Lattice Structures for Dynamics

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Rigorous Numerics in Dynamics Minitutorial - Part 1 of 3
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Iterate a map $f : X \to X$ on a locally compact metric space (not necessarily injective nor surjective).

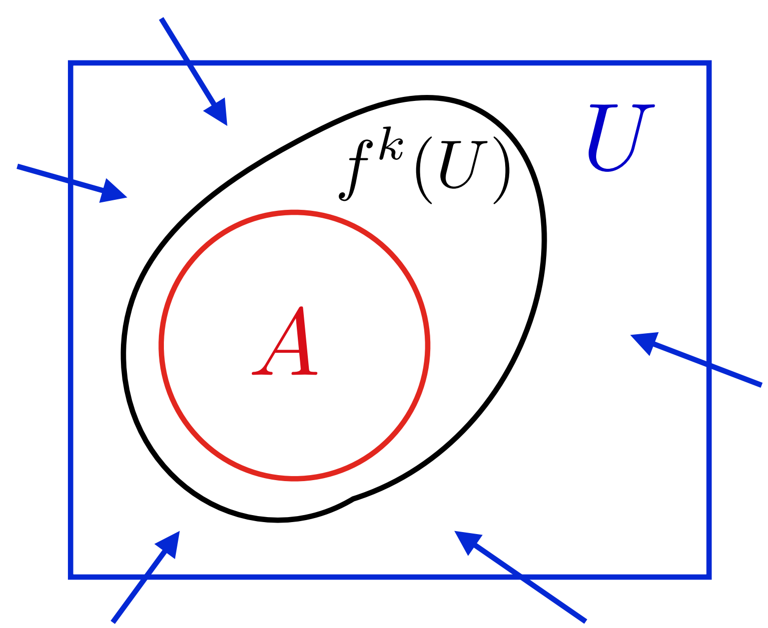
A set $U \subset X$ is an attracting neighborhood if there exists $k_0 > 0$ such that $f^k(\mathrm{cl}(U)) \subset \mathrm{int}(U)$ for all $k \geq k_0$.

A trapping region is forward invariant, ie. $f(U) \subset U$, attracting neighborhood.

An attracting block is a trapping region with $k_0 = 1$.

A set $A \subset X$ is an attractor if there exists an attracting neighborhood $A \subset U$ such that $A = \omega(U)$.

(Conley)



Observable dynamics: attractors

Computable dynamics: attracting blocks / trapping regions / attracting neighborhoods

$\vee = \cup, \ \wedge = \cap$ $\qquad$ $\mathsf{ABlock}(X, f)$

surjective lattice homomorphism between distributive lattices $\qquad$ $\downarrow \omega$

$\vee = \cup, \ \wedge = \omega(\cdot \cap \cdot)$ $\qquad$ $\mathsf{Att}(X, f)$

Lattices

A *bounded, distributive lattice* is a set L with the binary operations $\vee, \wedge : \mathsf{L} \times \mathsf{L} \to \mathsf{L}$ satisfying the following axioms:

(i) (idempotent) $a \wedge a = a \vee a = a$ for all $a \in \mathsf{L}$,

(ii) (commutative) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ for all $a, b \in \mathsf{L}$,

(iii) (associative) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$ for all $a, b, c \in \mathsf{L}$,

(iv) (absorption) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ for all $a, b \in \mathsf{L}$.

(v) (distributive) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in \mathsf{L}$.

(vi) (neutral elements) $\exists\, 0, 1 \in \mathsf{L}$ such that $0 \wedge a = 0$, $0 \vee a = a$, $1 \wedge a = a$, and $1 \vee a = 1$ for all $a \in \mathsf{L}$.

All sublattices contain 0 and 1, and all homomorphisms preserve 0 and 1.

Think sets!

