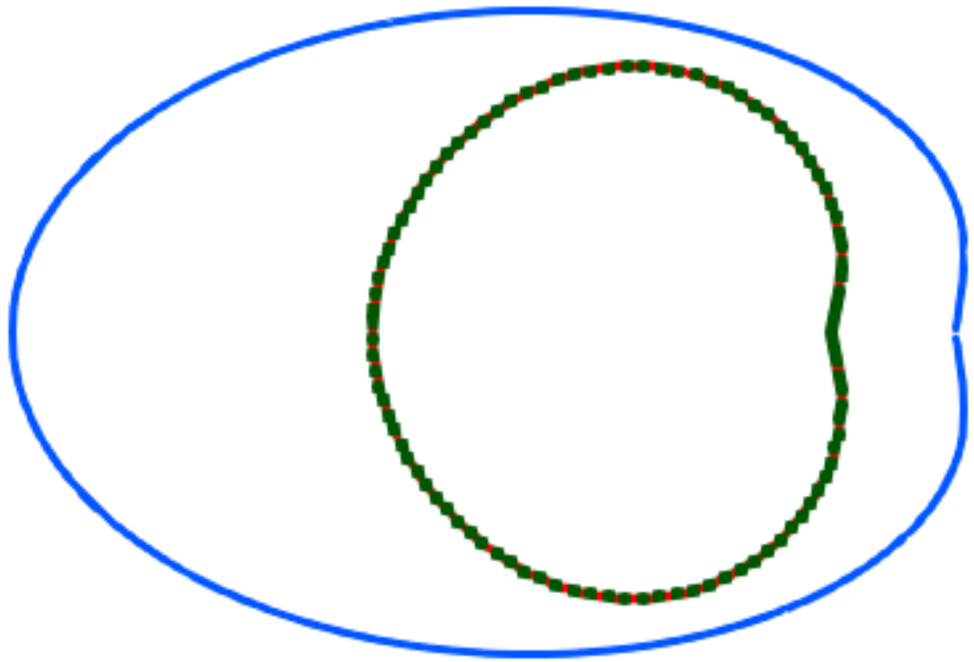


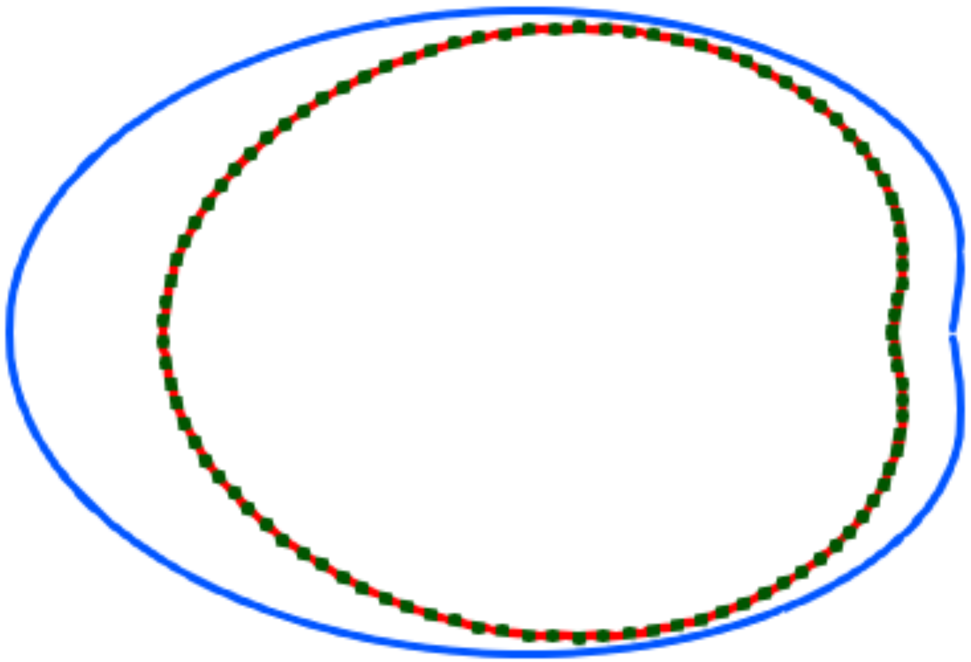


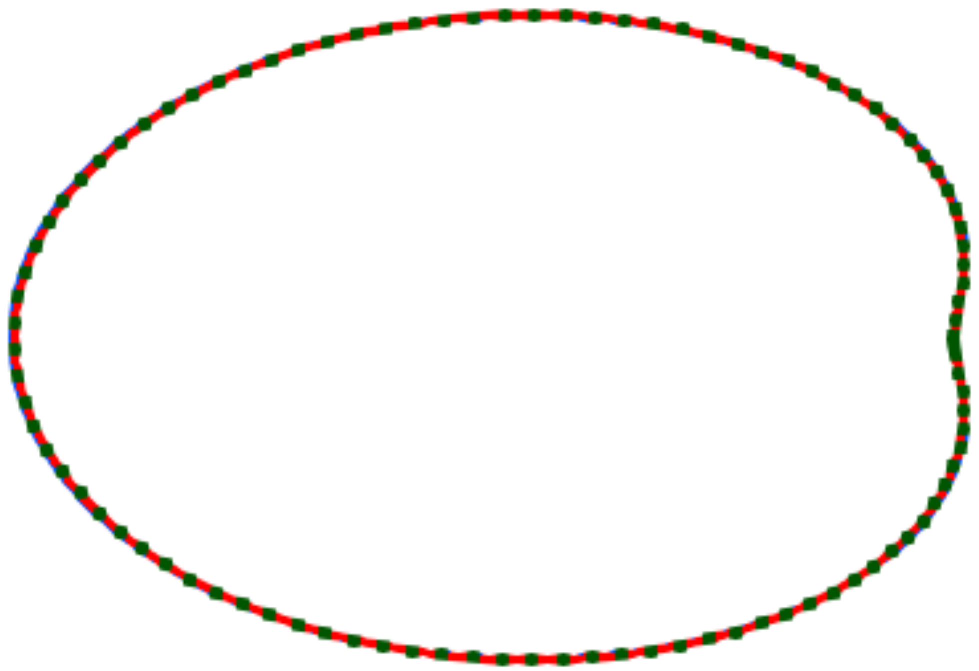


H_LDDMM with H'_invariant norm

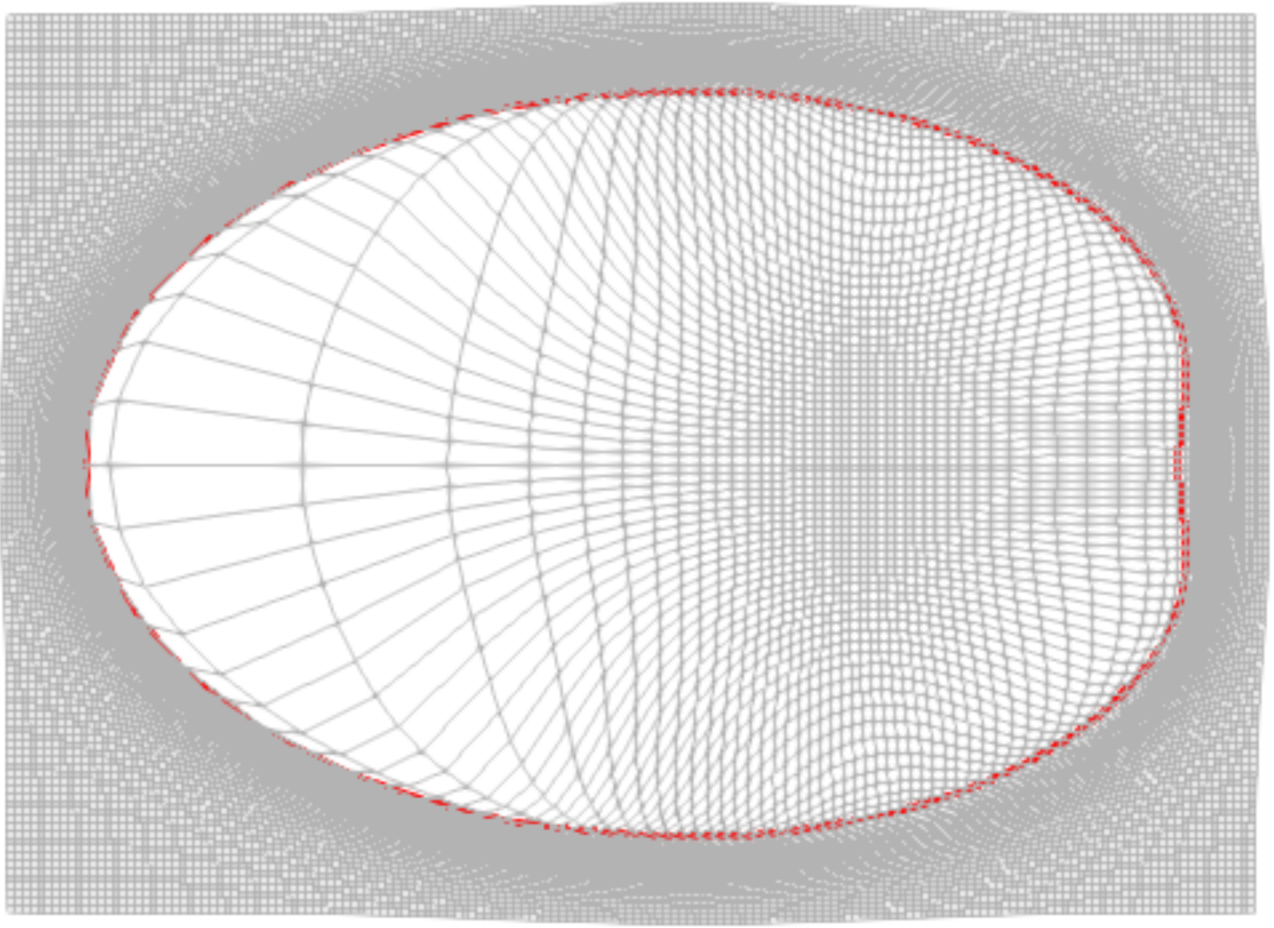




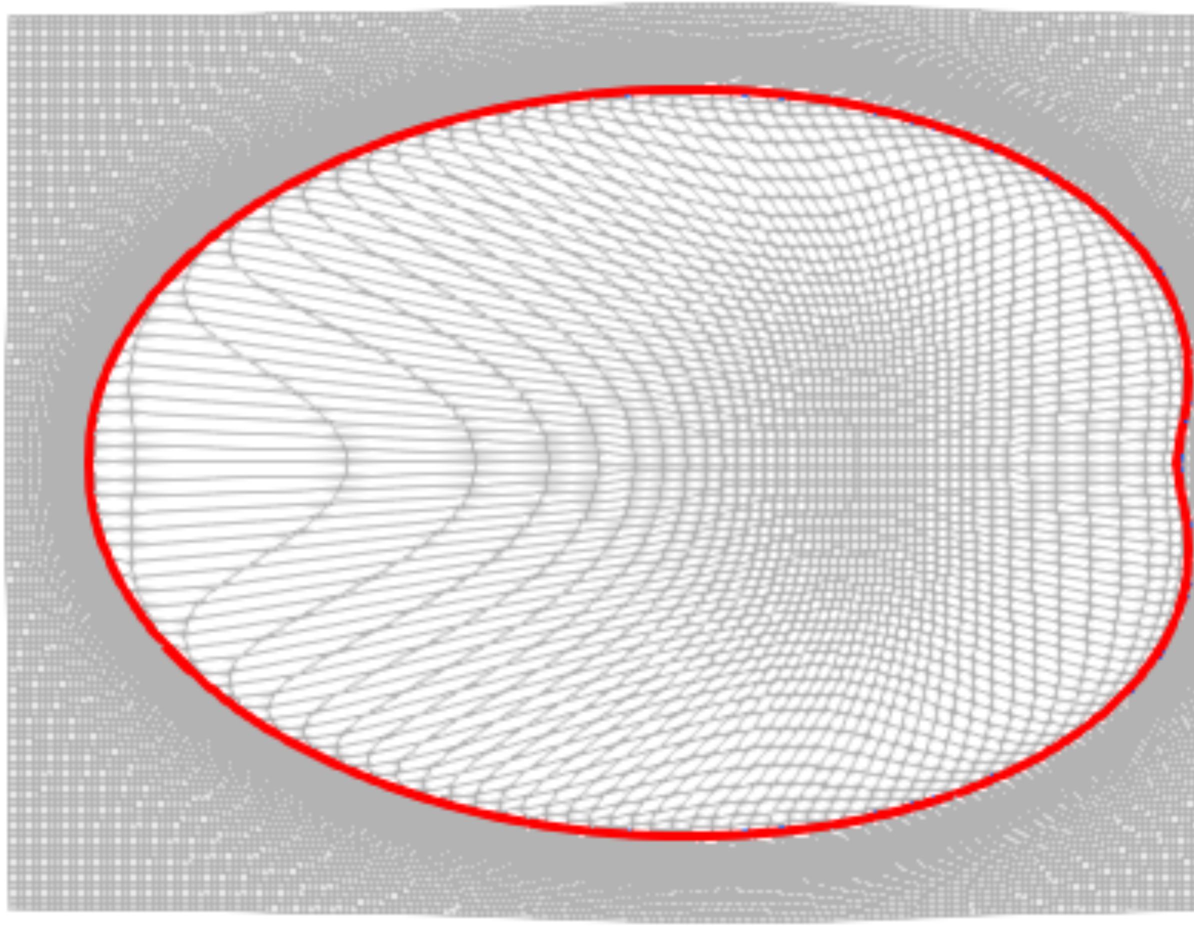




Estimated transformations

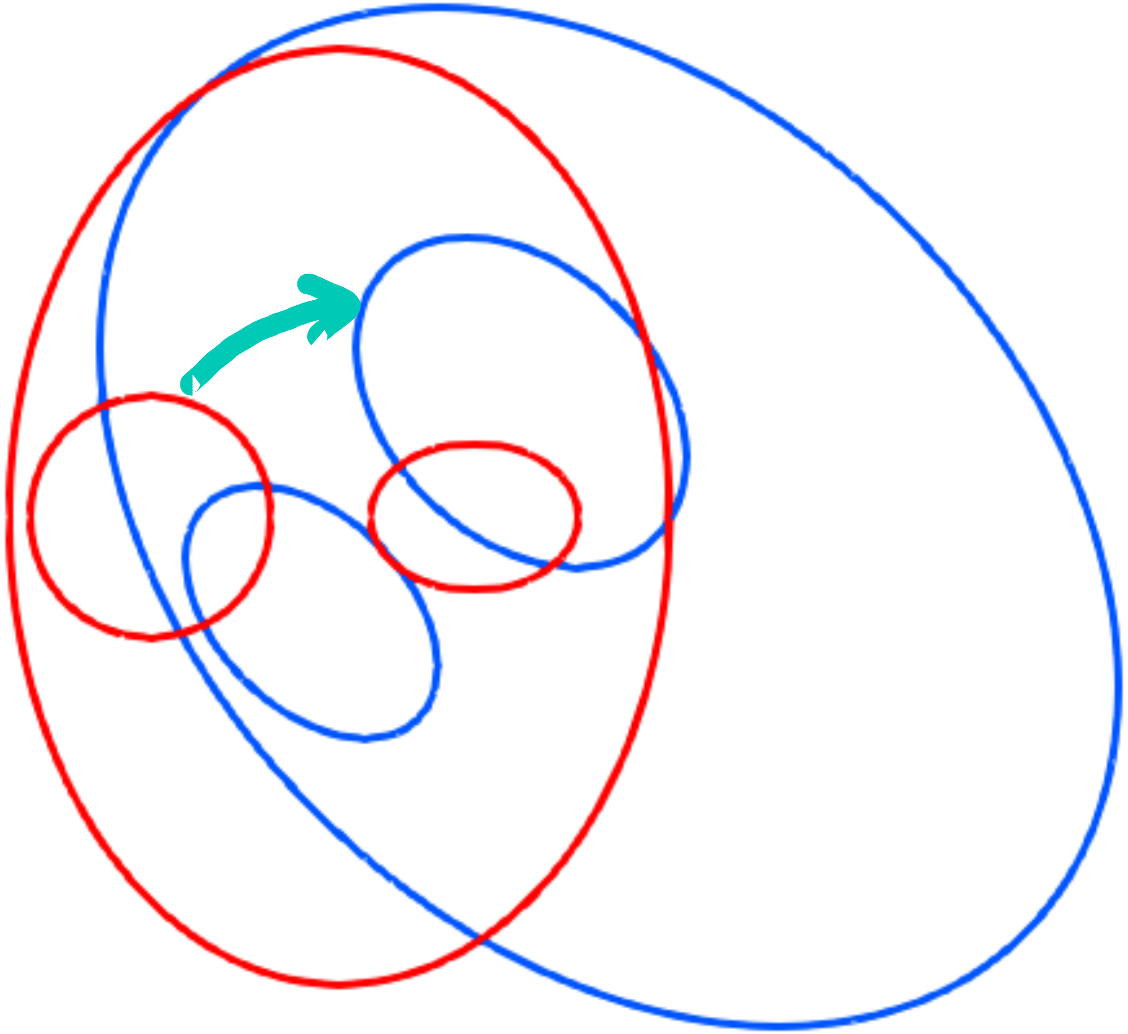


LDDMM



H-LDDMM

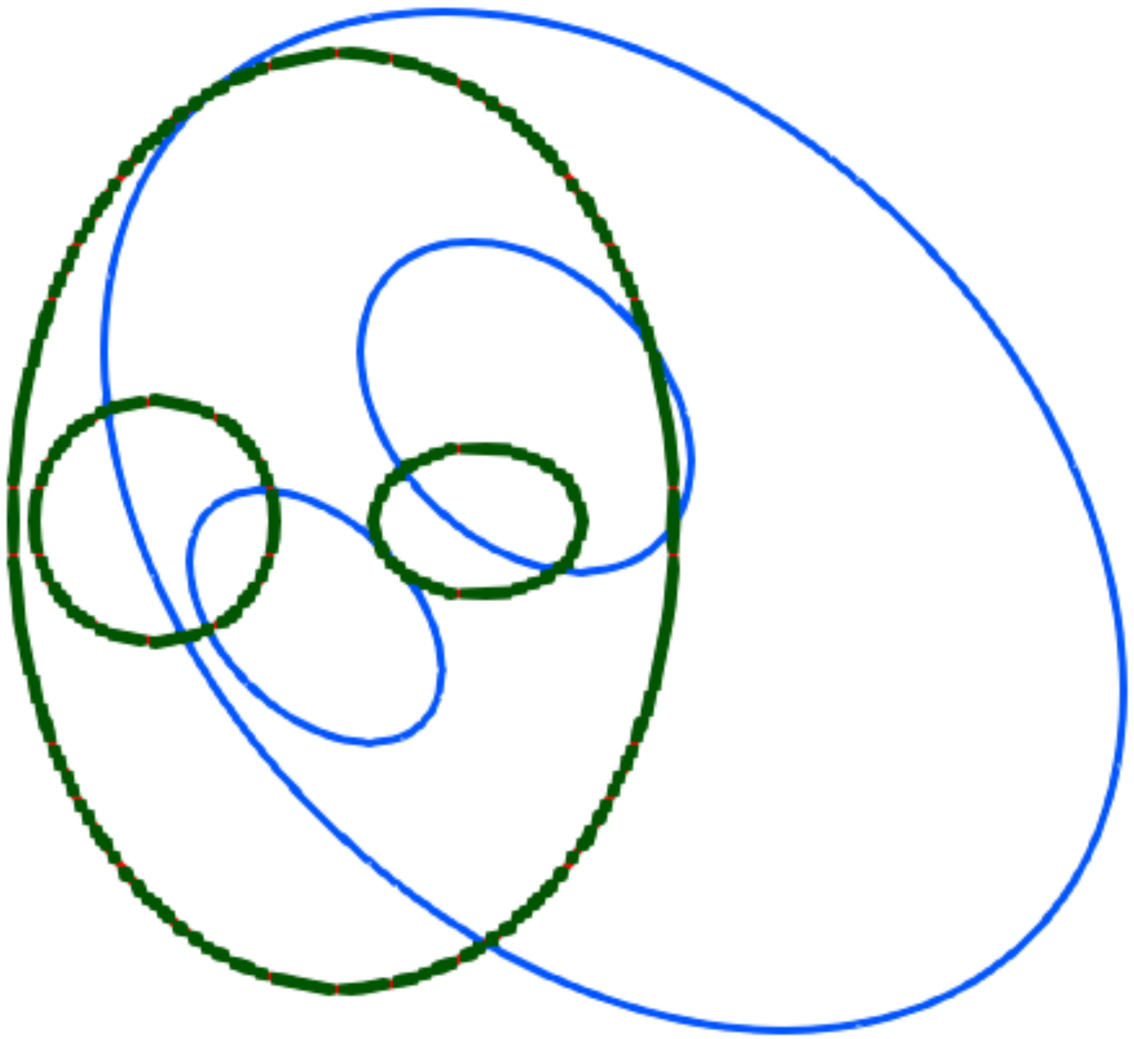
ELLIPSES...