For computations:

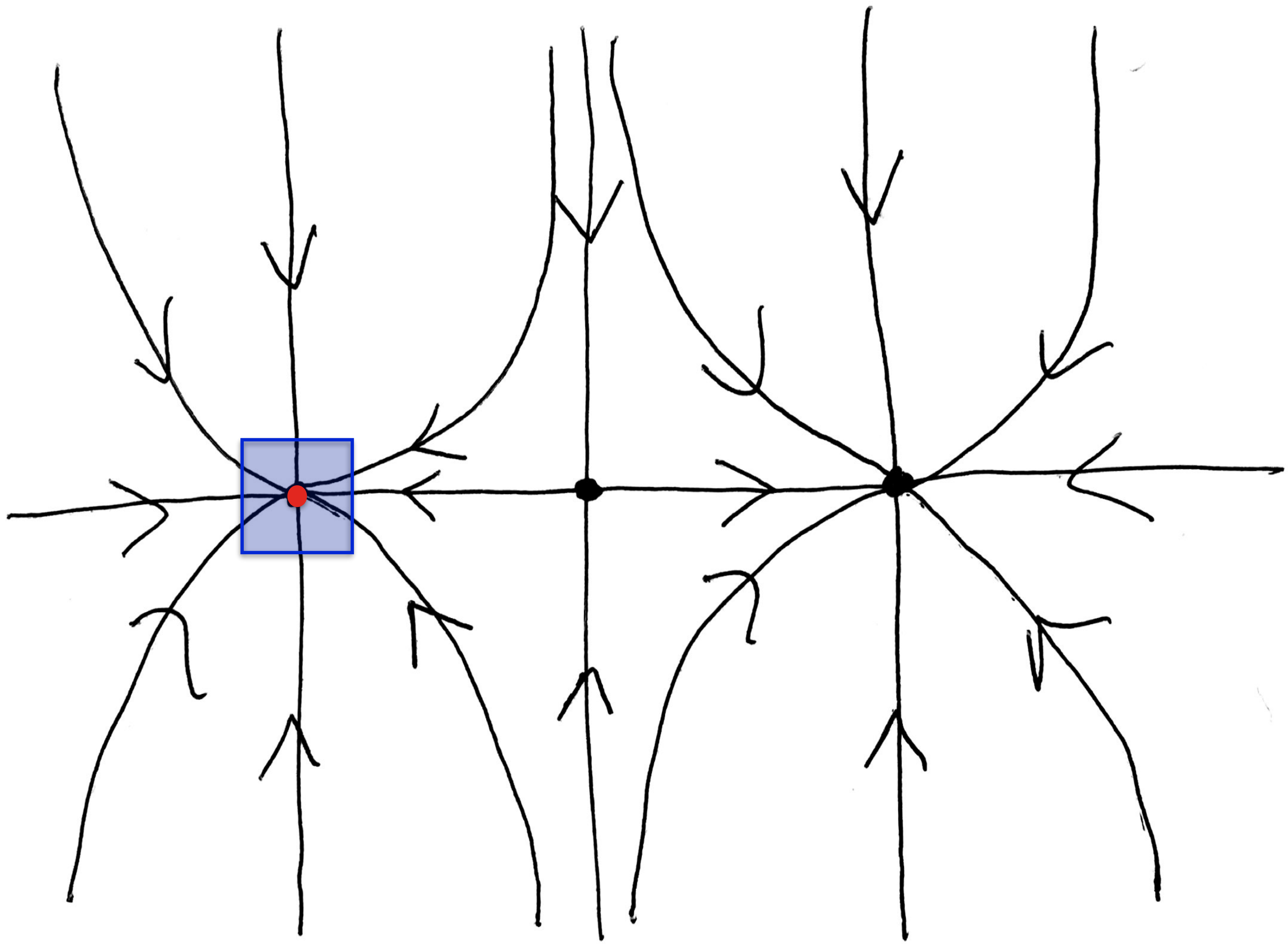
(1) consider a finite sublattices of attractors
(2) combinatorialize the phase space

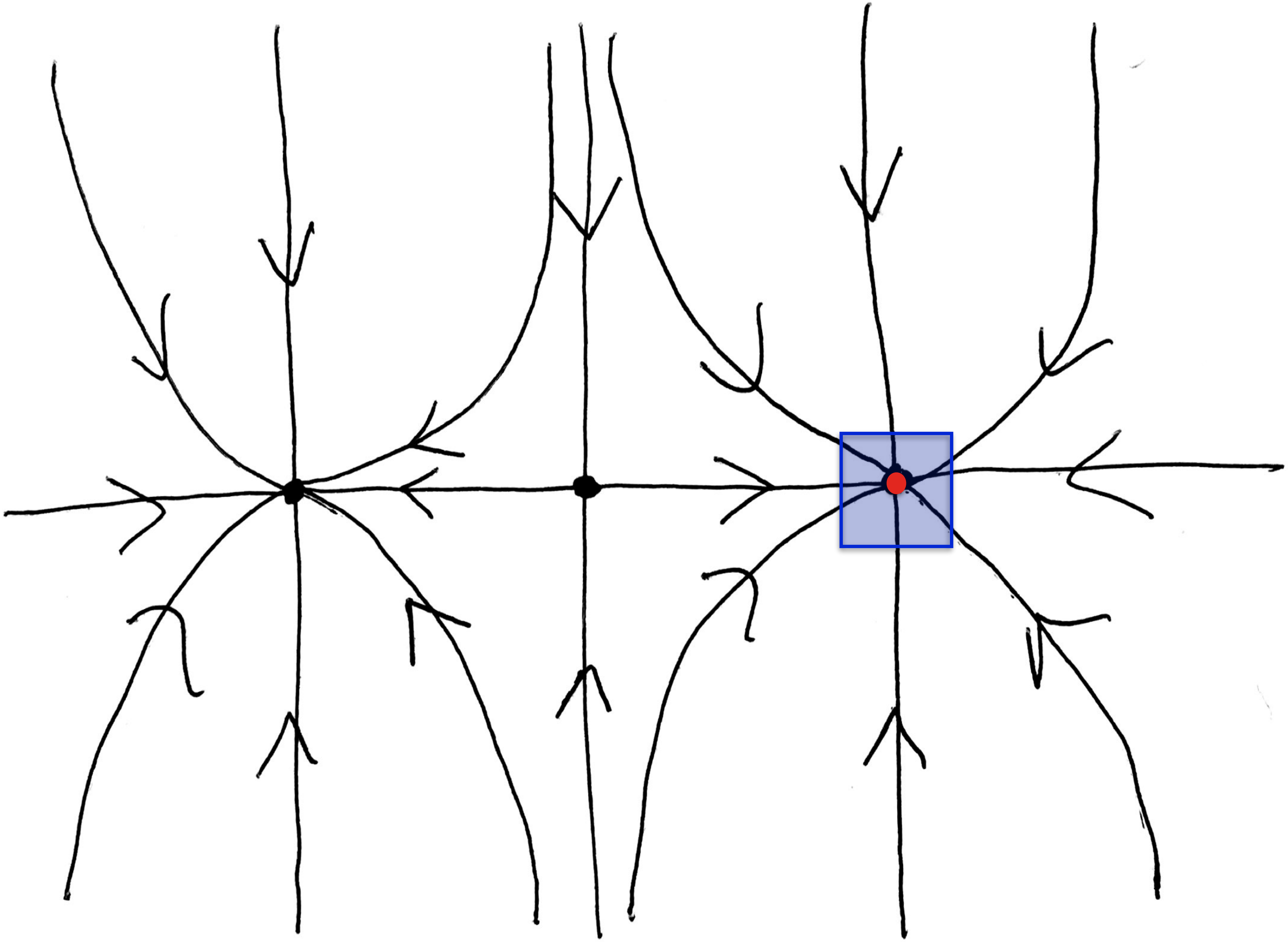
Consider the lattice of regular closed subsets, ie.