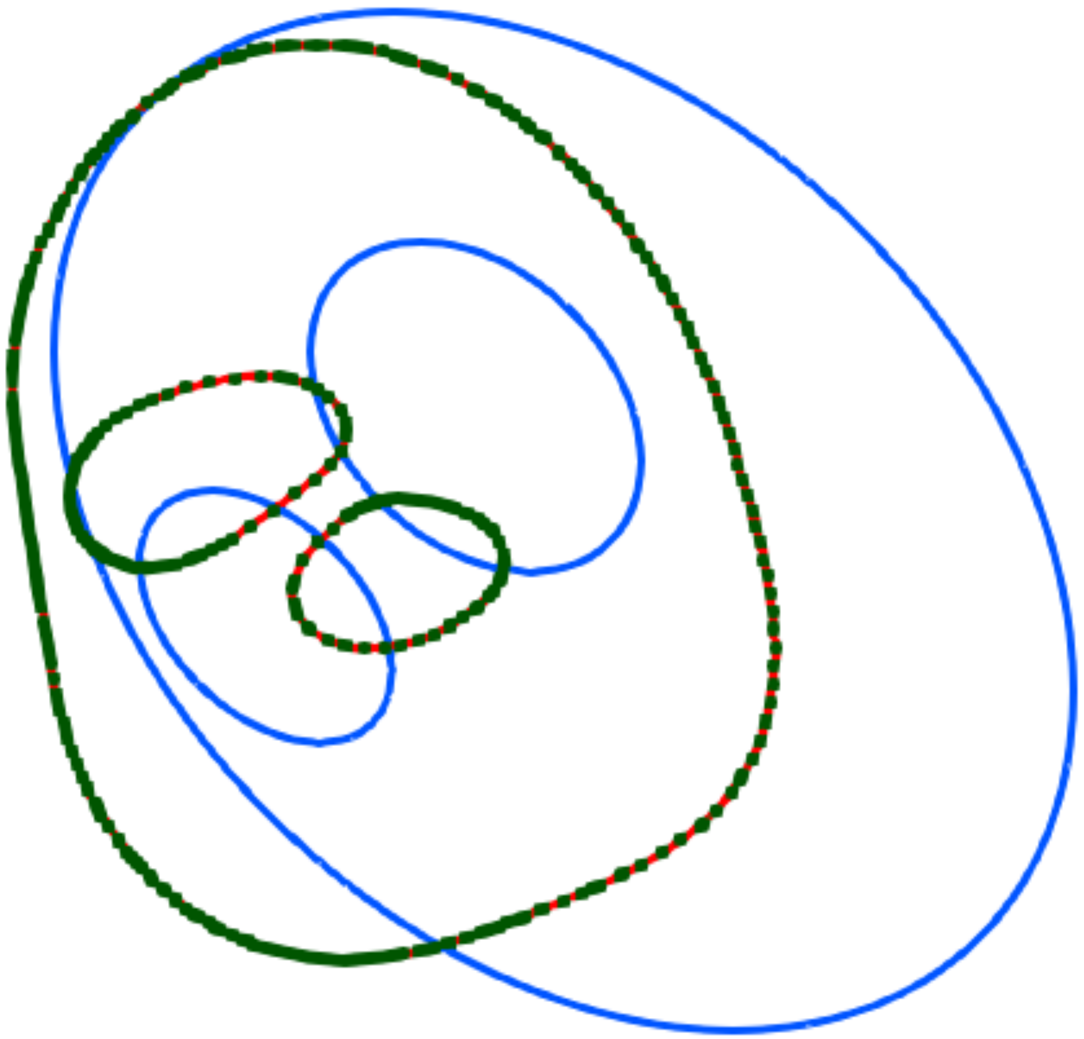


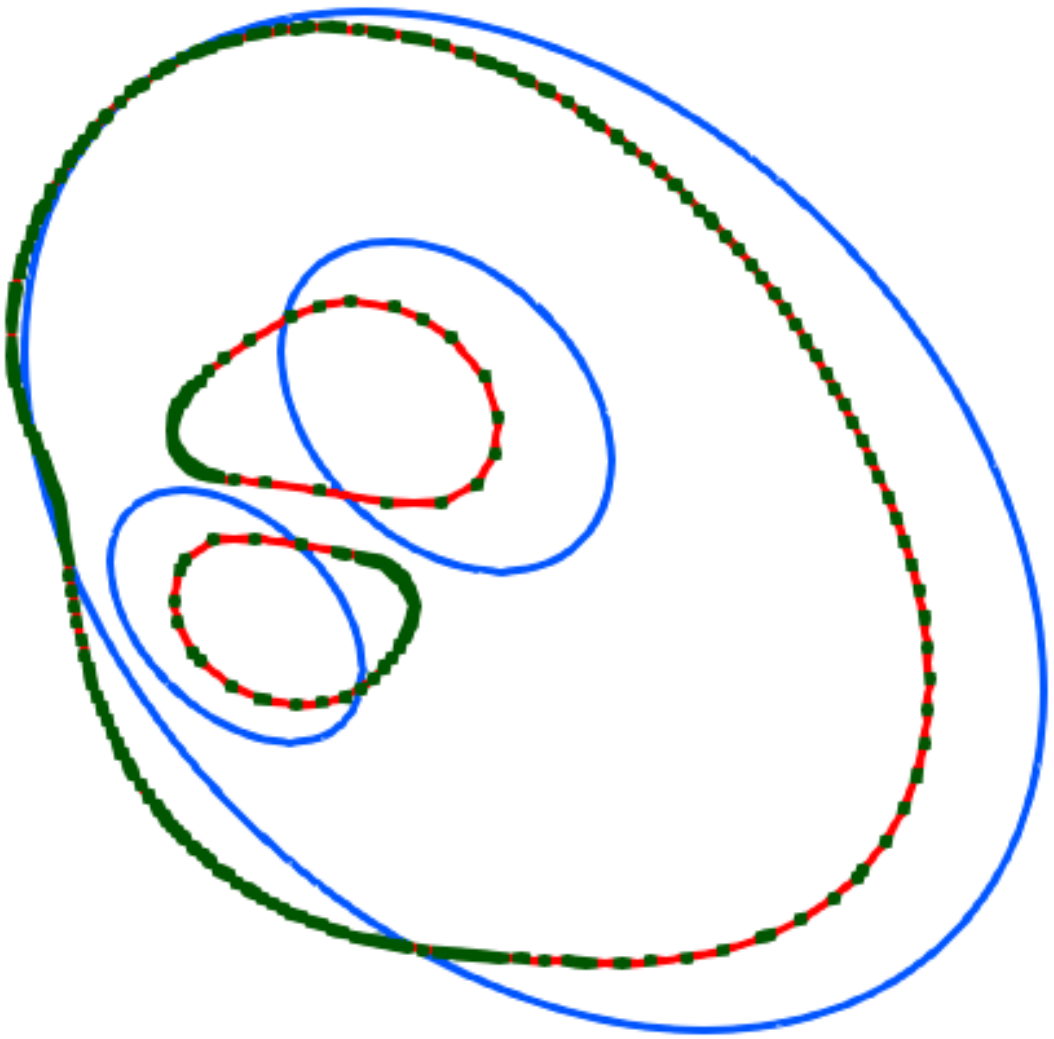
Remark: For this multiple-curve example only: the cost function ensures that homologous curves are mapped on each other (i.e., the curves are labeled).

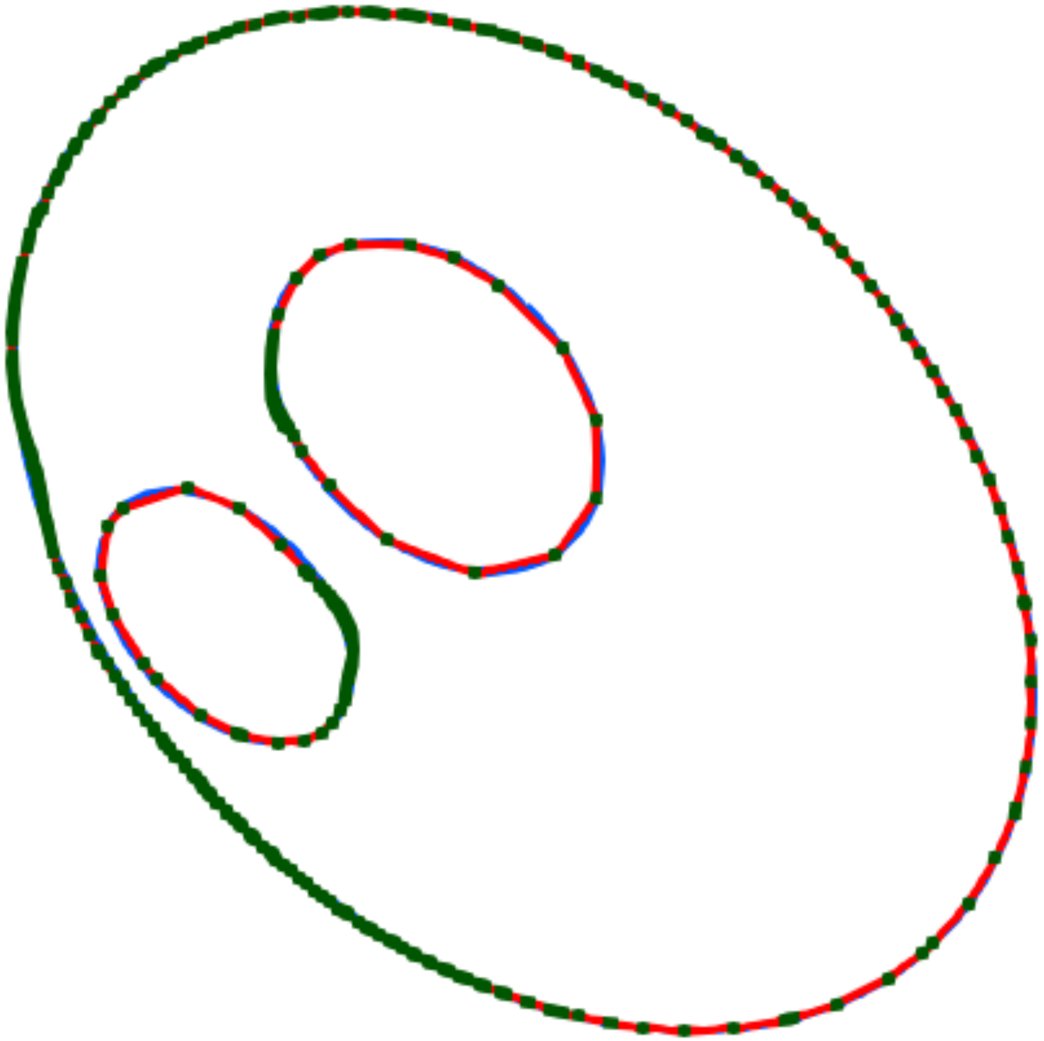
In all subsequent examples, the cost function is blind to curve relabelling.

LDDMM



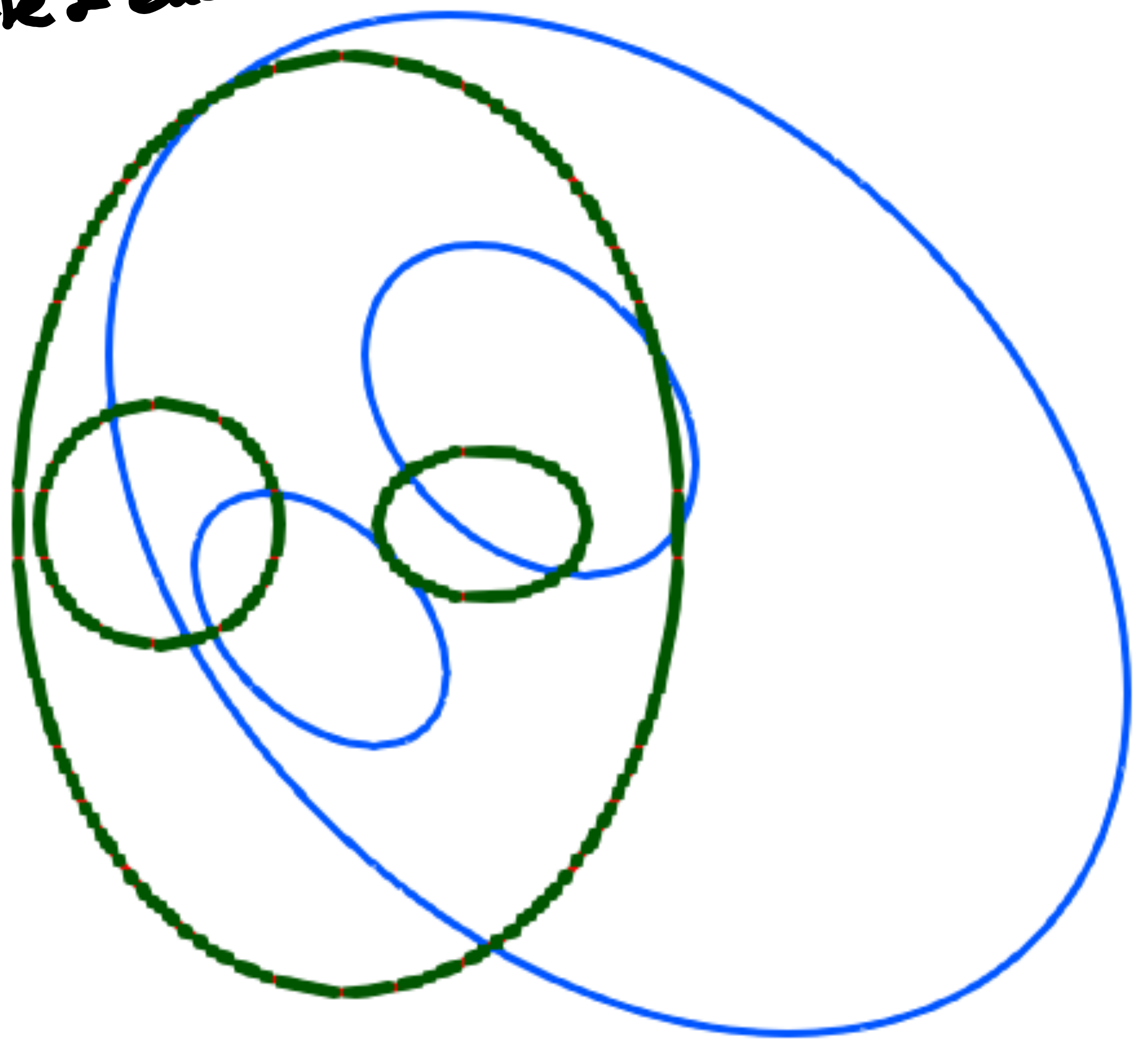


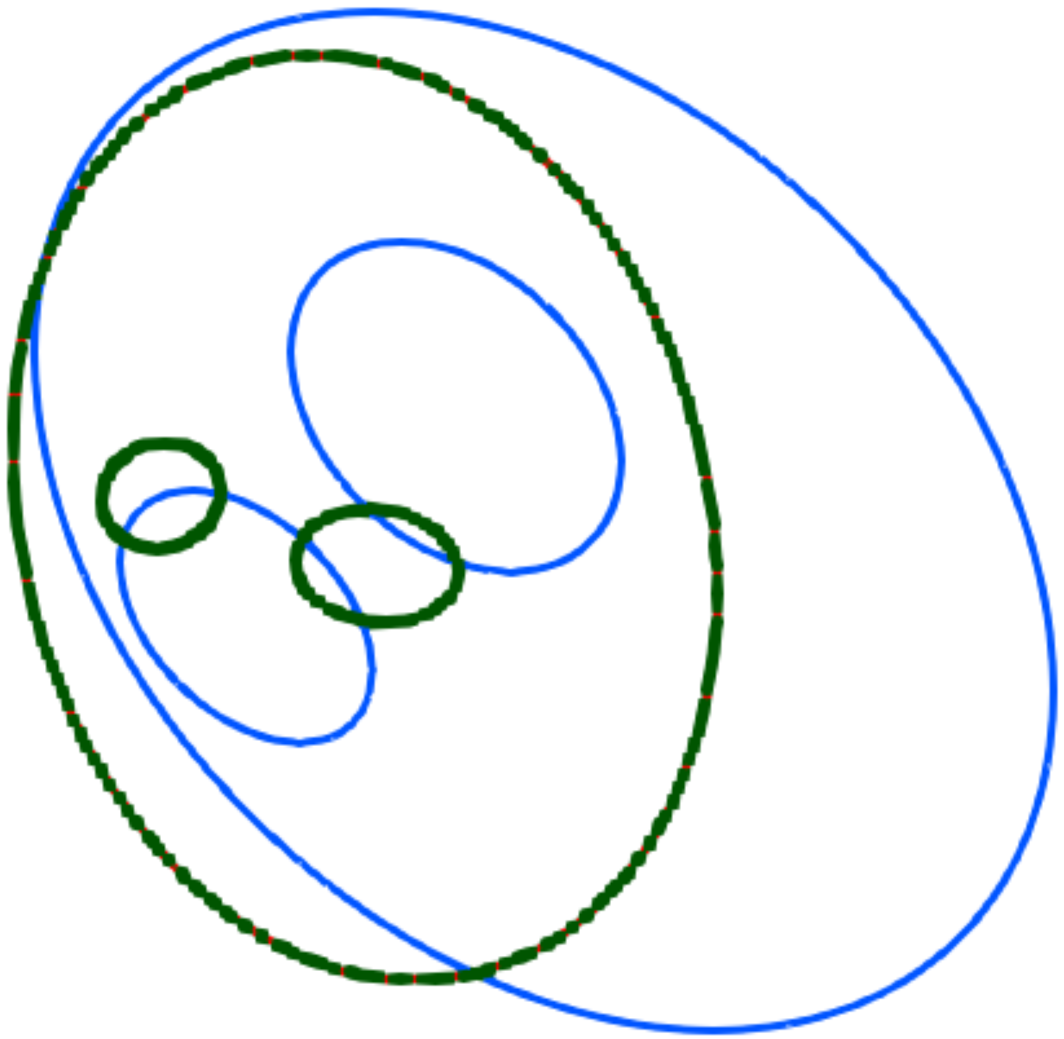


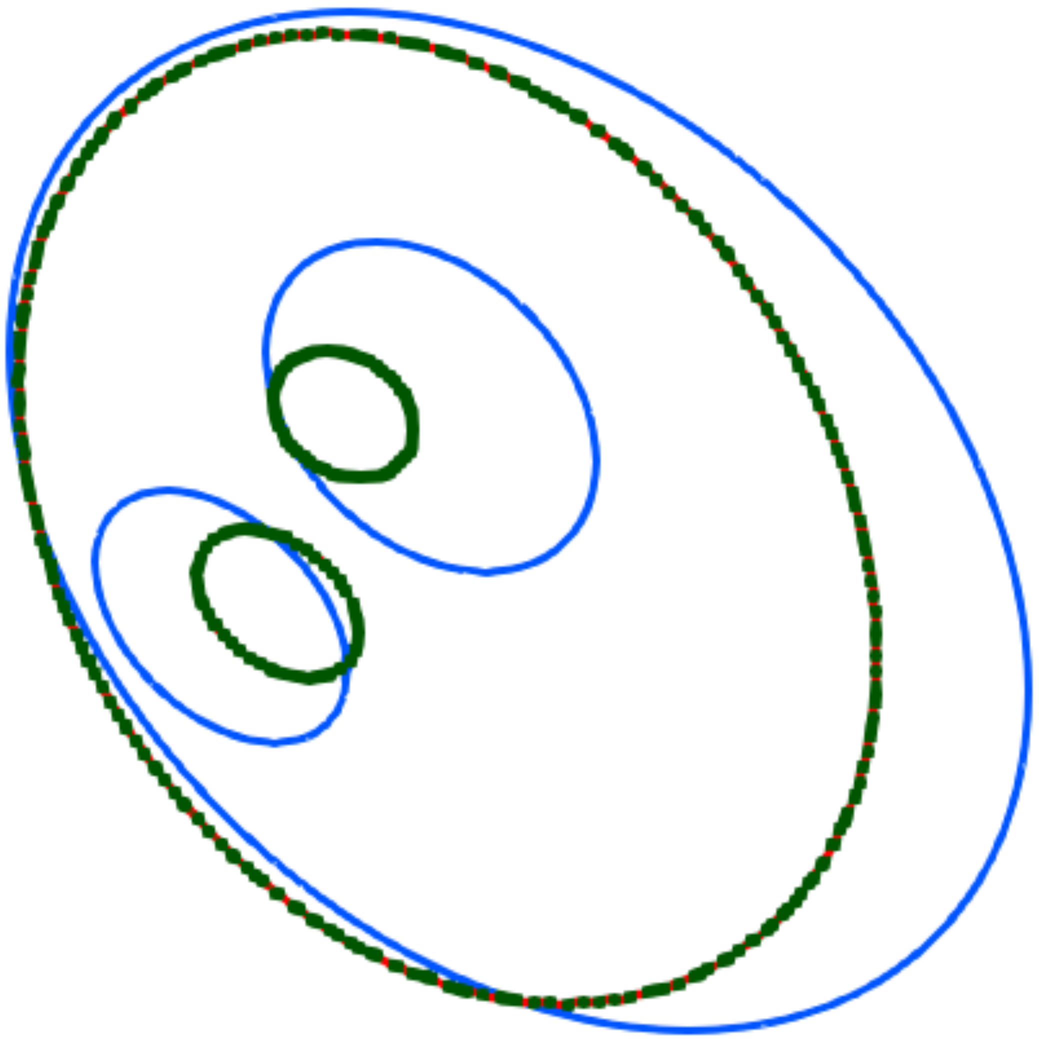


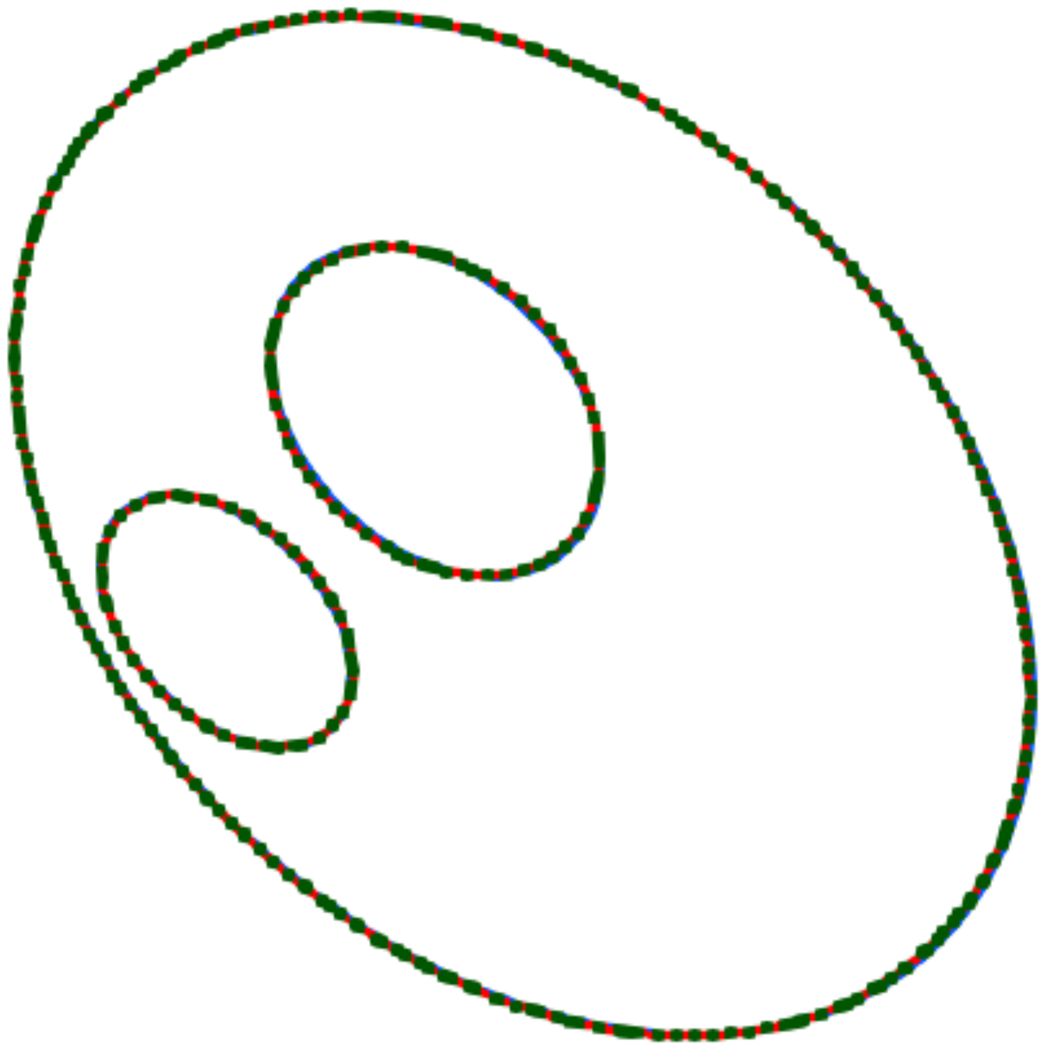
H - LOOMM

H' - invariant (scale + euclidean)