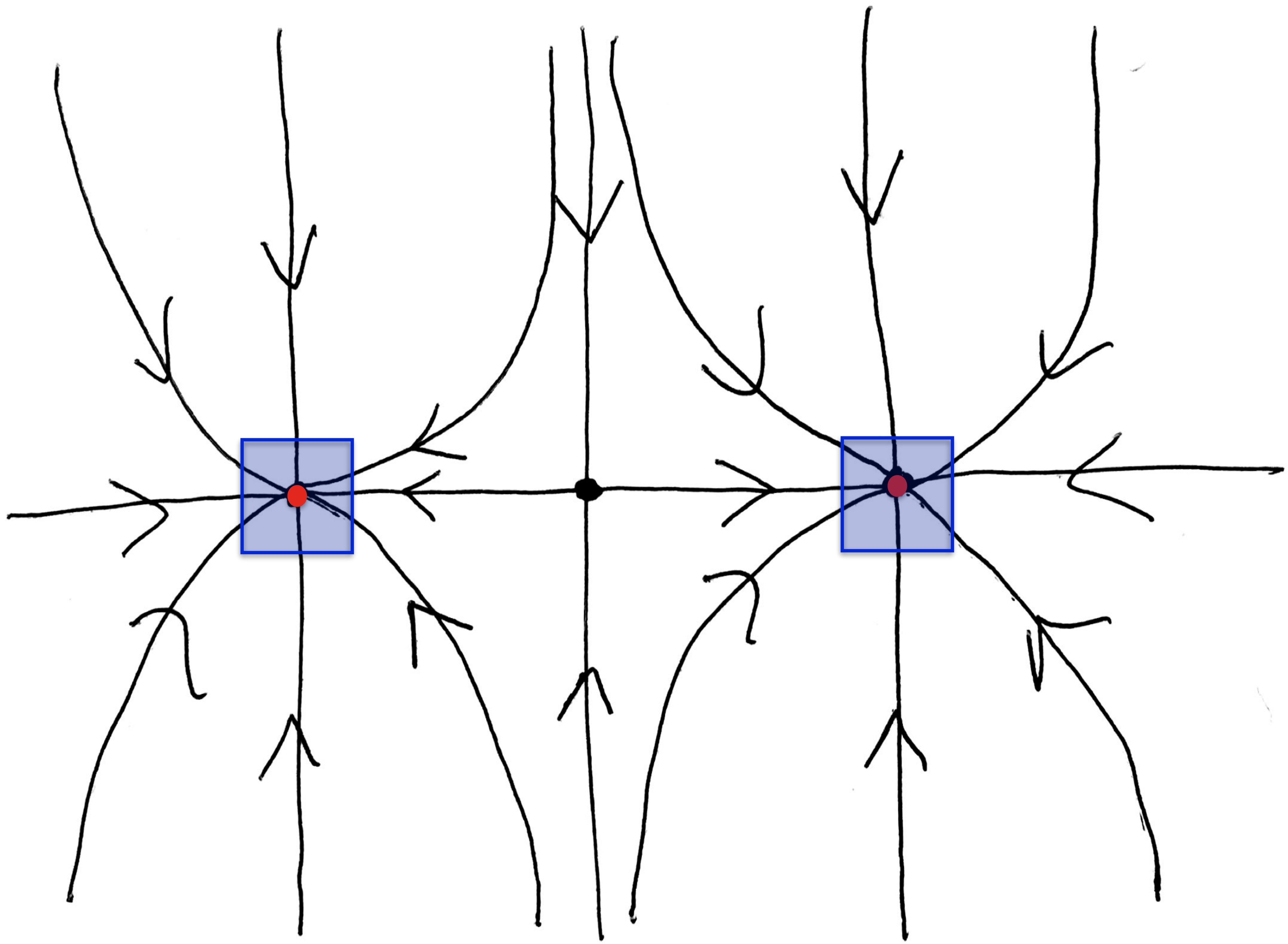
$$S = \mathrm{cl}(\mathrm{int}(S)) \qquad \vee = \cup, \ \wedge = \mathrm{cl}(\mathrm{int}(\cdot \cap \cdot))$$

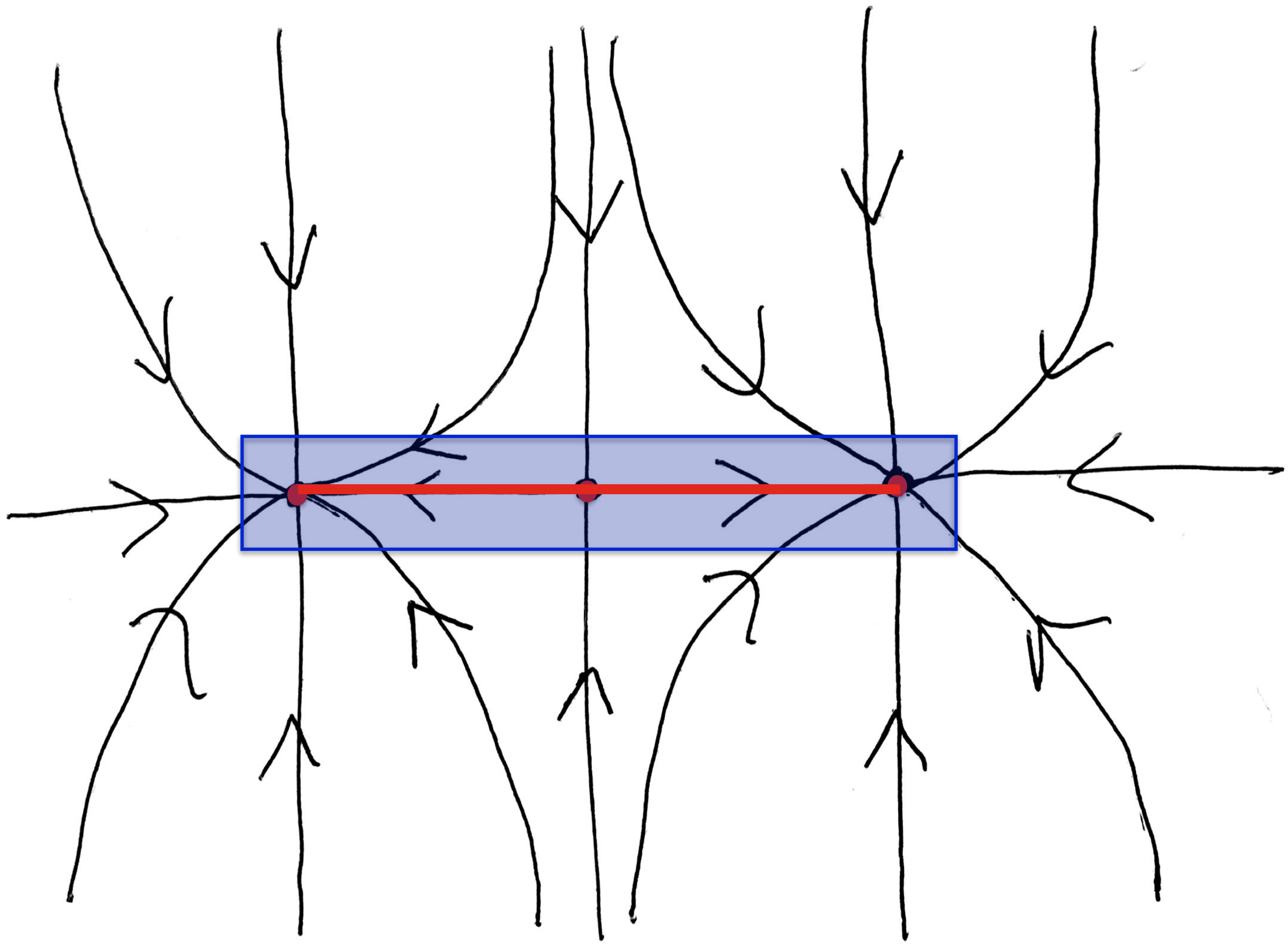
Example: the power set of a finite, full simplicial / cubical complex generates a finite sublattice of regular closed subsets.