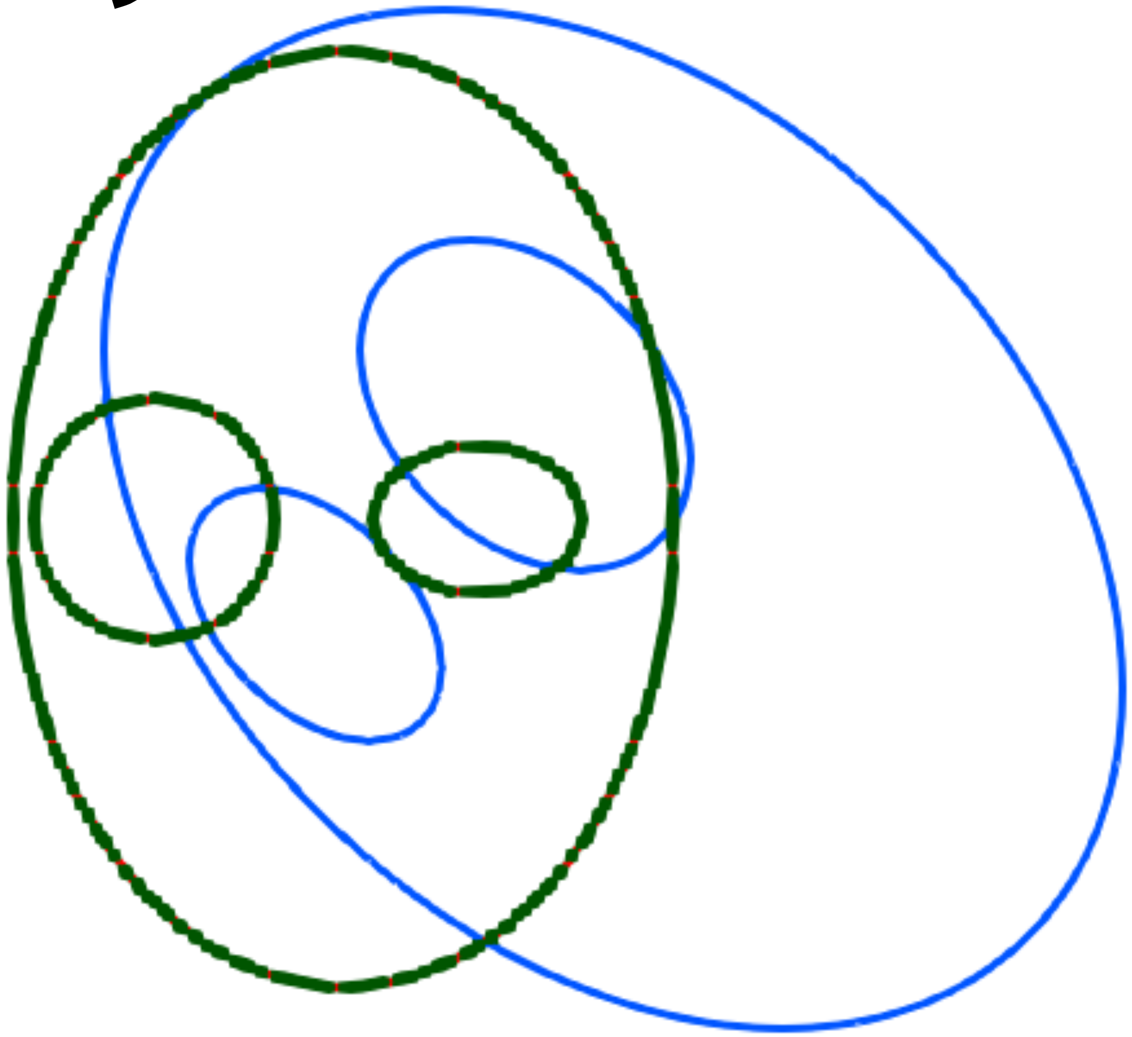


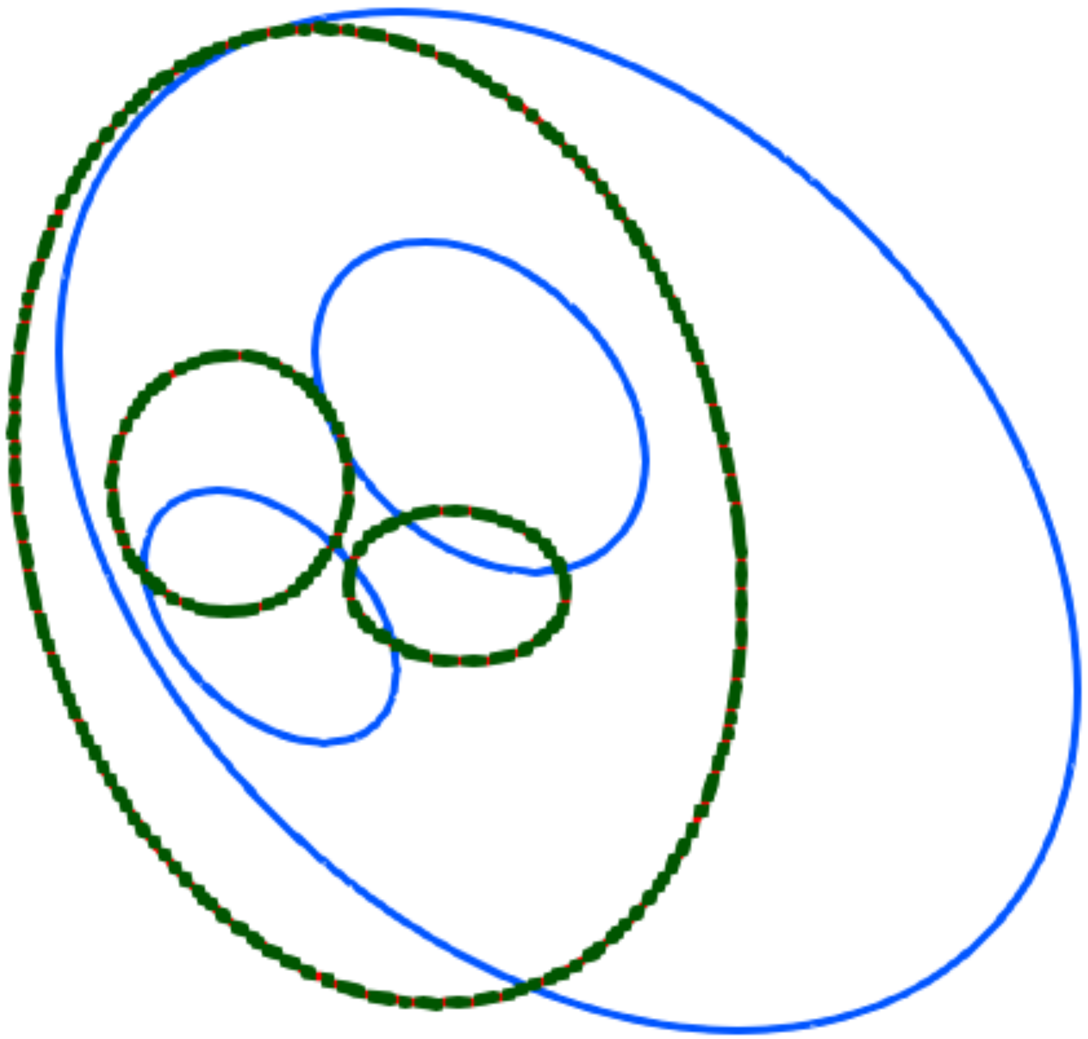


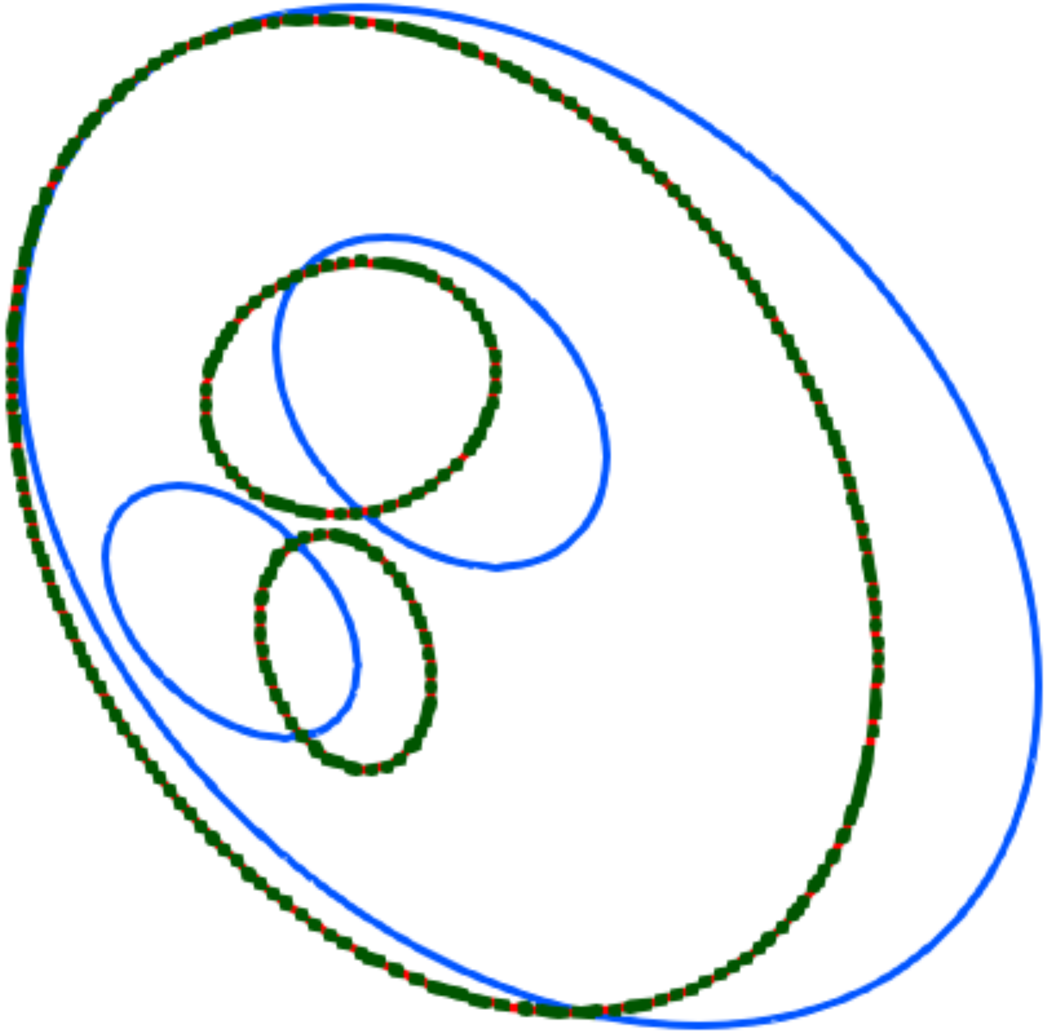


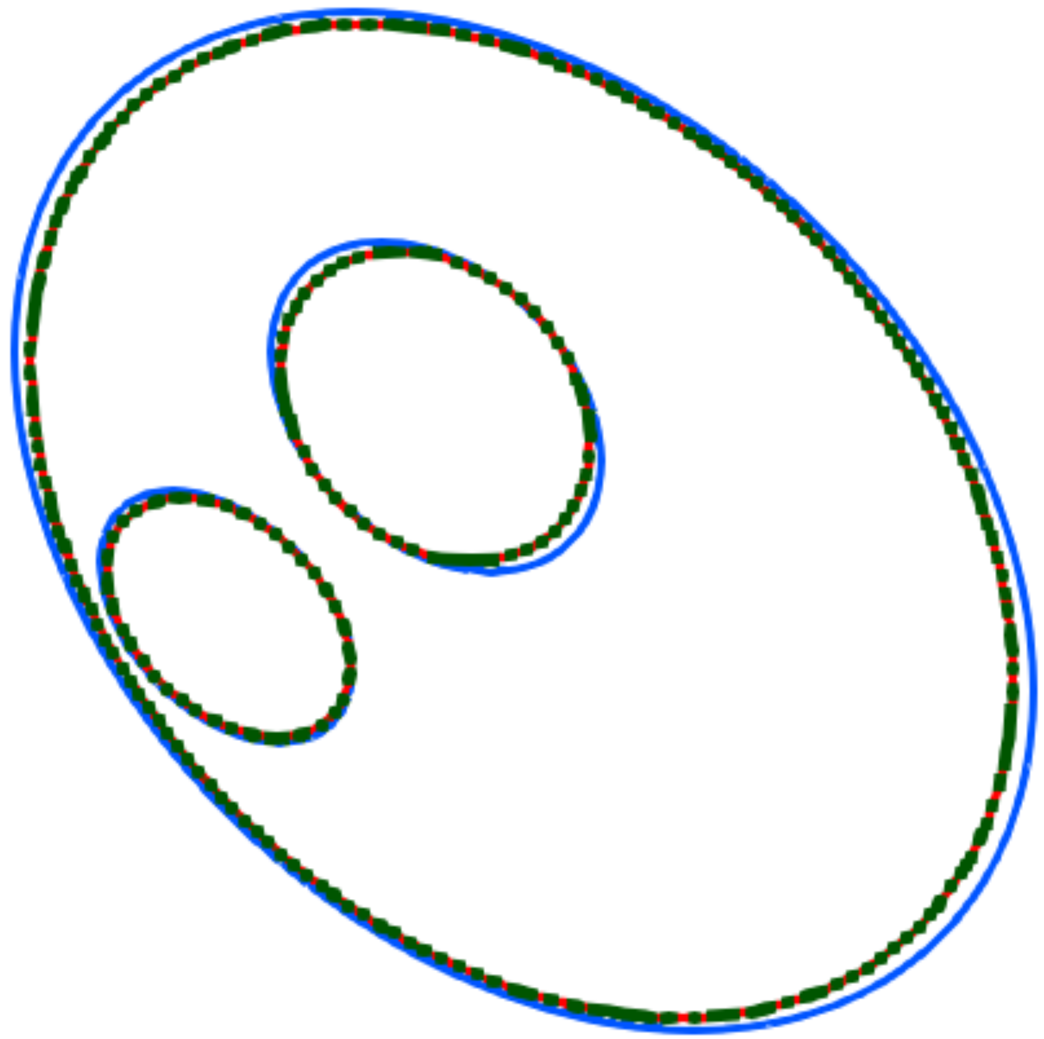


H_LOOMM
H'- invariant (euclidean)

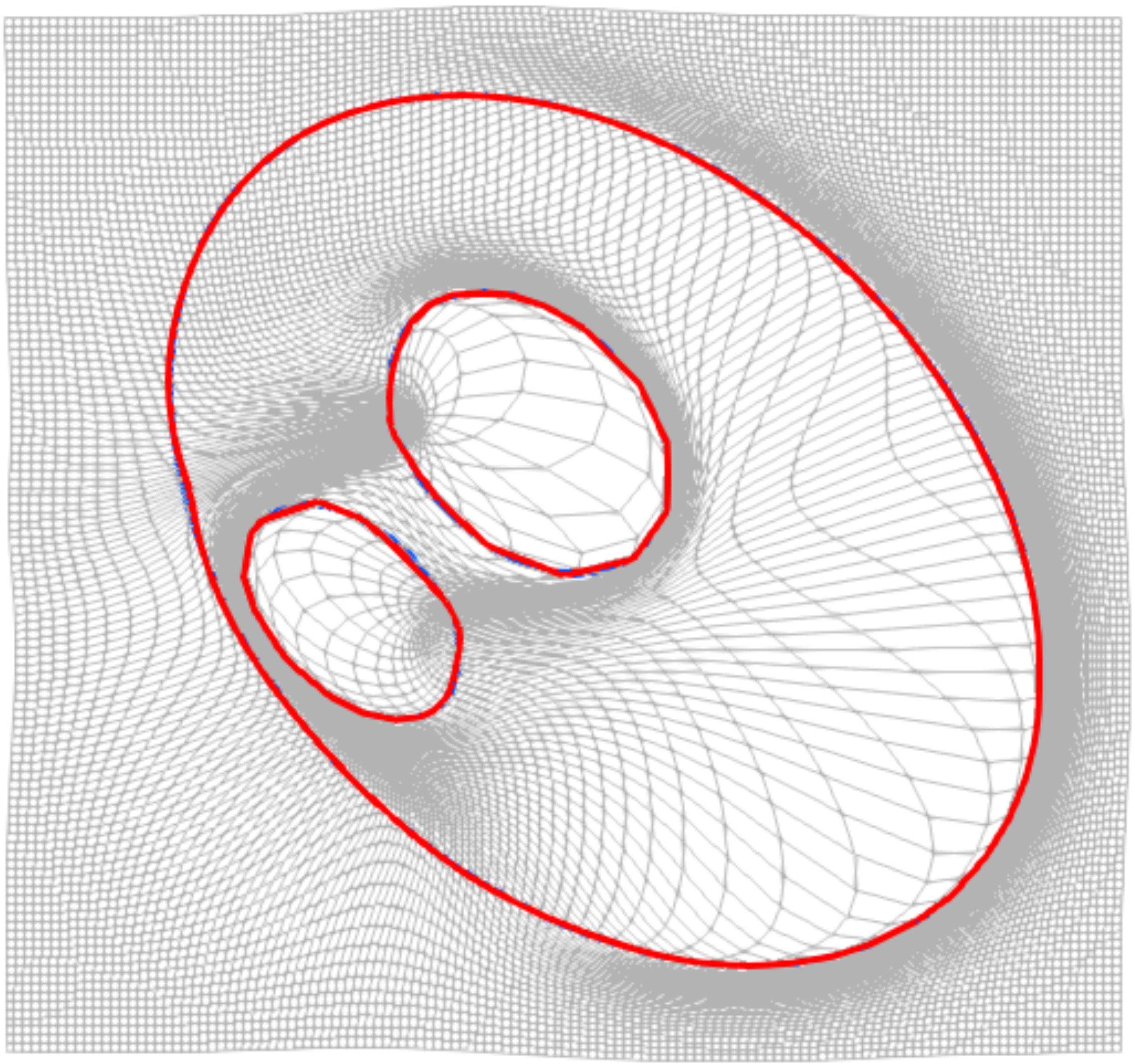






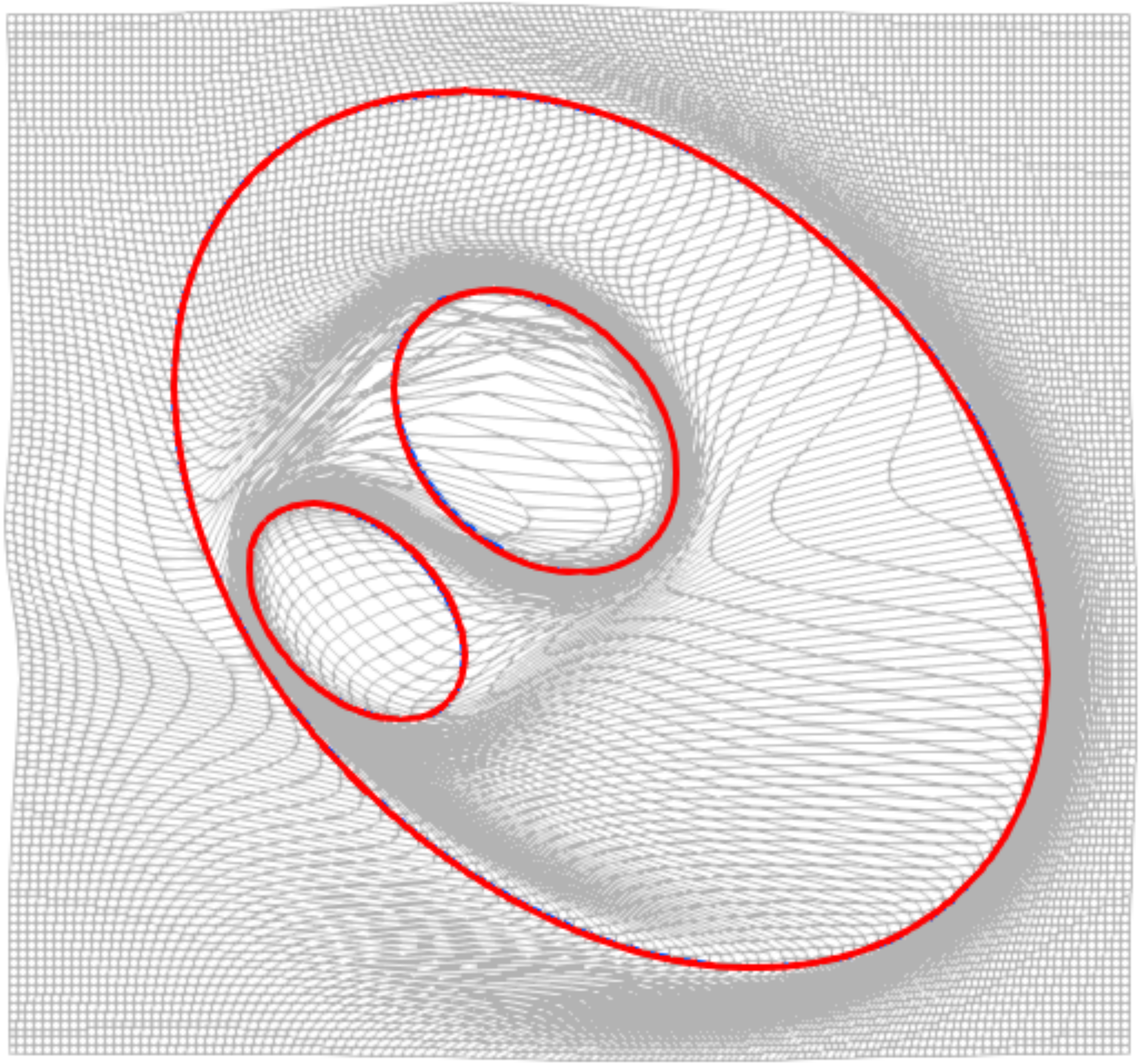


LDDMM



H_LDDMM

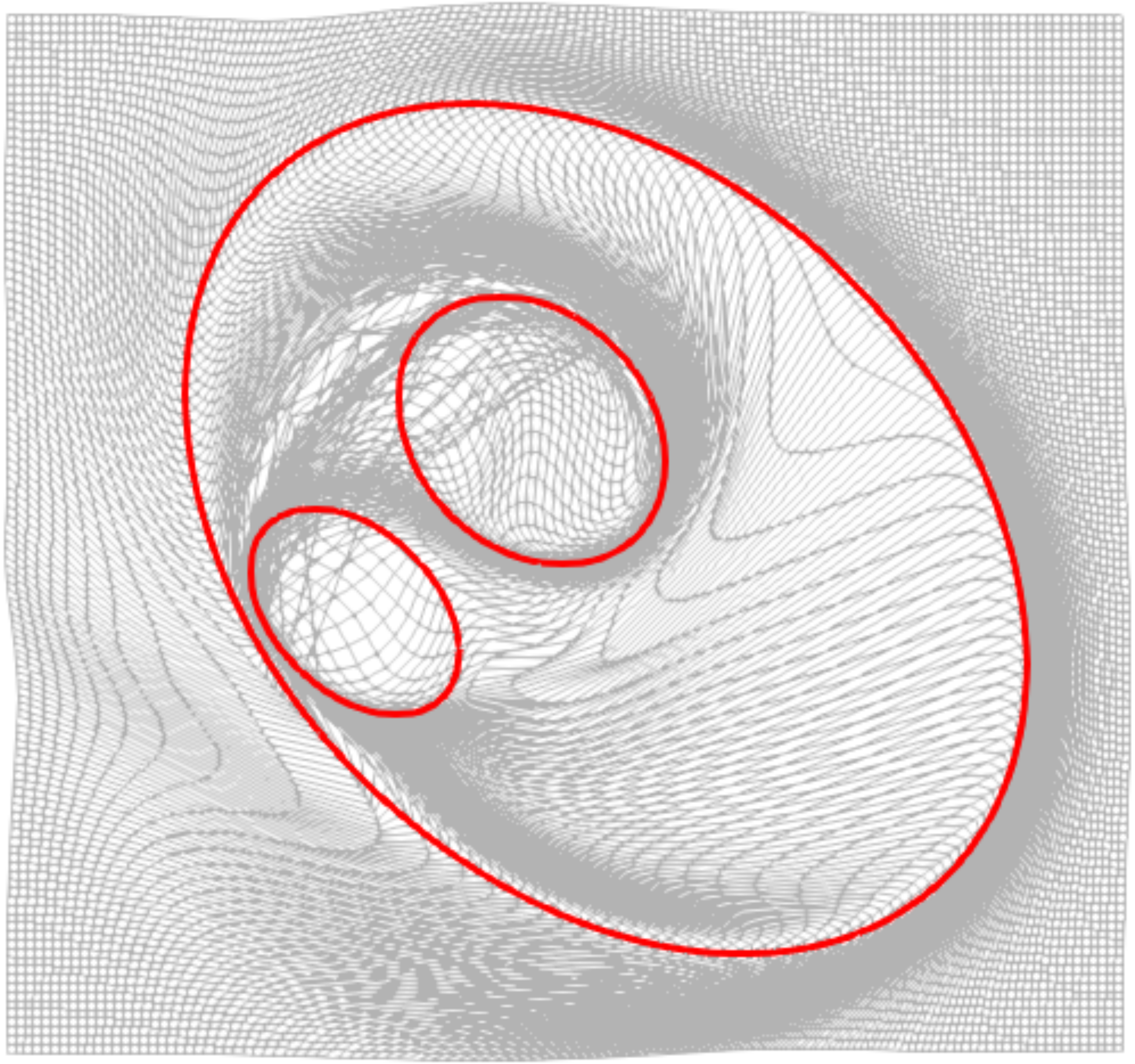
H¹-invariant (scale + euclidean)

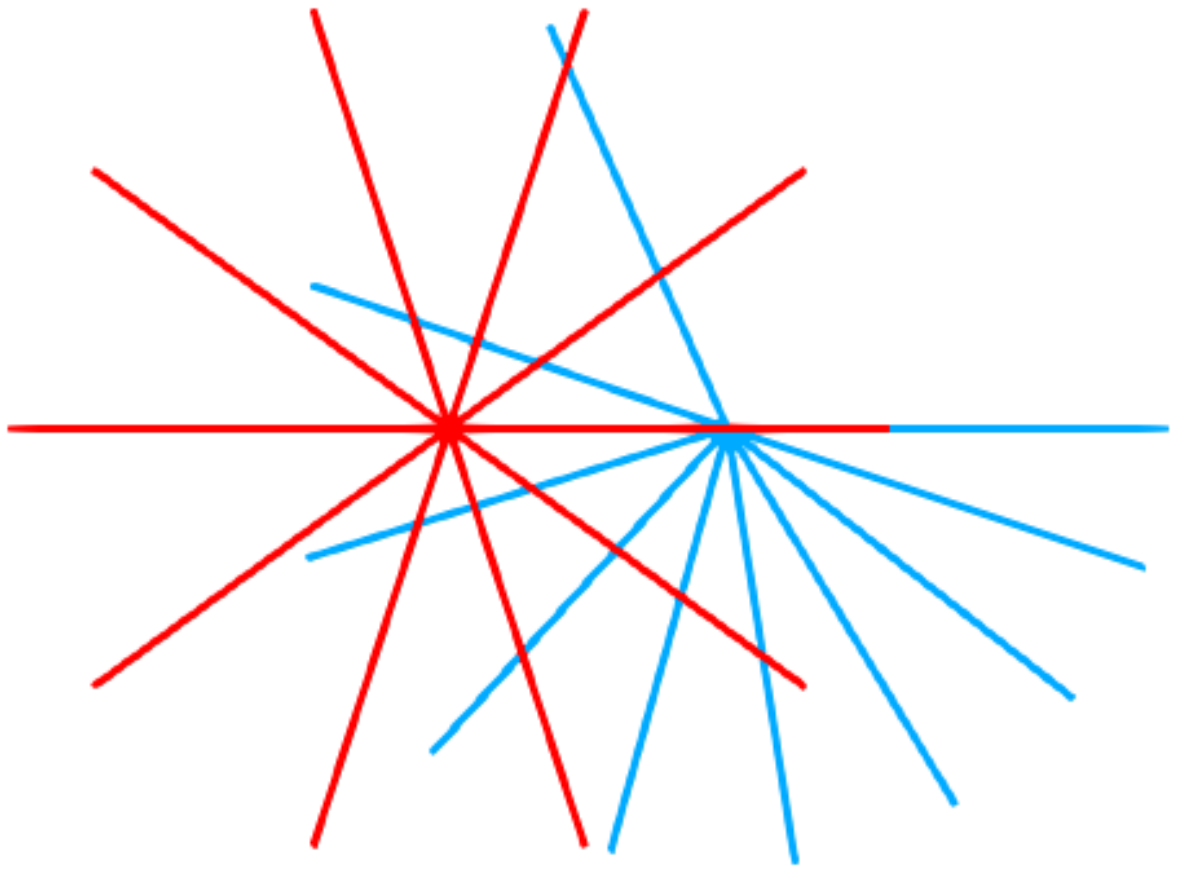


H_LDDMM

H'_invariant

(euclidean)





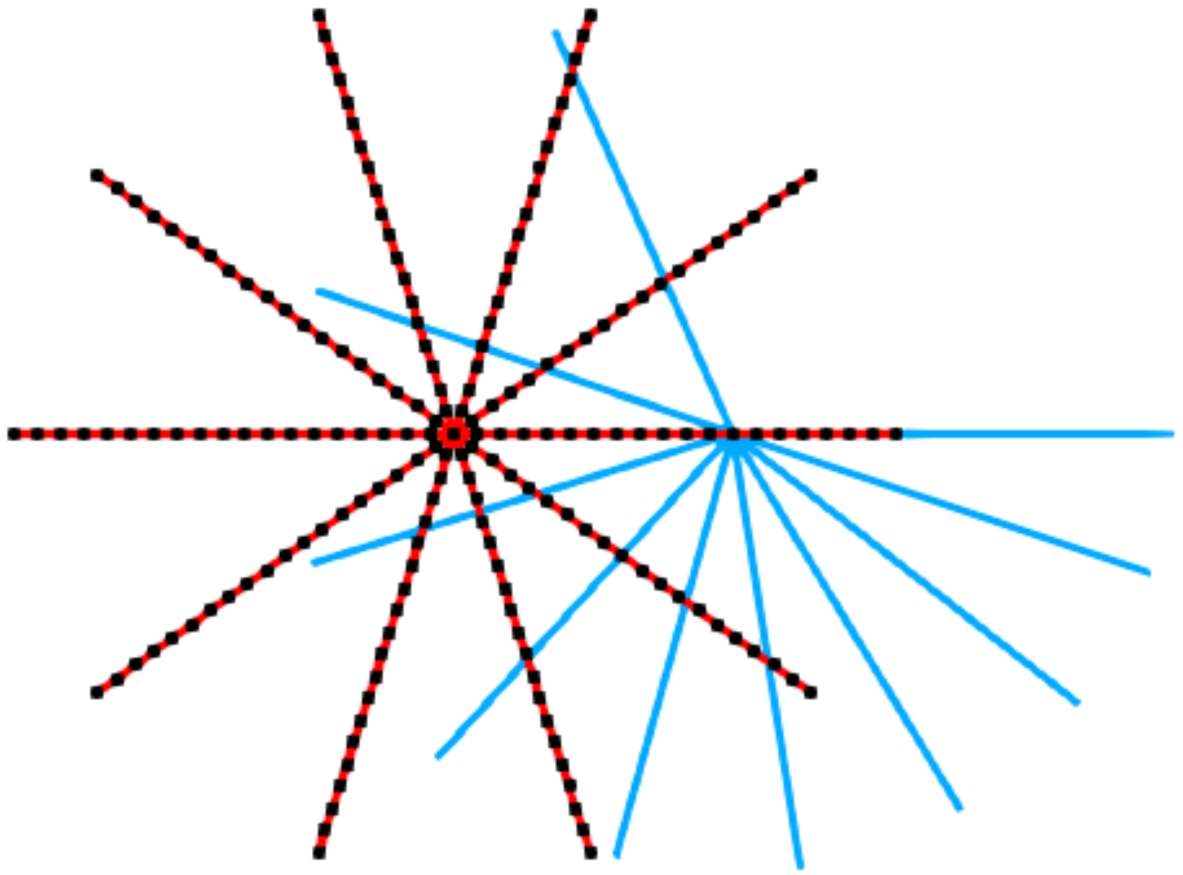
→ No explicit ray labelling

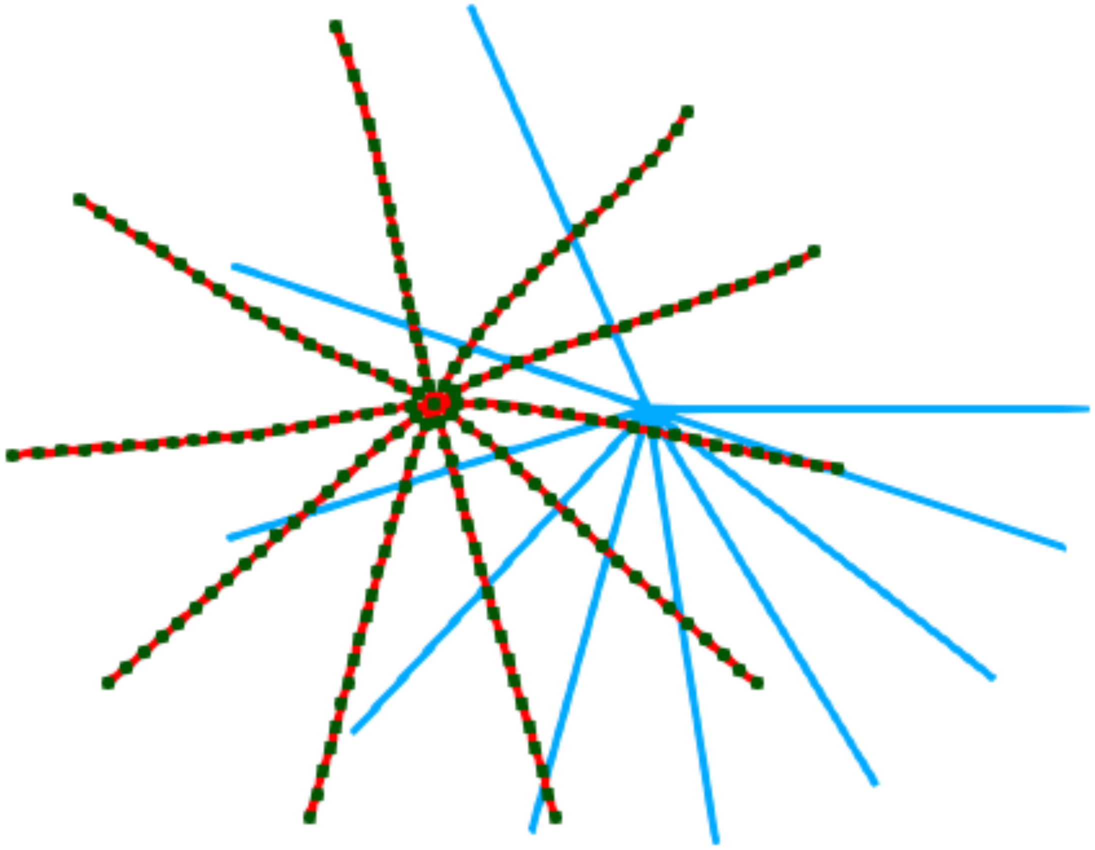
→ Mapping challenging ↔ many possible local minima

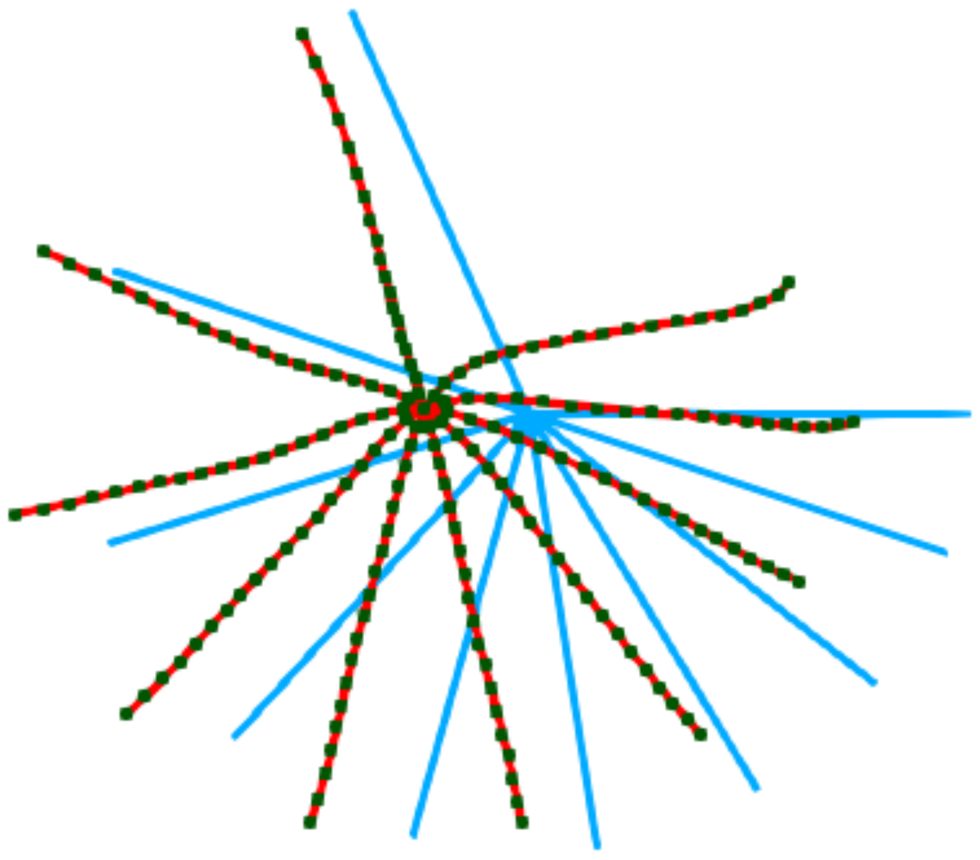
→ LDDMM with small kernel fails

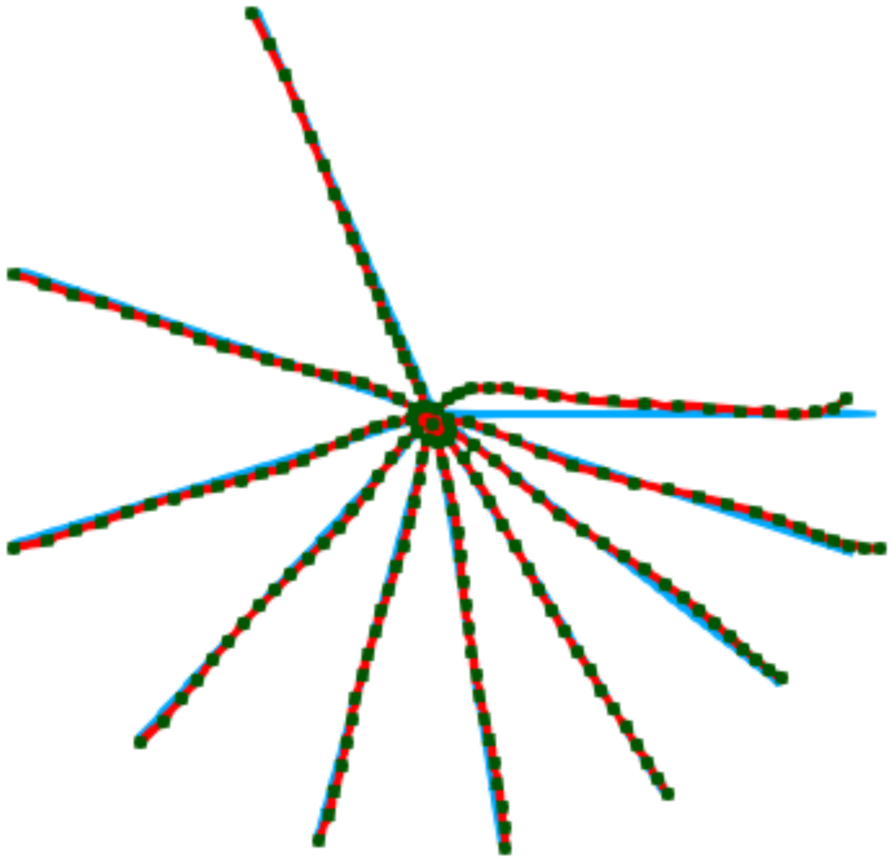
→ example use "smoother" kernel.