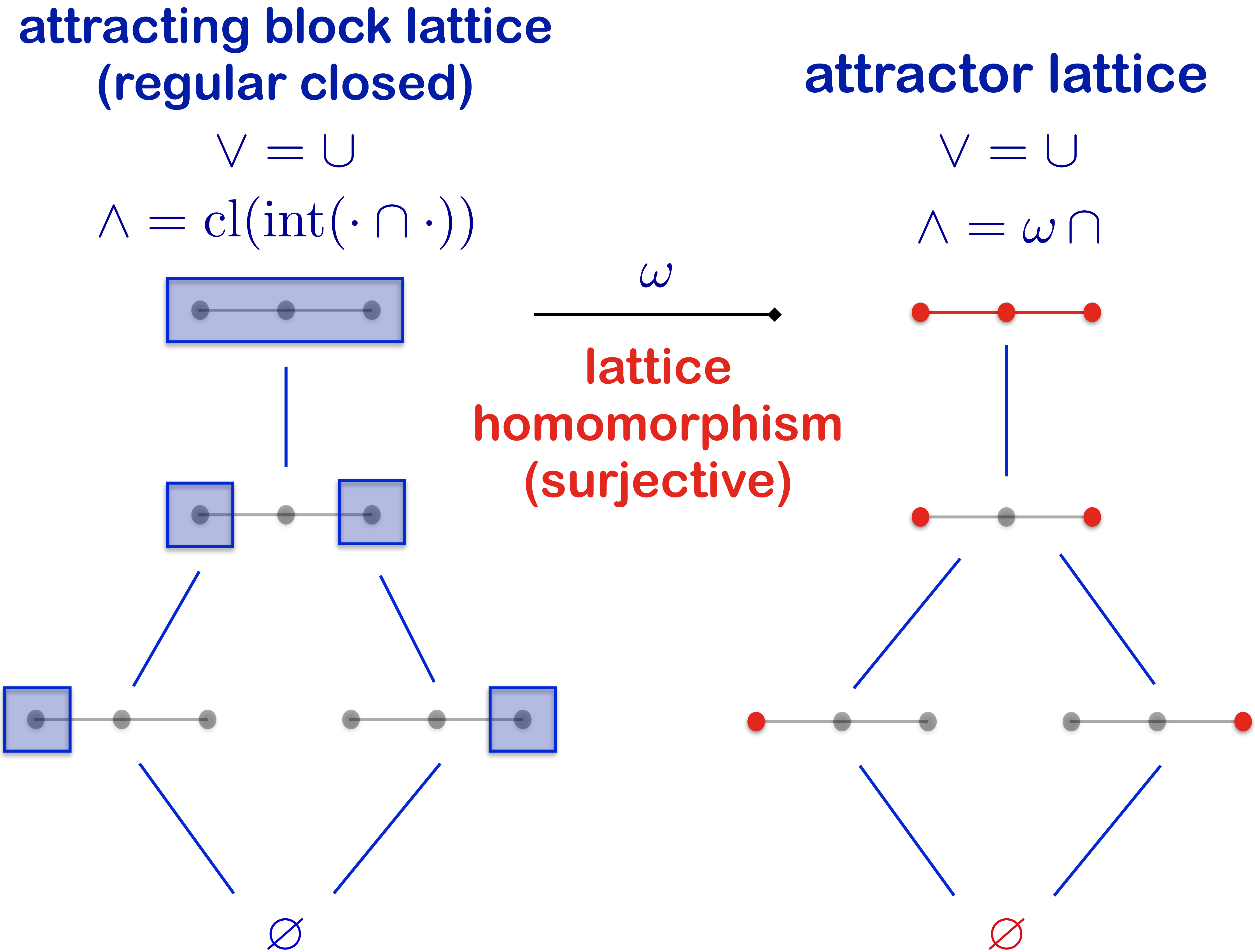
An example: time-T map of $\dot{x} = x - x^3, \ \dot{y} = -y$










attracting block lattice
(regular closed)
$\vee = \cup$
$\wedge = \mathrm{cl}(\mathrm{int}(\cdot \cap \cdot))$
$\varnothing$
ω
lattice
homomorphism
(surjective)
attractor lattice
$\vee = \cup$
$\wedge = \omega \cap$
$\varnothing$

J - functor

A lattice L has a naturally induced partial order as follows.

Given $a, b \in$ L define

$$a \leq b \quad \Leftrightarrow \quad a \wedge b = a.$$

Given a lattice L, an element $0 \neq c \in$ L is *join-irreducible* if

$$c = a \vee b \text{ implies } c = a \text{ or } c = b \text{ for all } a, b \in \text{L}.$$

The set of join-irreducible elements in L is denoted by J(L).

c is join-irreducible iff there exists a unique element $a \in$ L such that $a < c$ and there is no z such that $a < z < c$.

J is a contravariant functor from finite distributive lattices to finite posets.