LDDMM



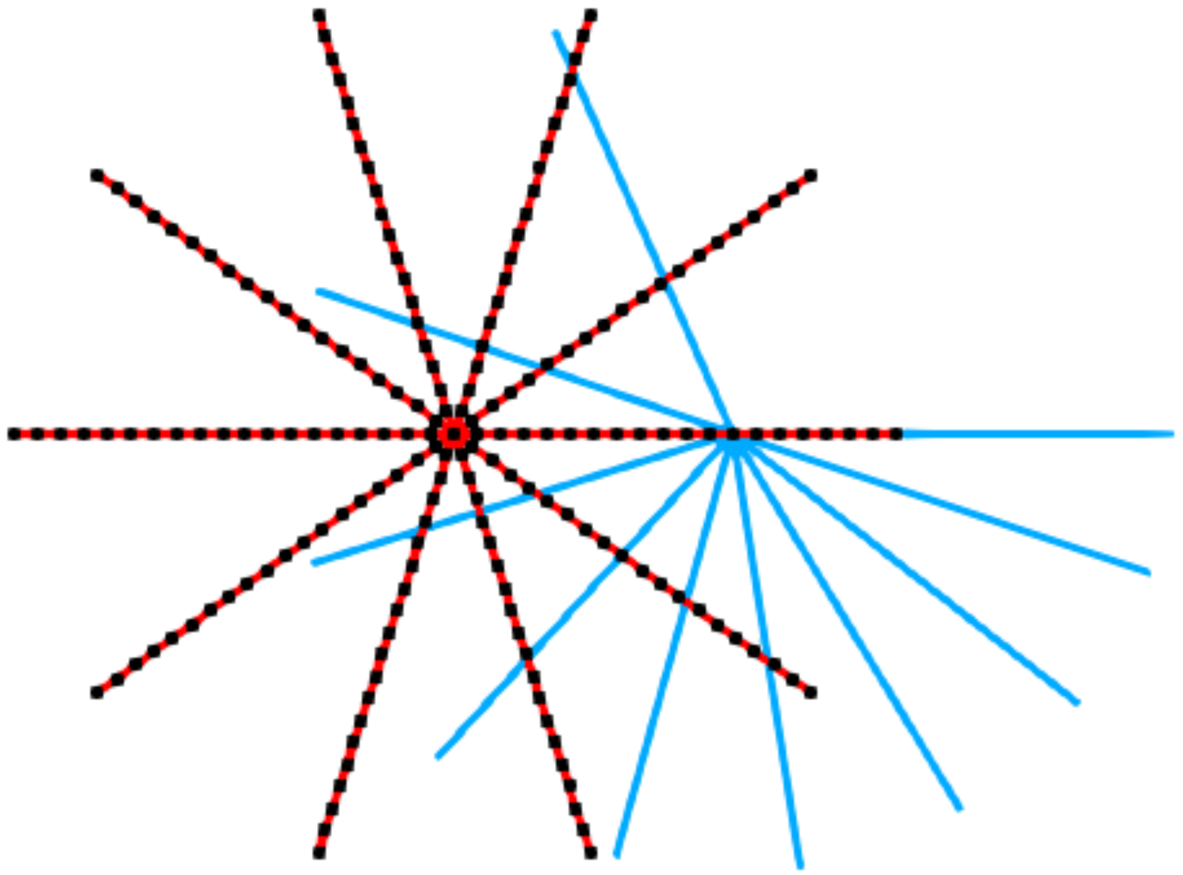


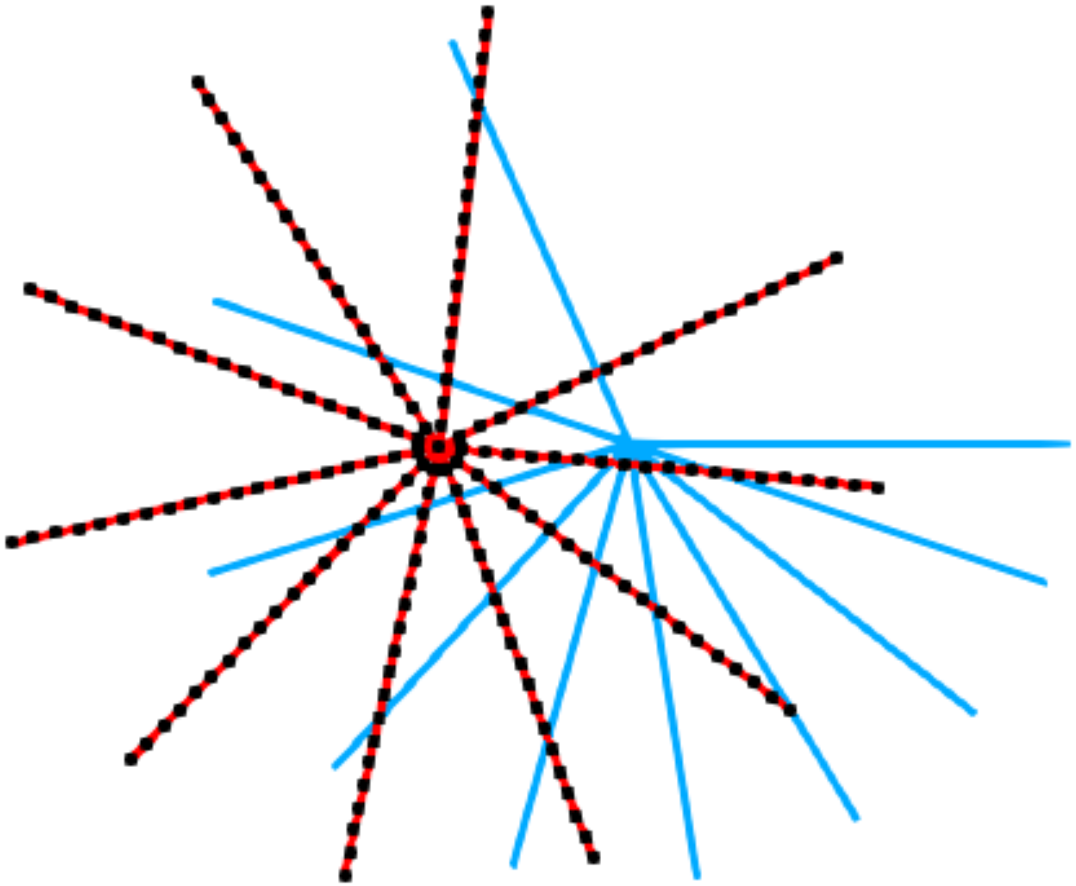


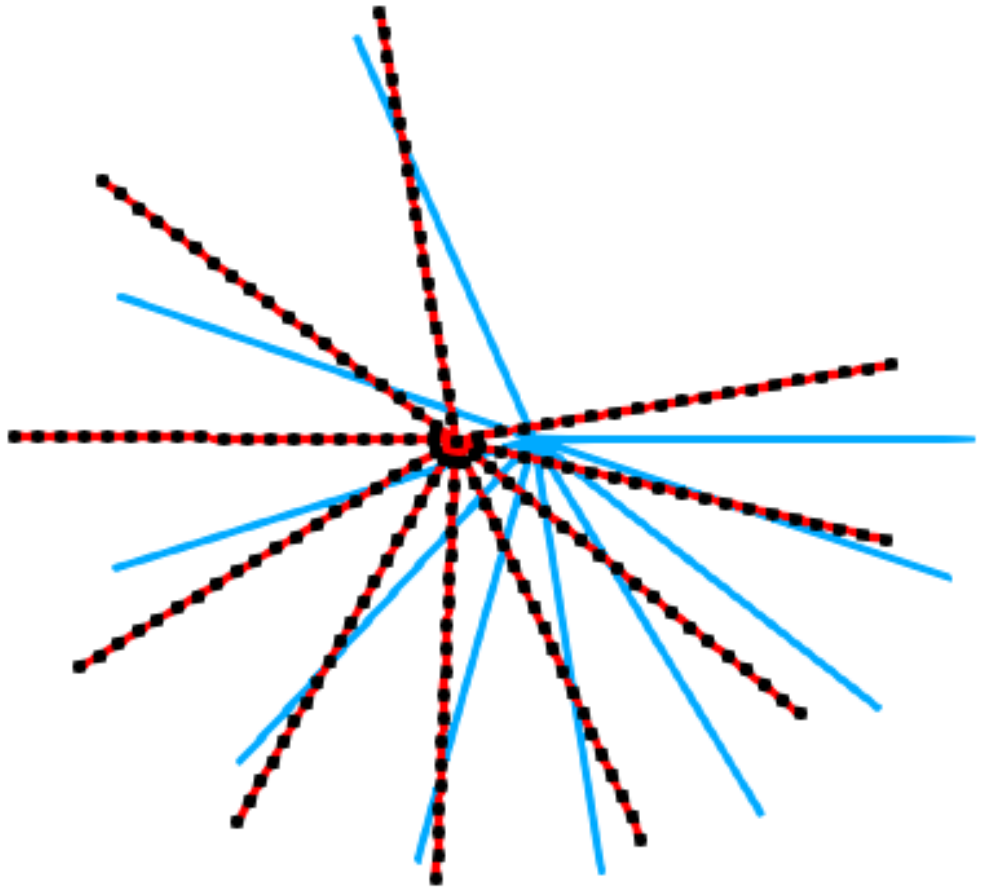


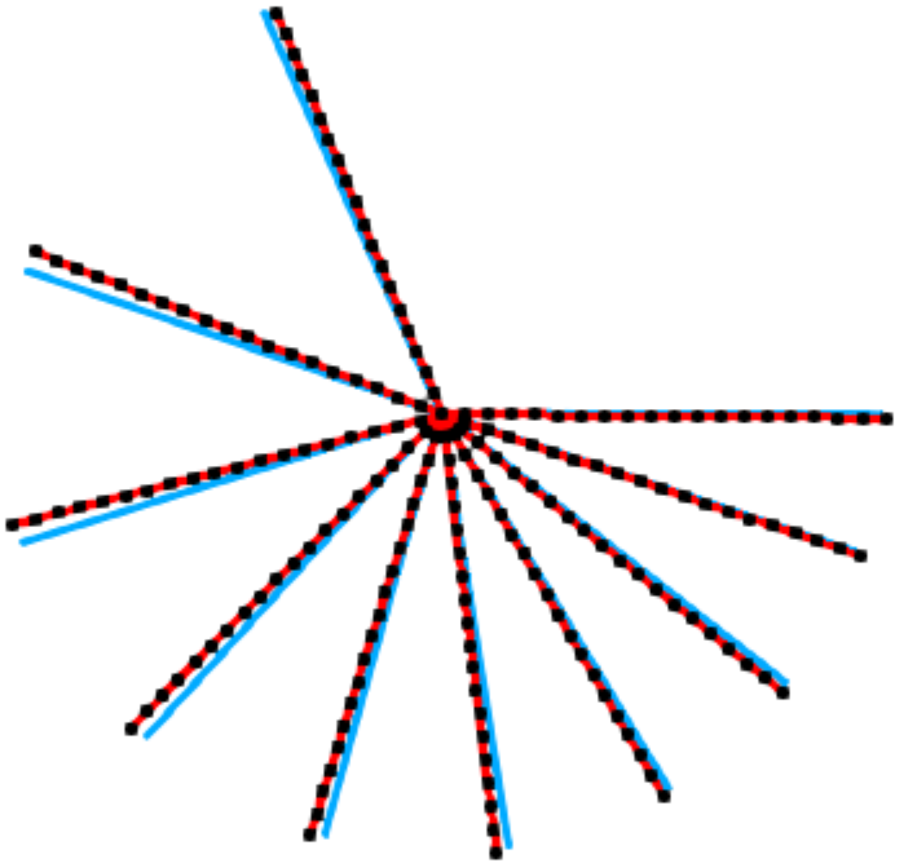
H_LDDMM

H'_invariant

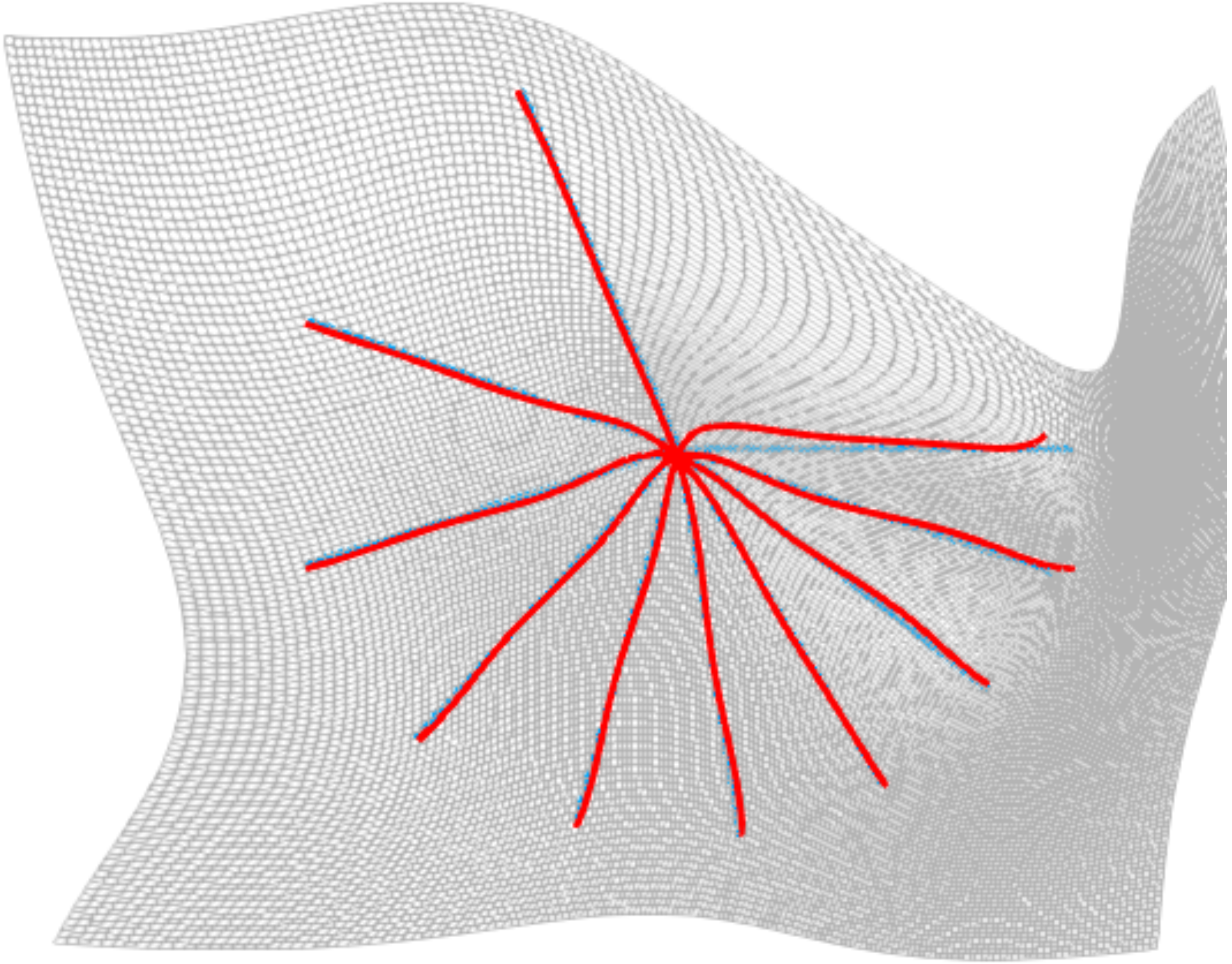




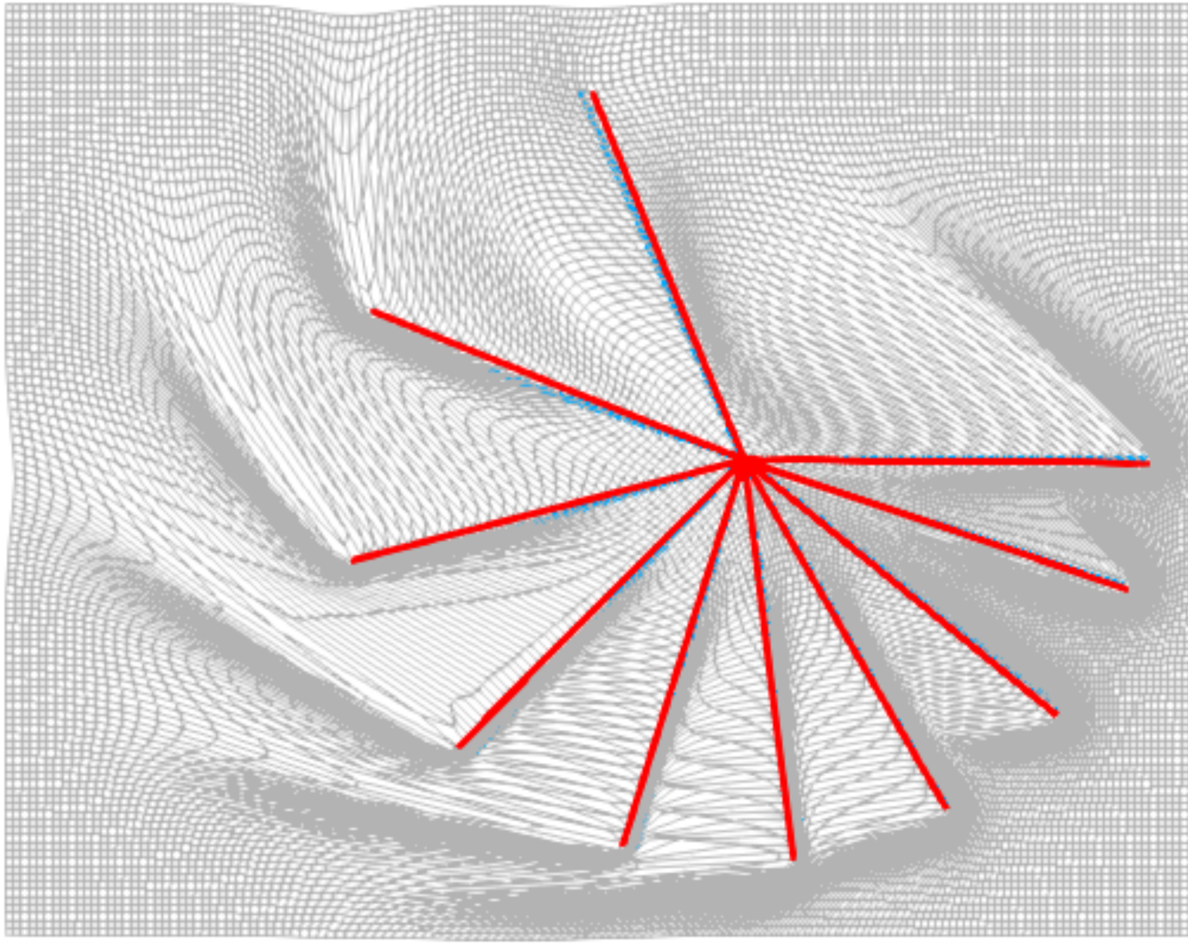




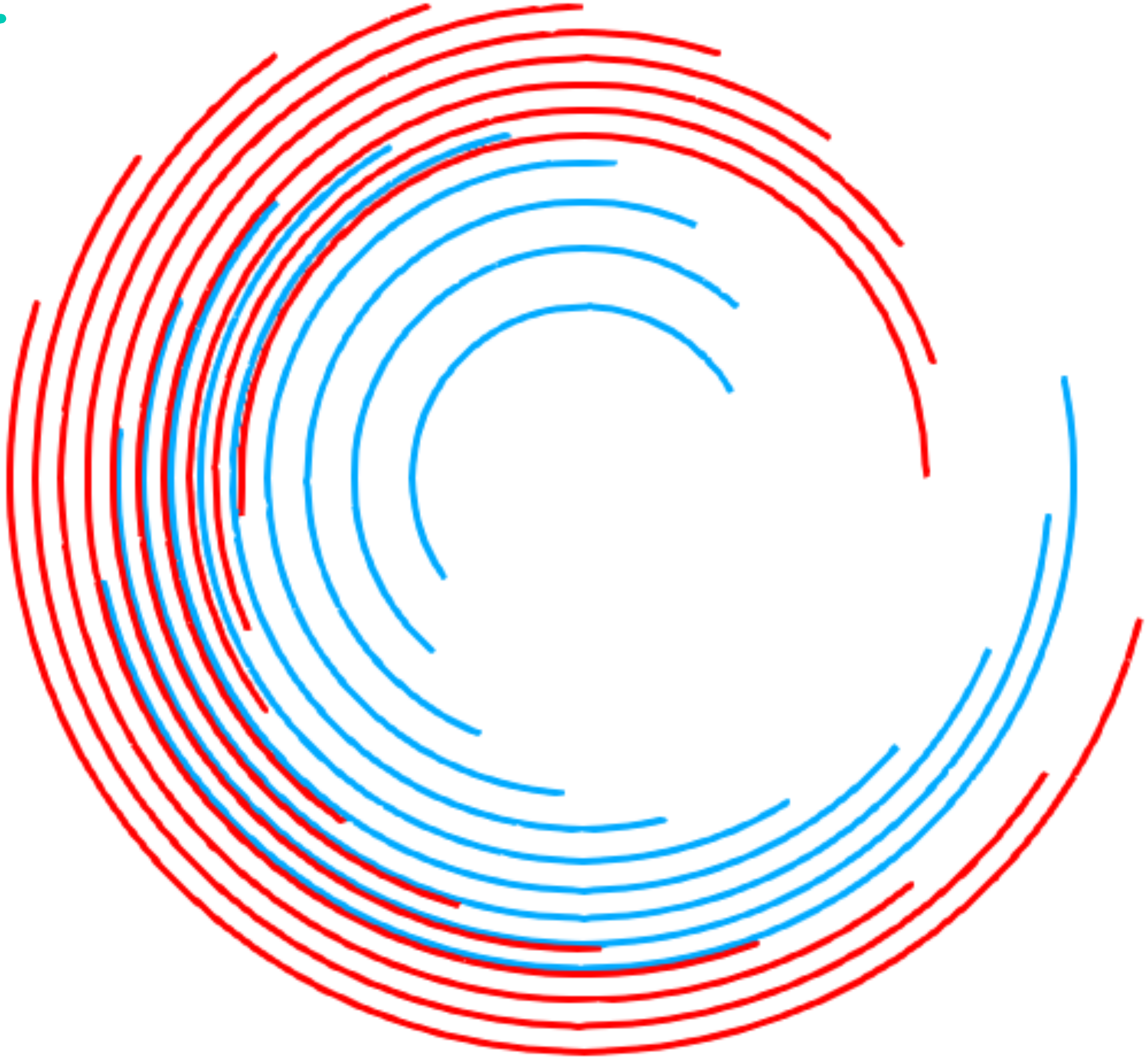
LDDMM



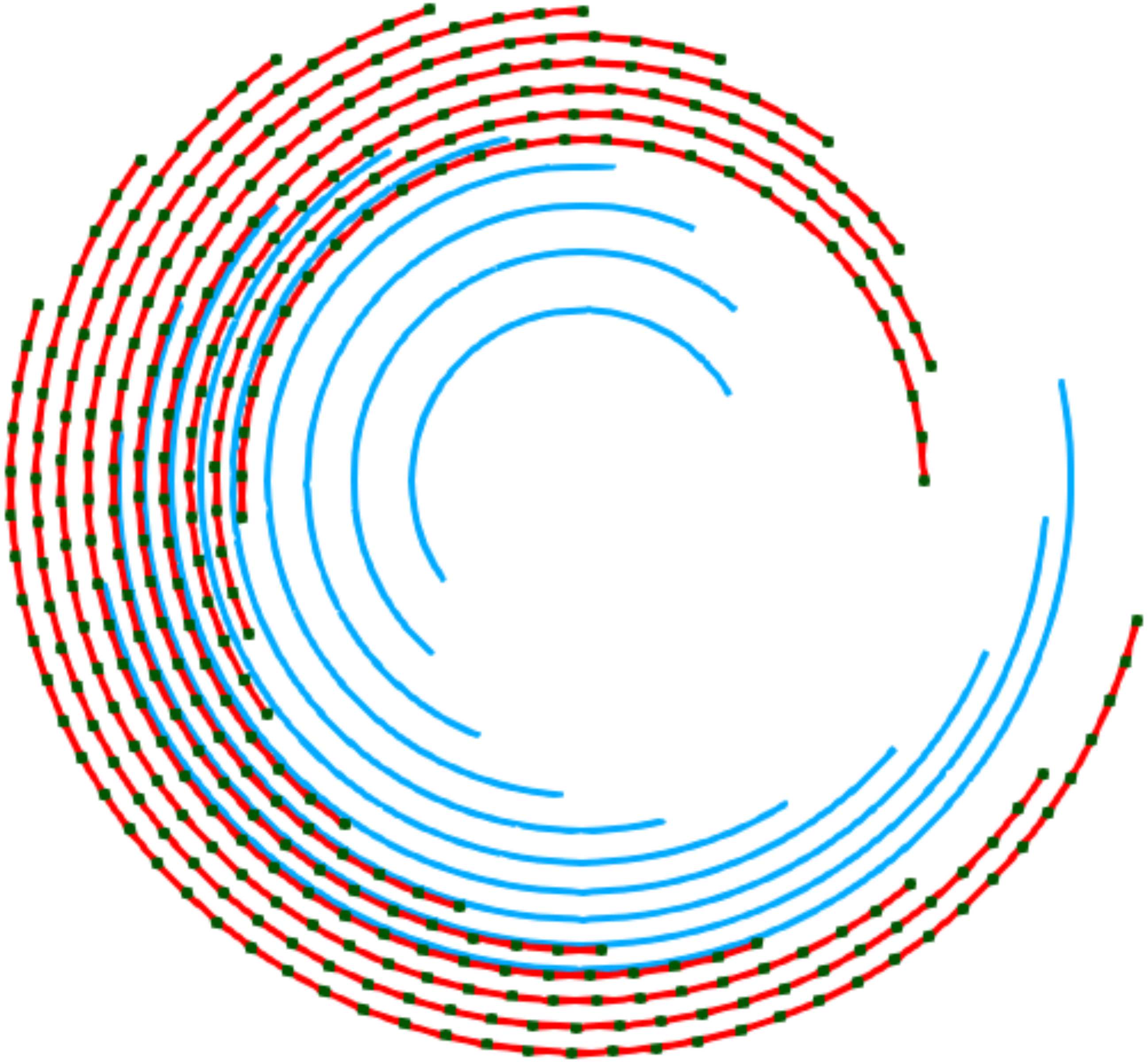
H_LDDMM

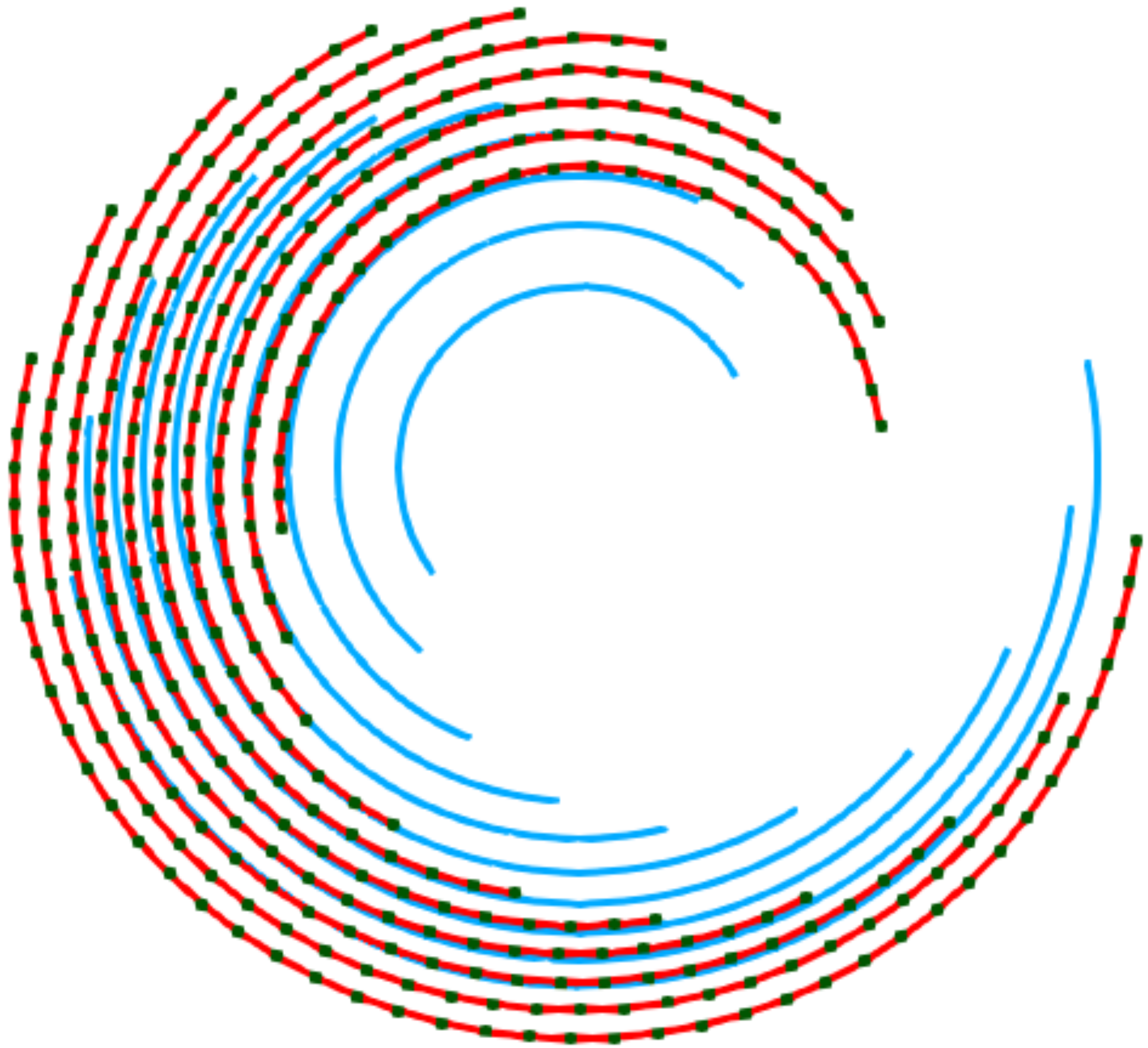


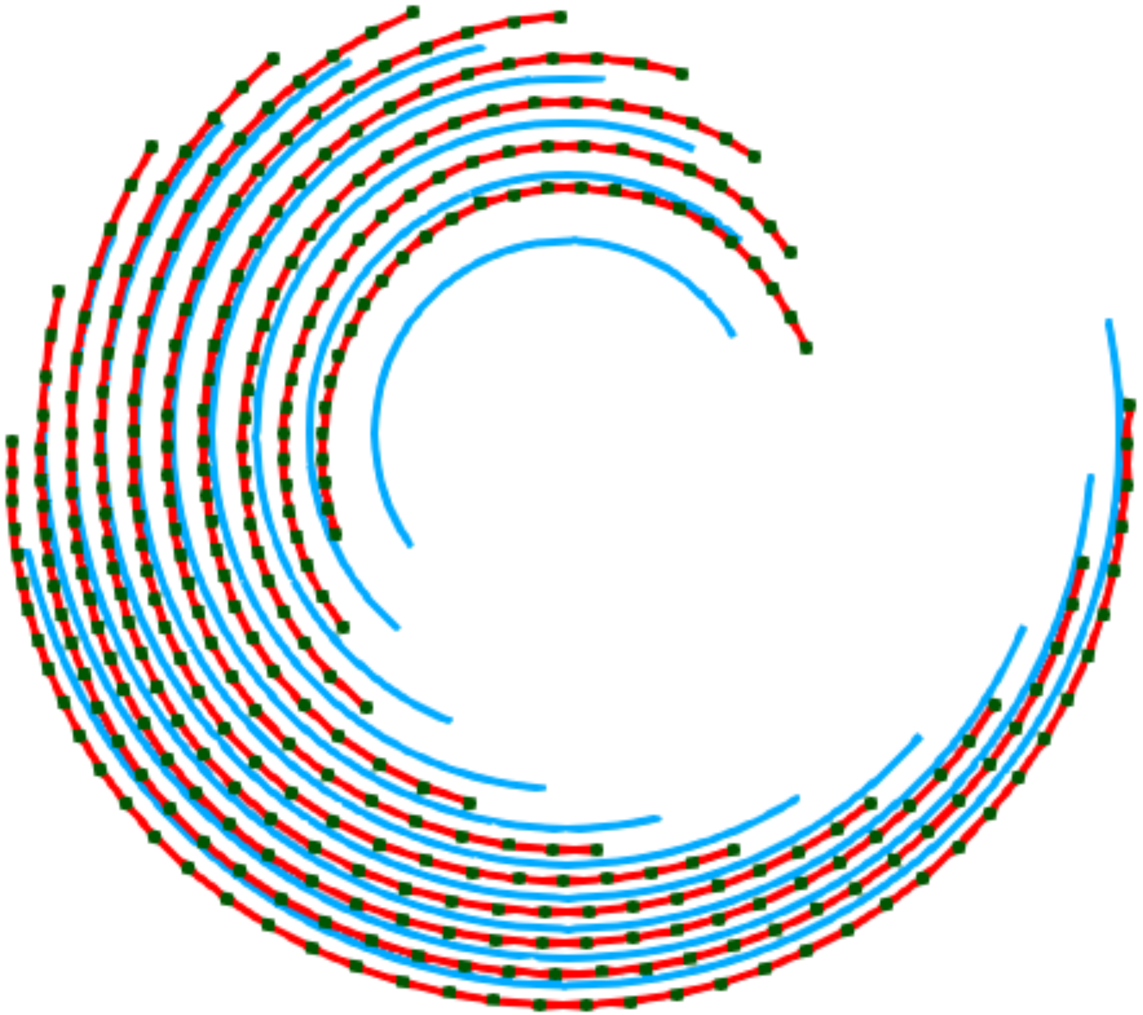
Arcs of circle

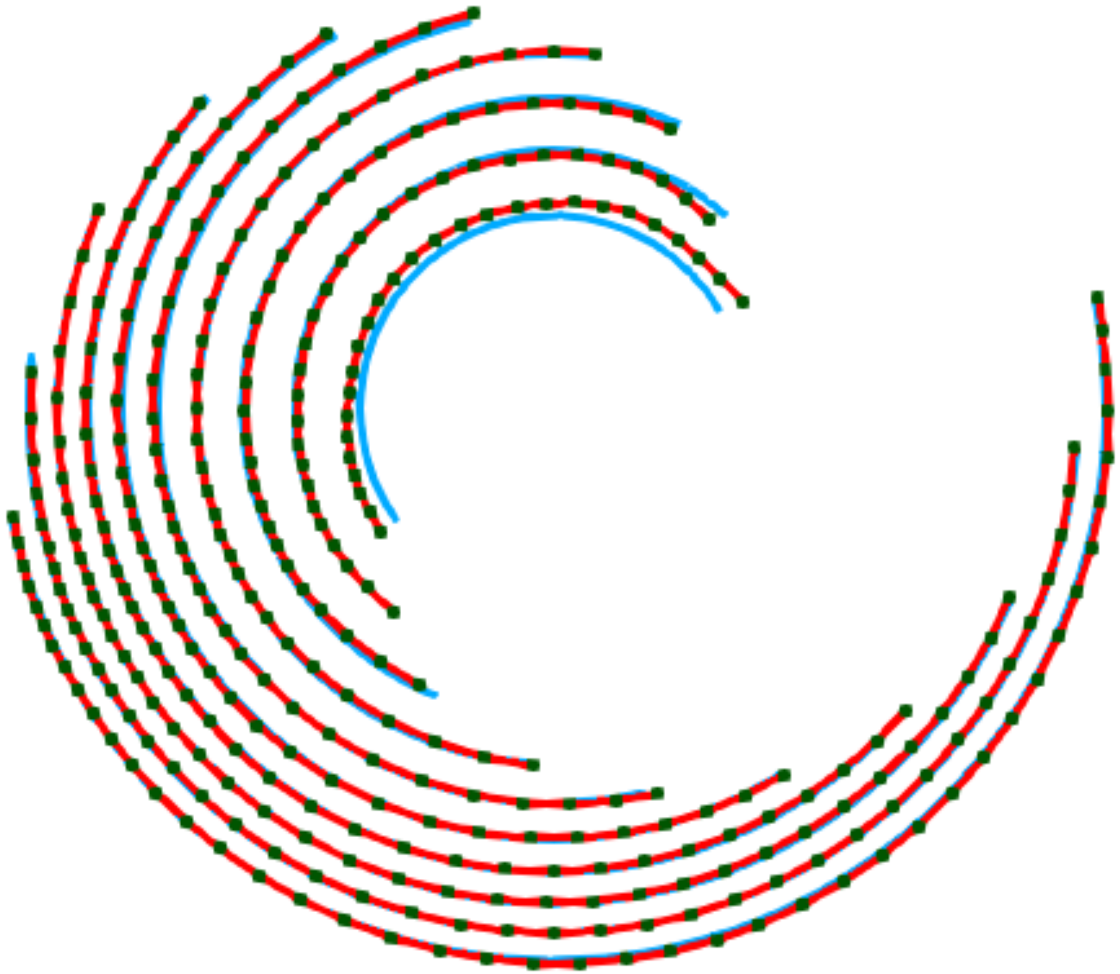


LDDMM

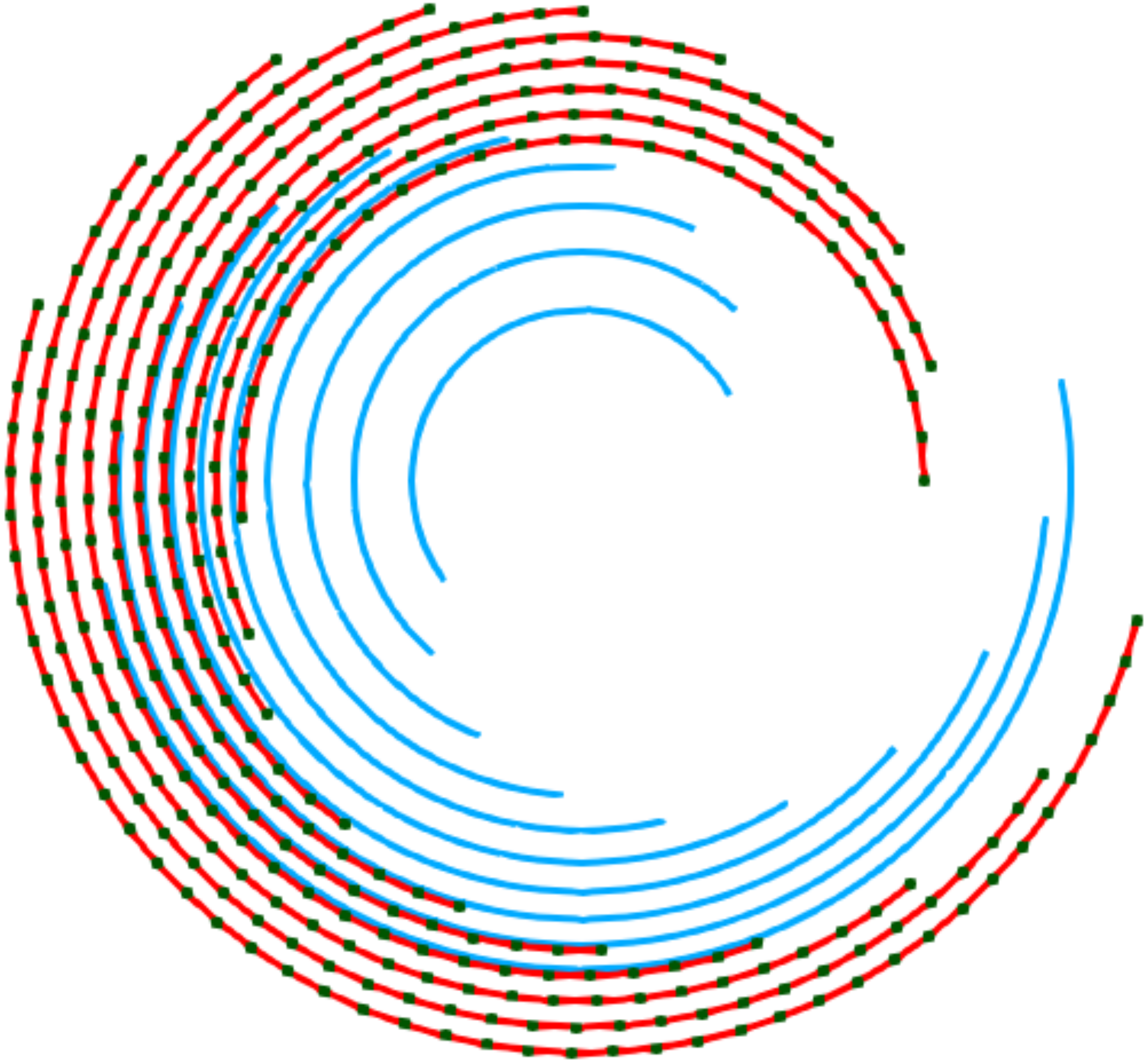


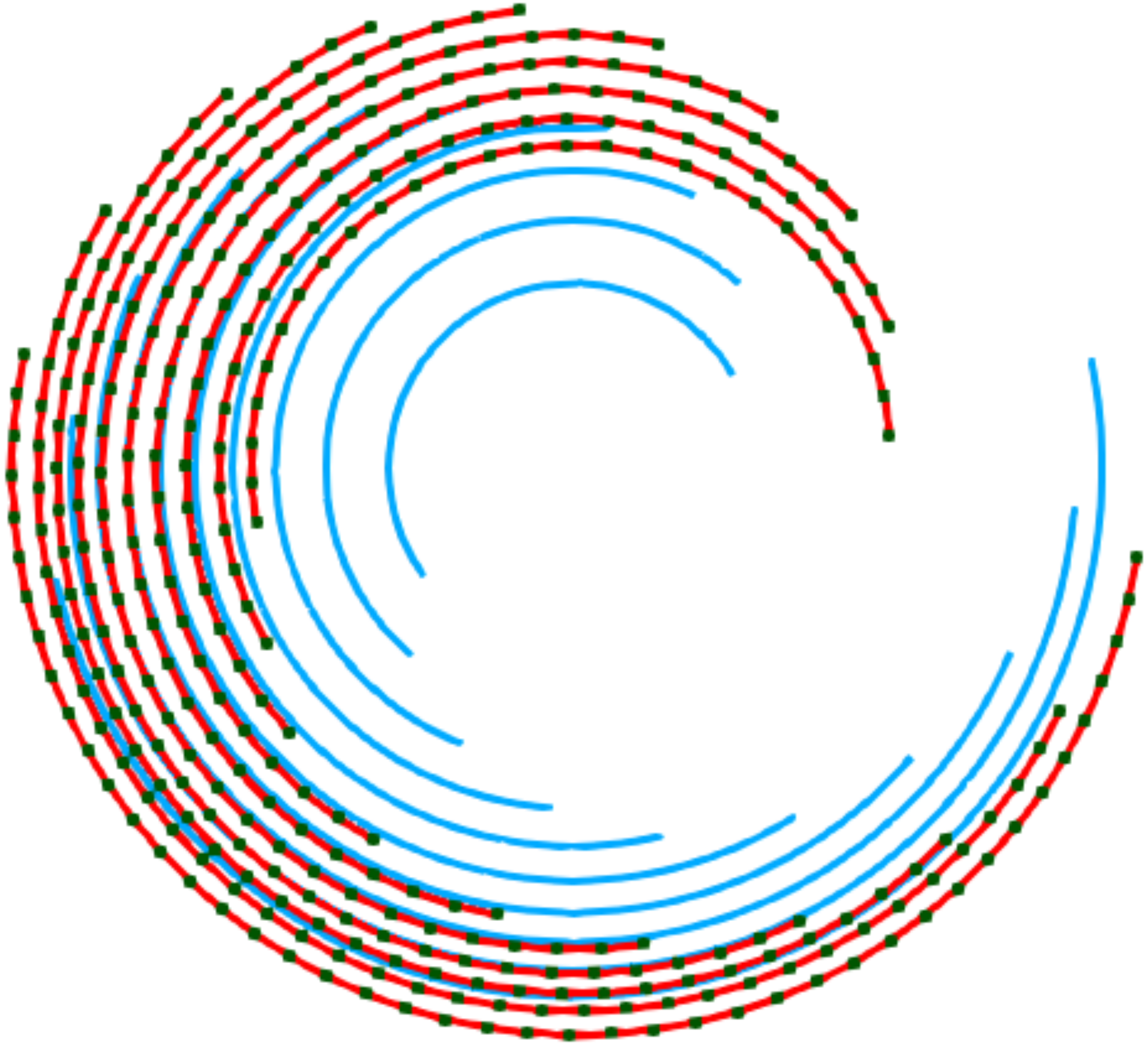


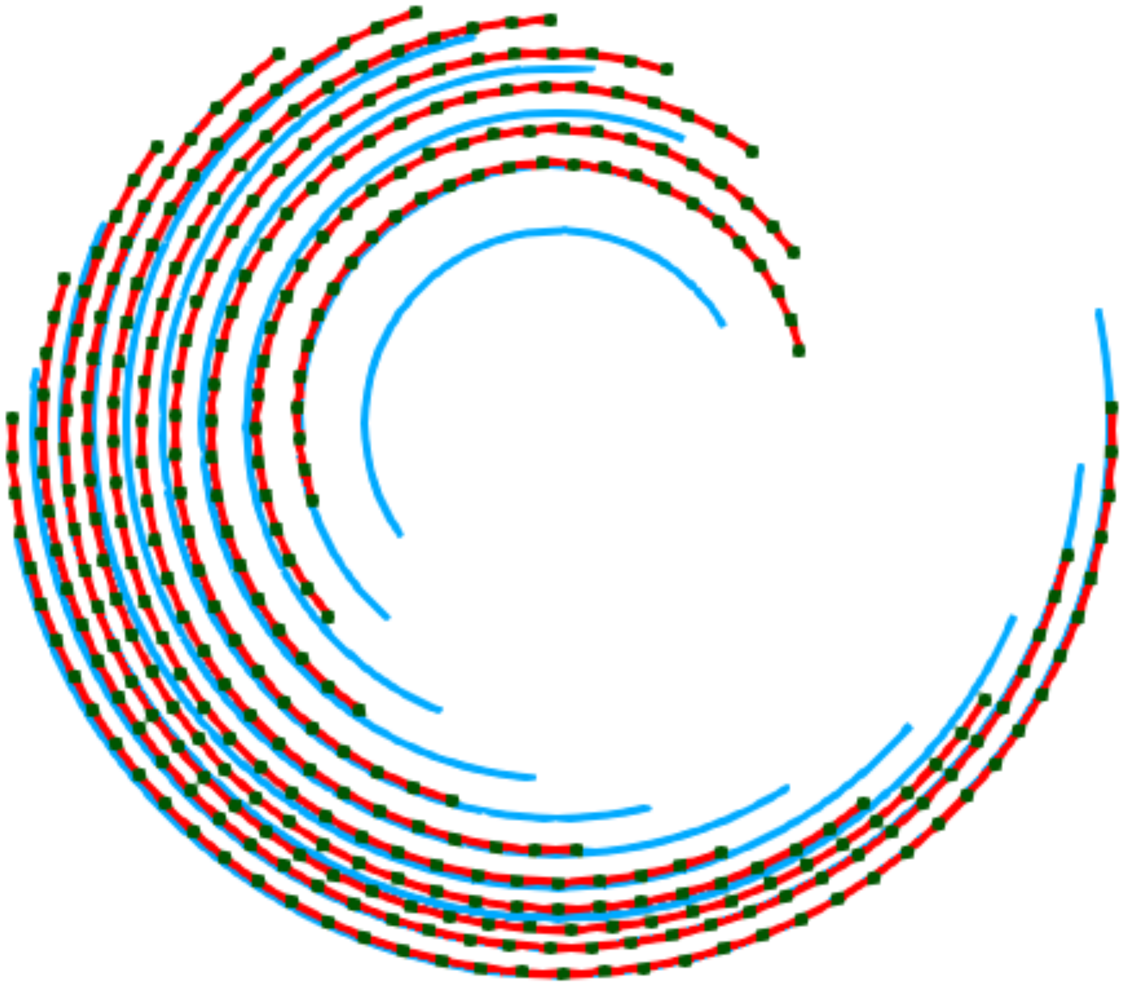


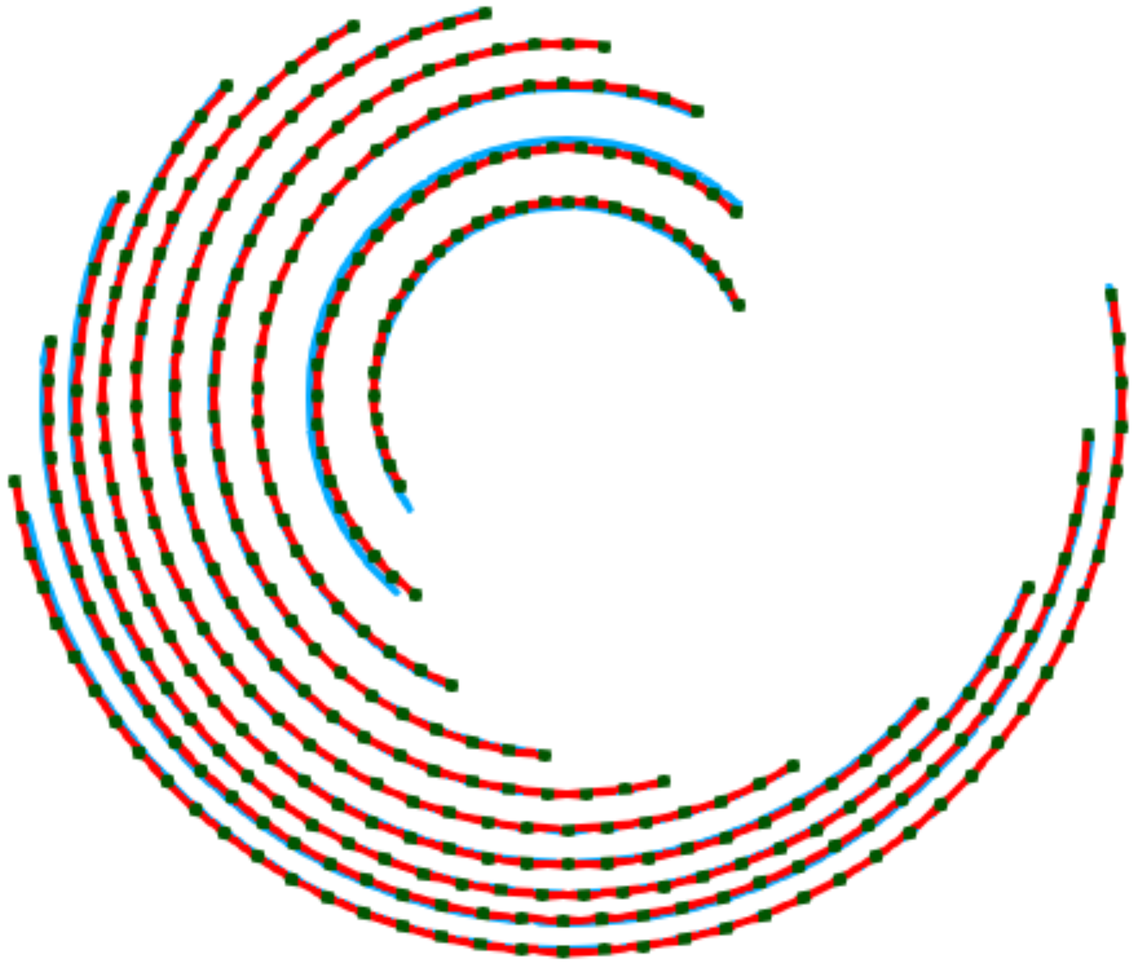


H_LODNM

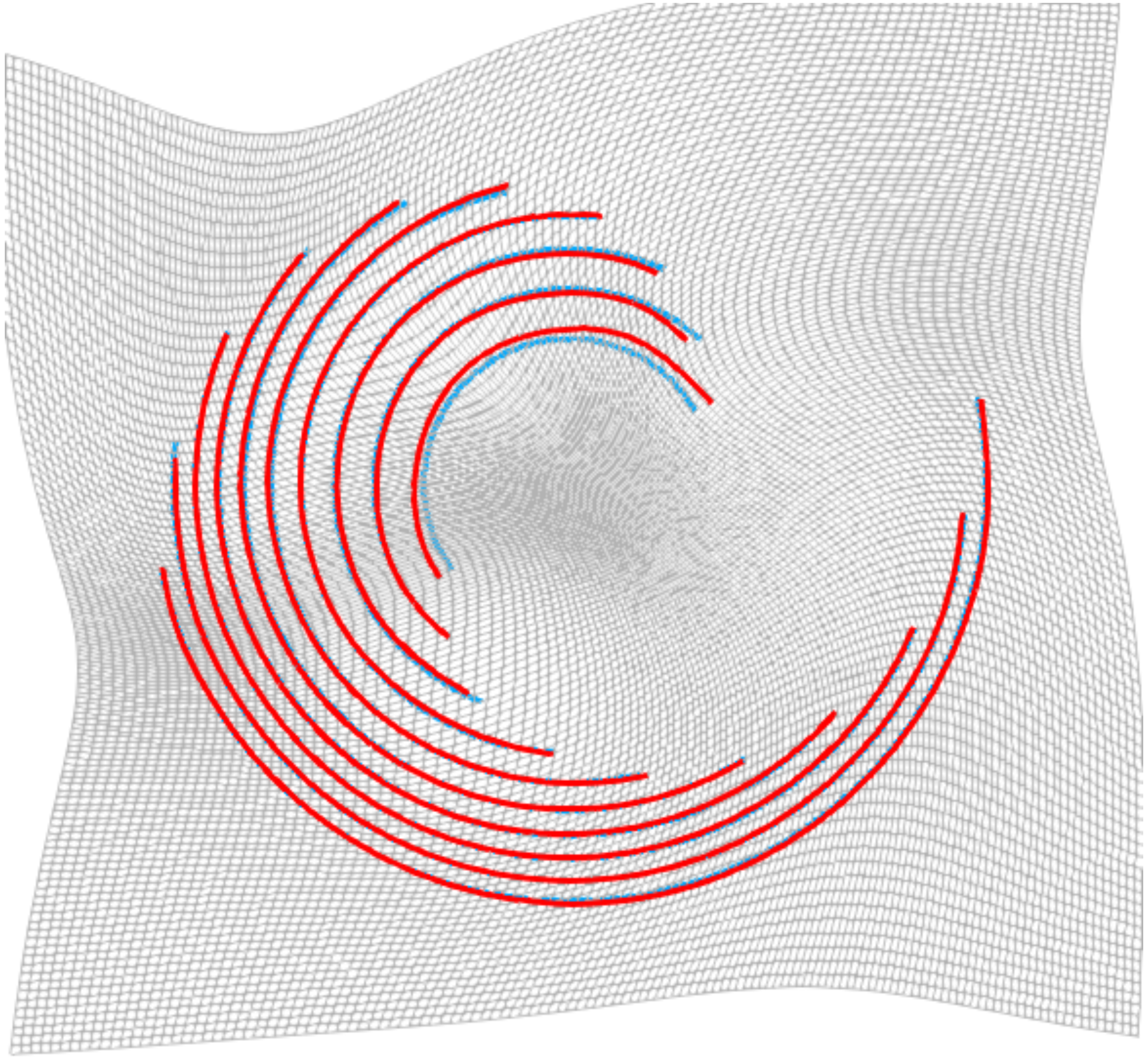




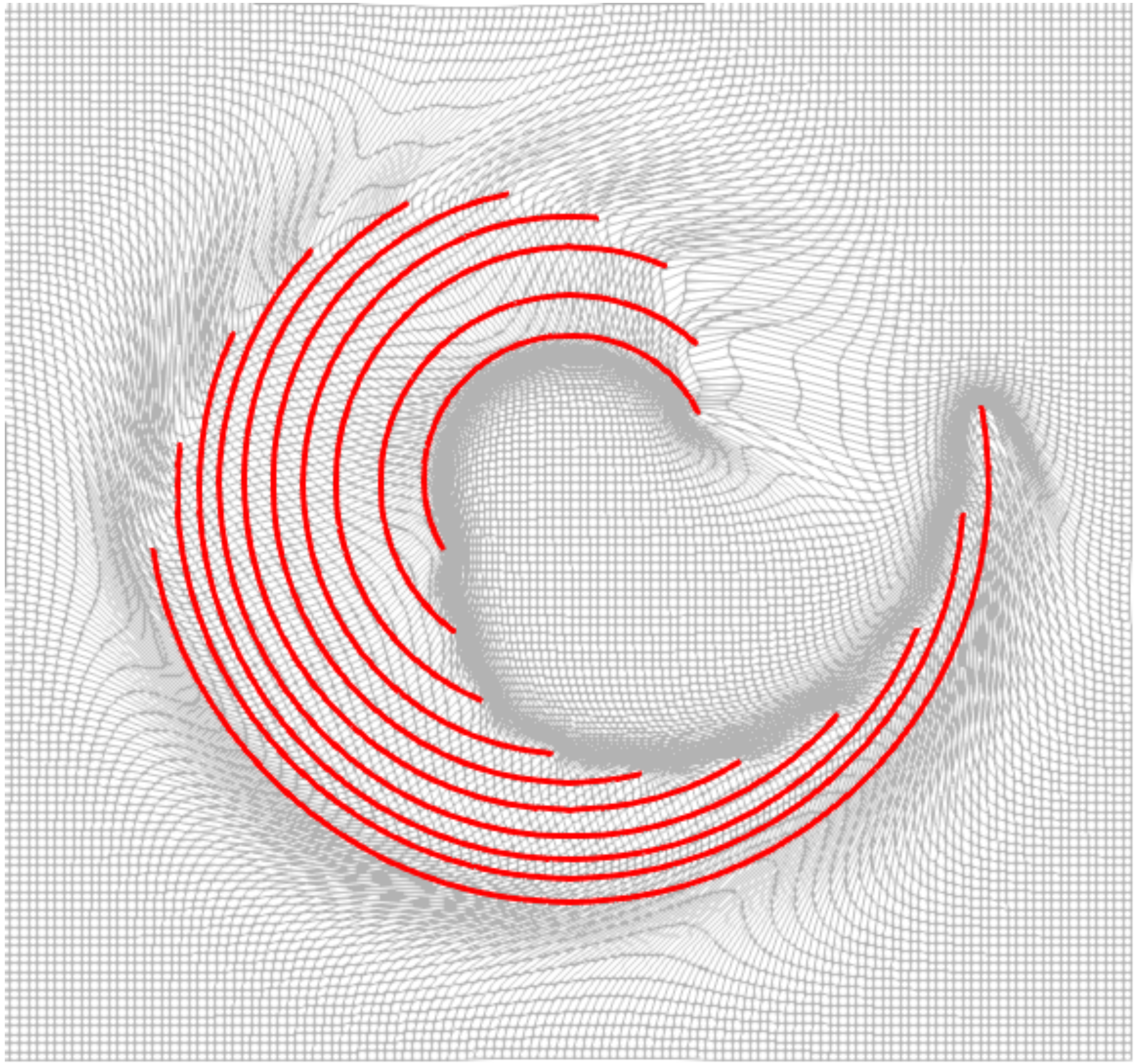




LDDMM



H_LDDMM

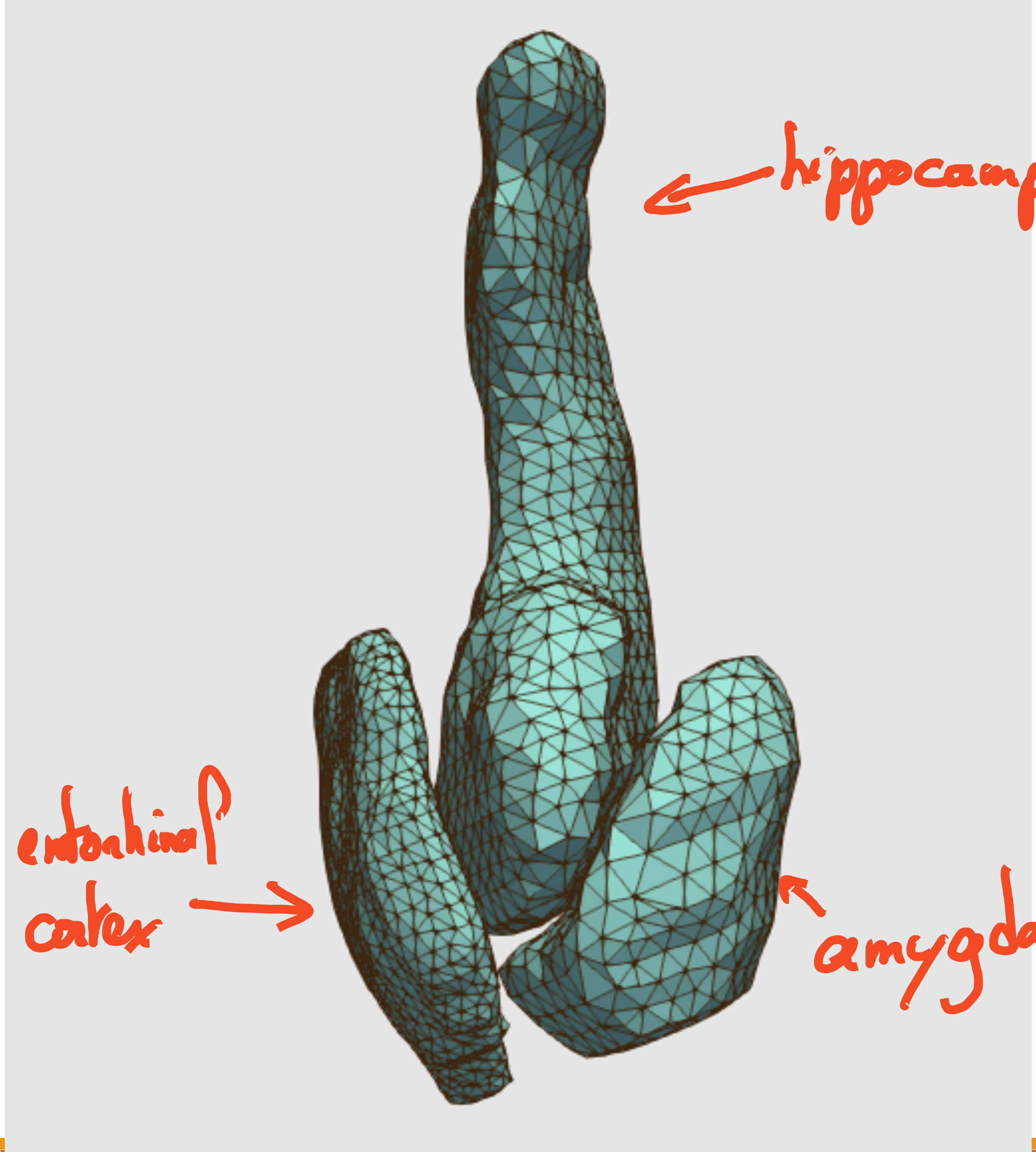


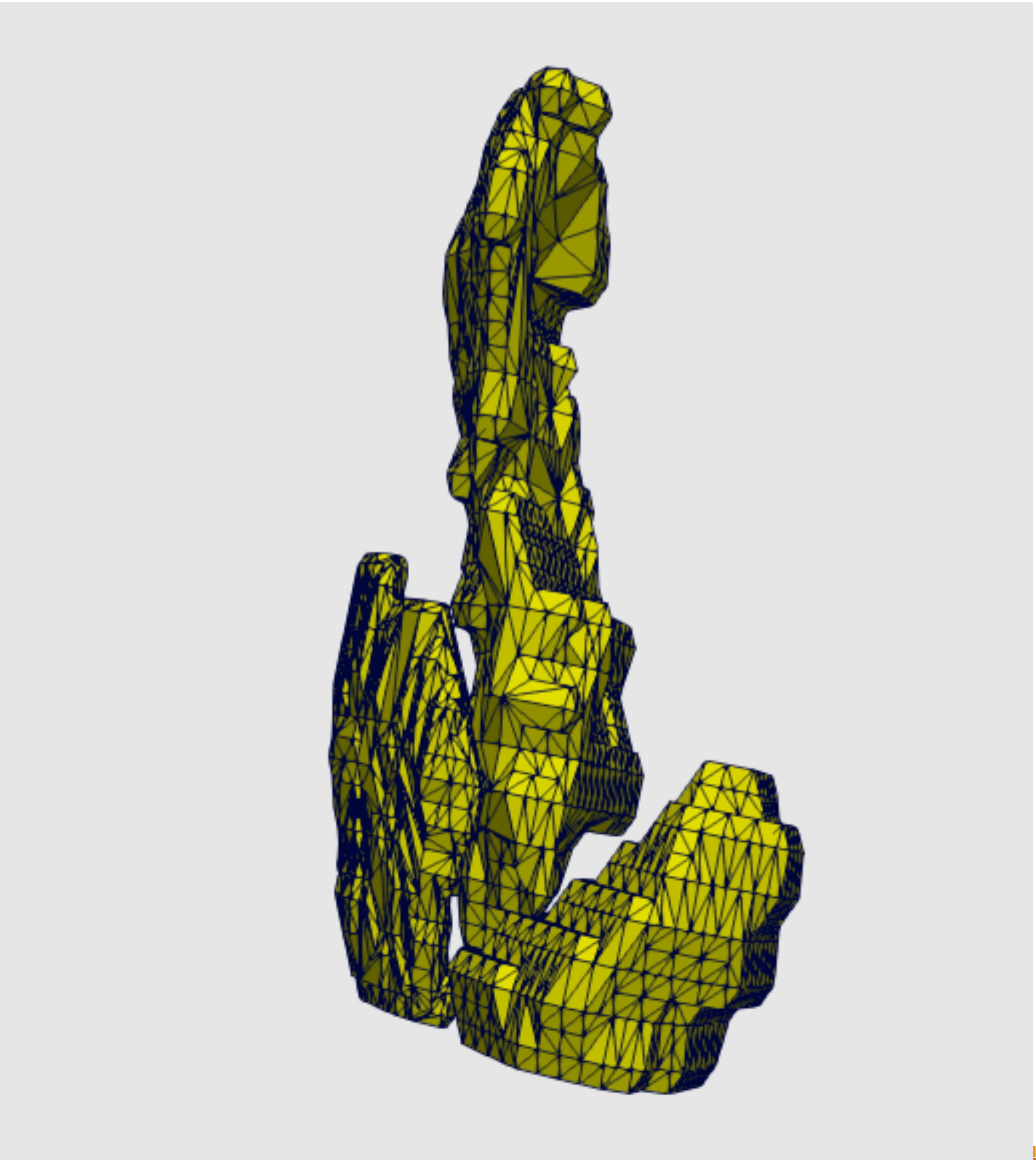
Multi - surfaces

Hippocampus

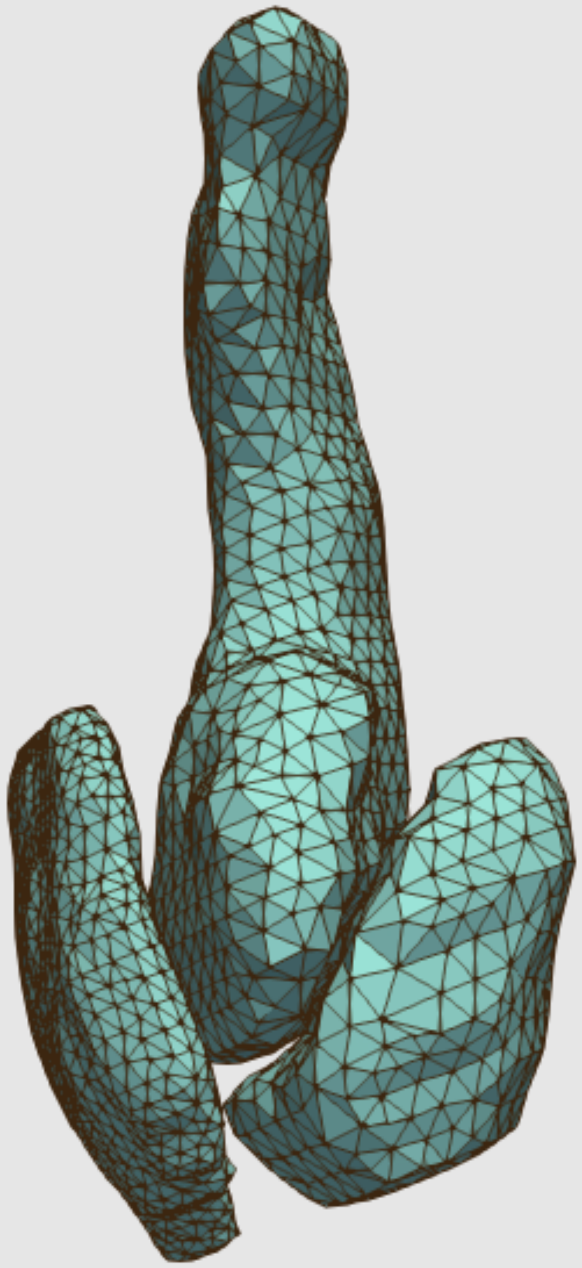
Amygdala

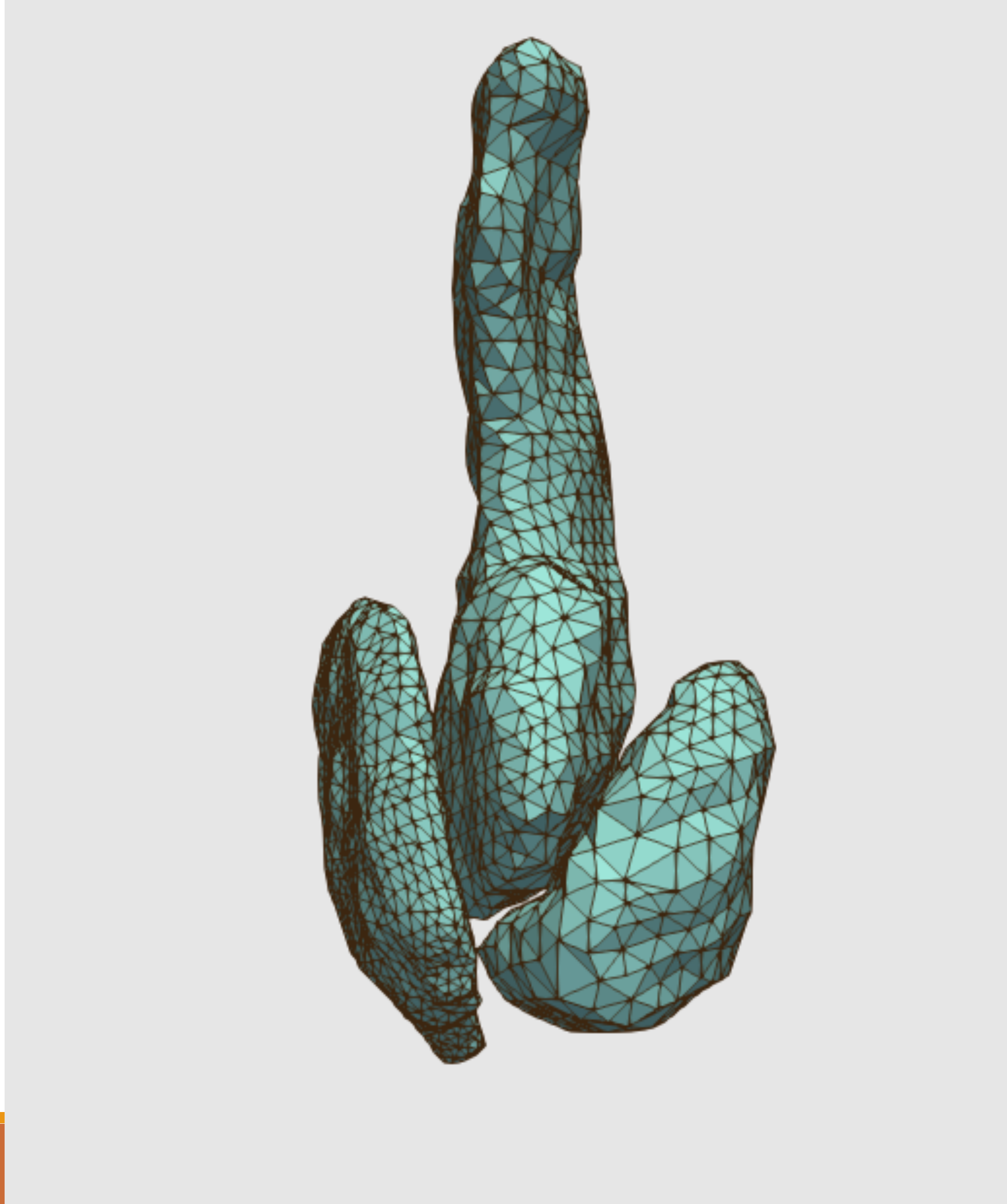
Entorhinal cortex

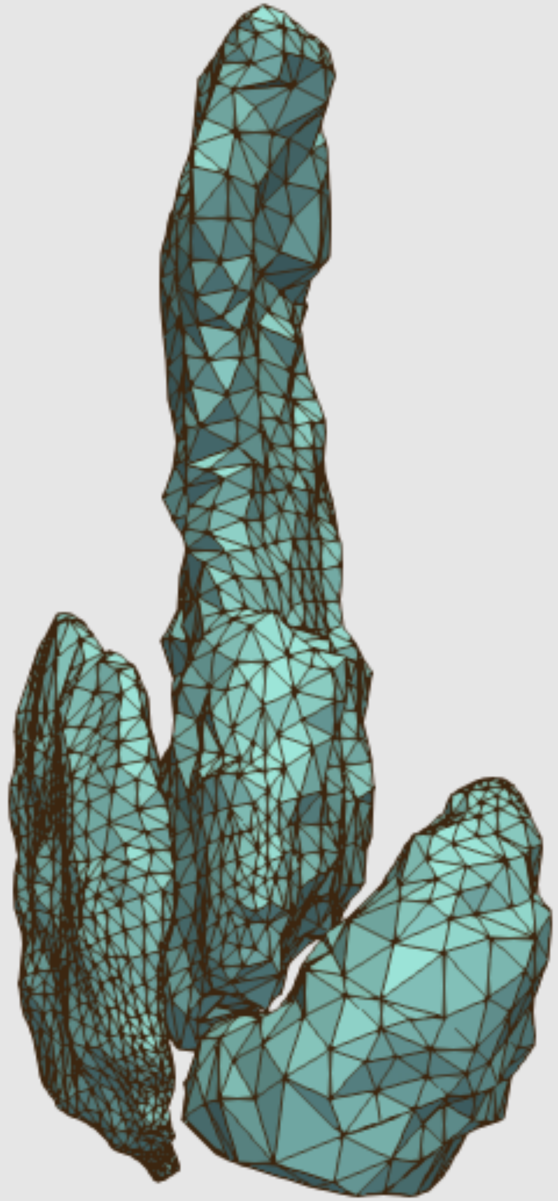

hippocampus
entorhinal
cortex
amygdala

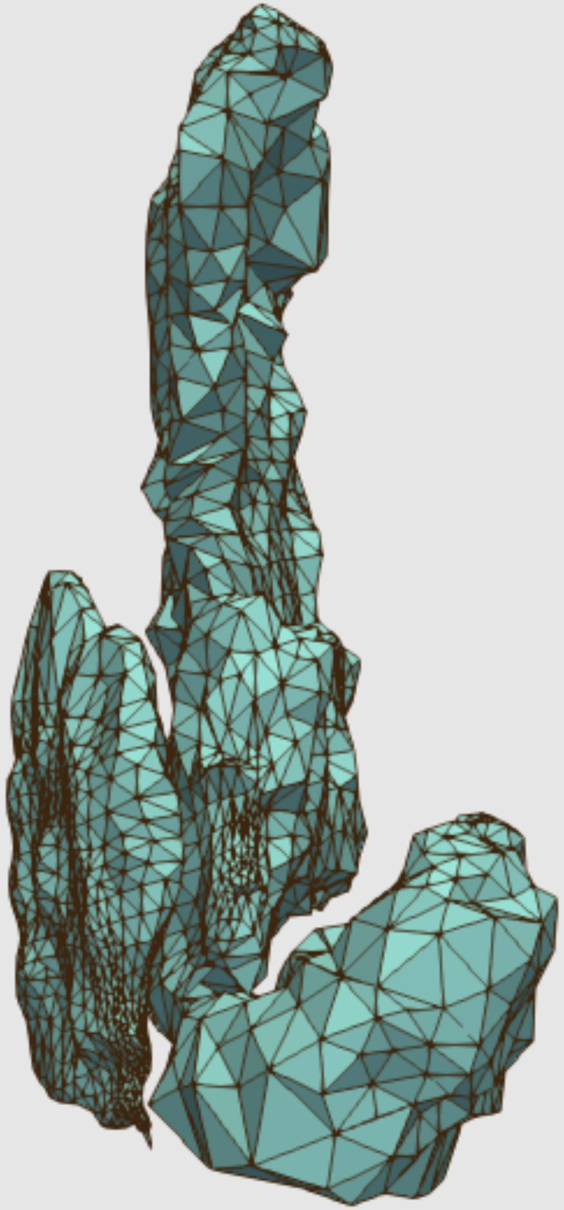


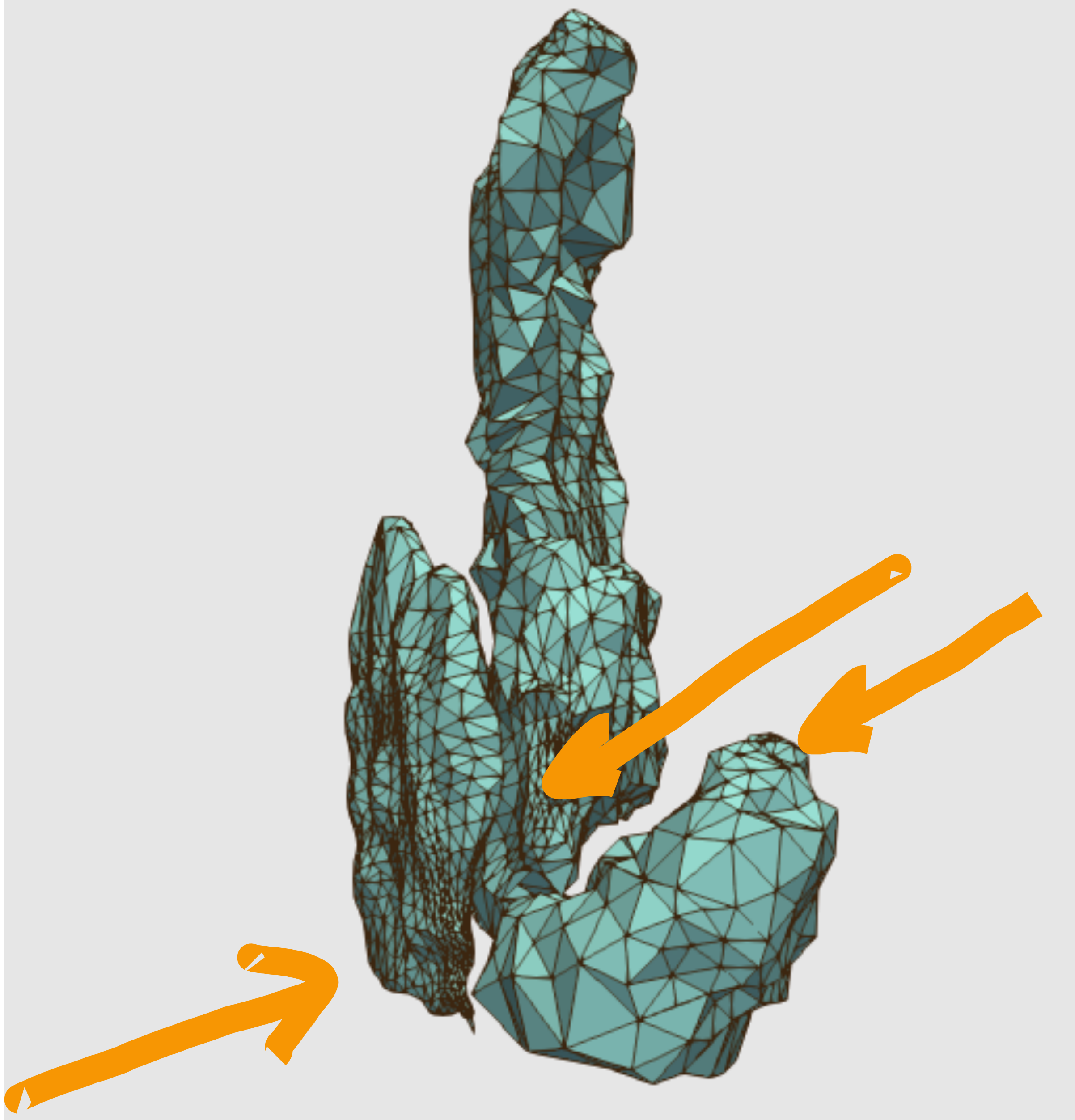
LDDMM
(small kernel width)





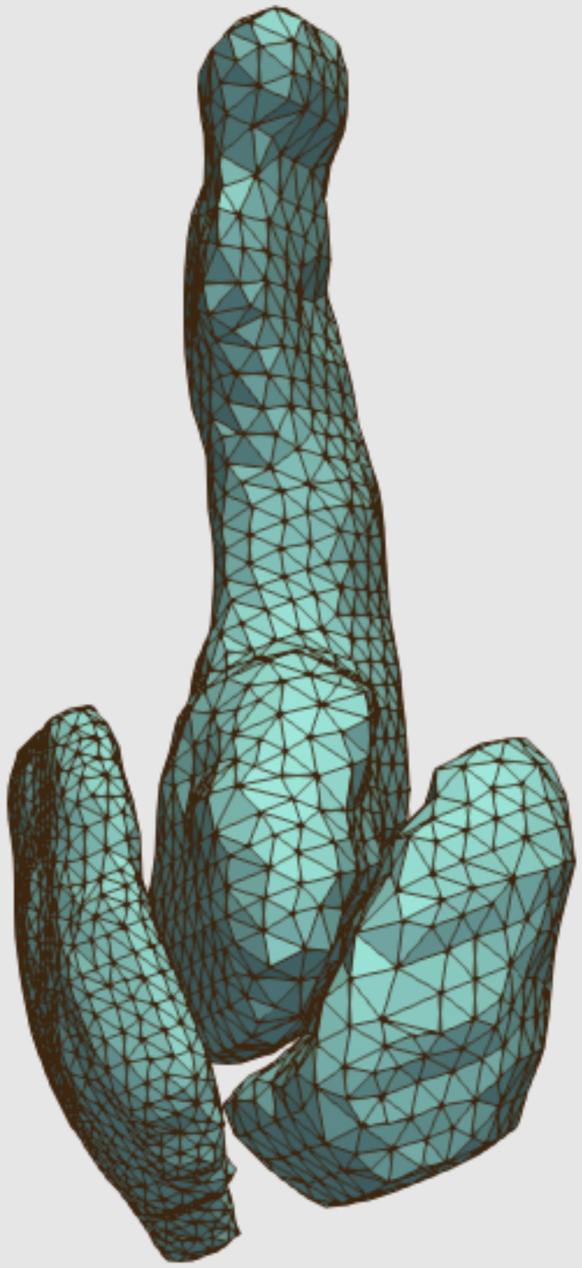


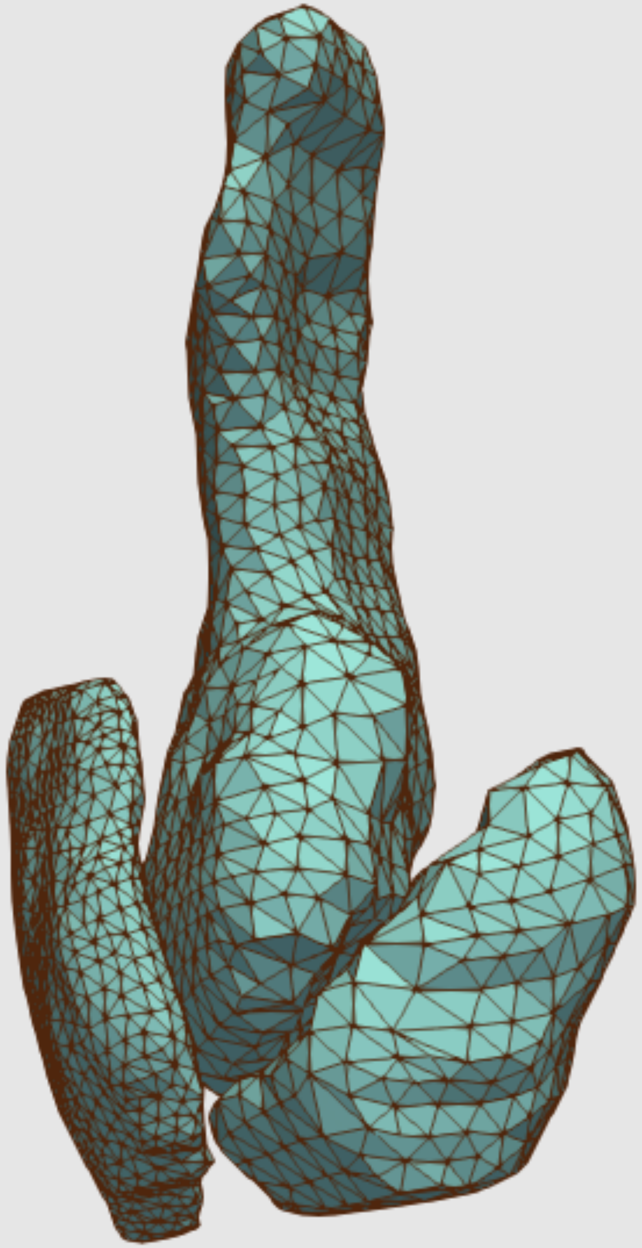


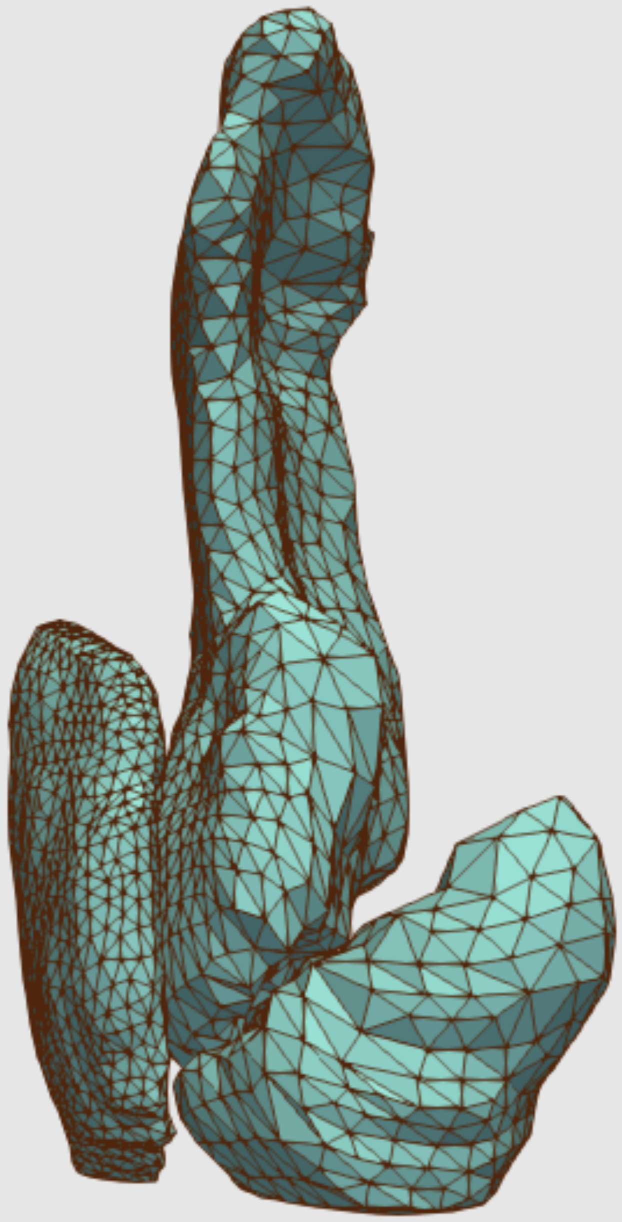


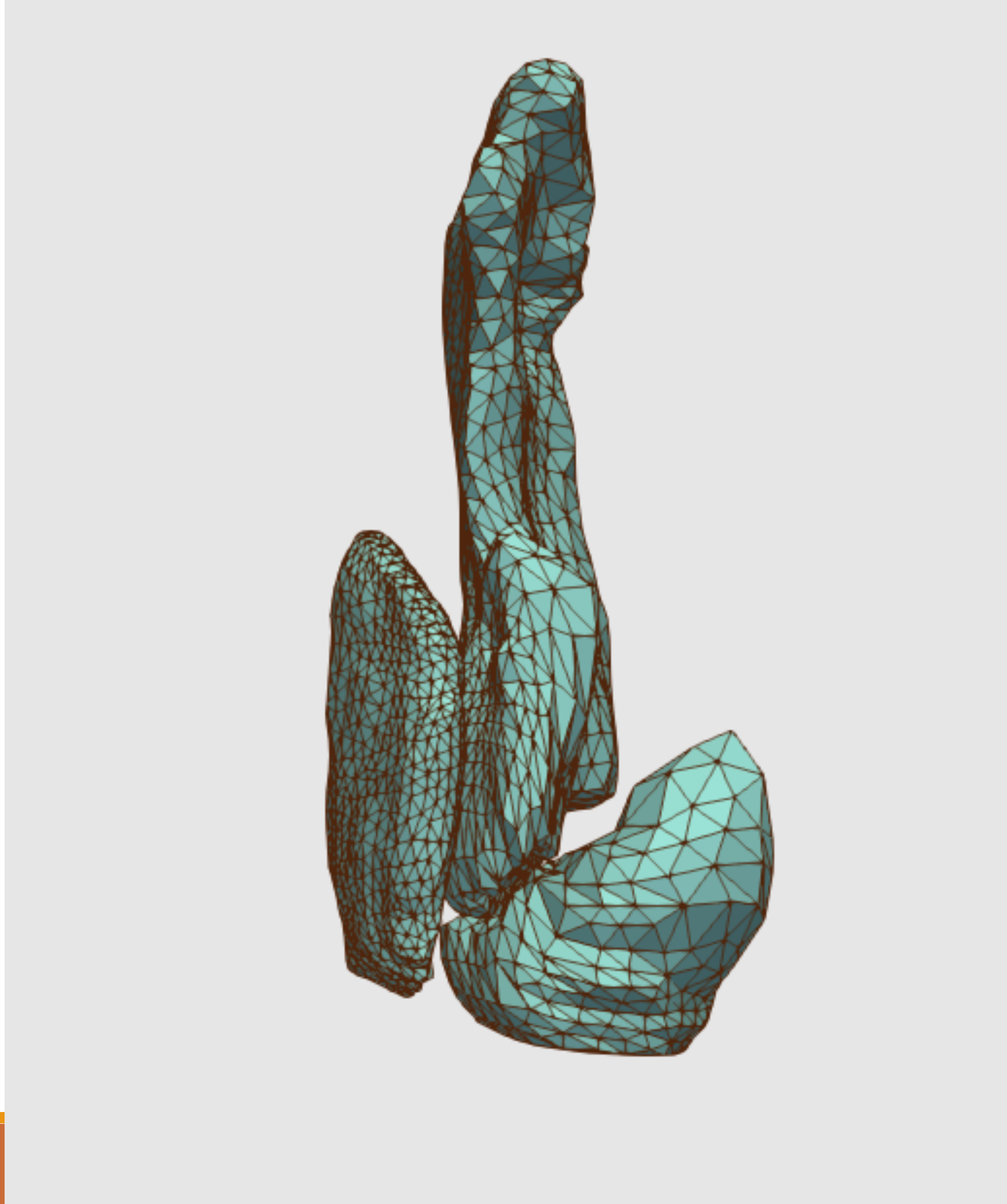
LDDMM

(larger kernel size)









Remark: When using LDDMM in computational anatomy, each structure is studied separately.

Deformation artifacts are avoided but relative structure positions are ignored.

(May be okay in that case.)

HLDDΠM

H' norm