In all lattices we consider, the order $\leq$ is induced by inclusion.

attracting block lattice

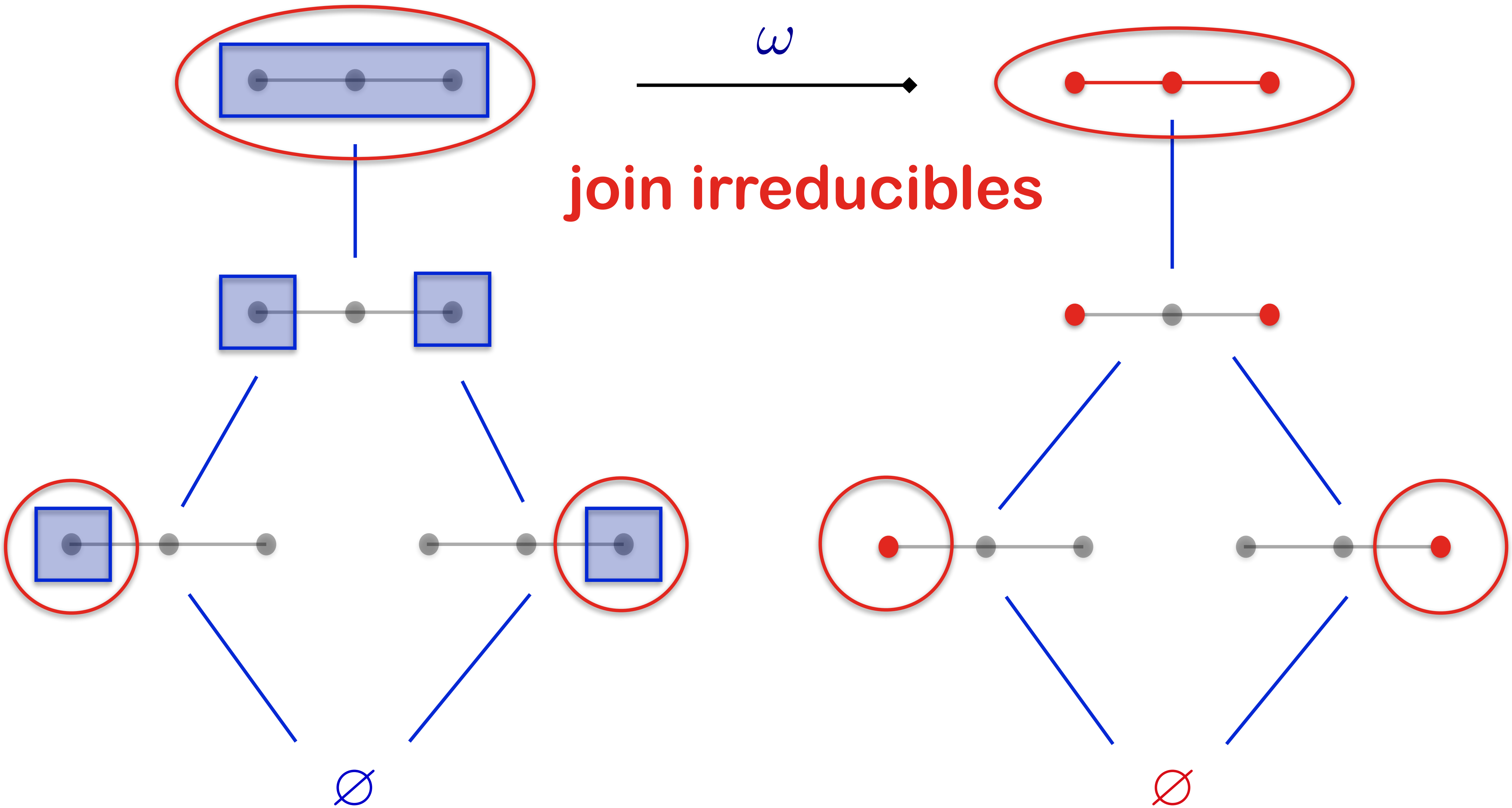
$$\vee = \cup$$

$$\wedge = \mathrm{cl}(\mathrm{int}(\cdot \cap \cdot))$$

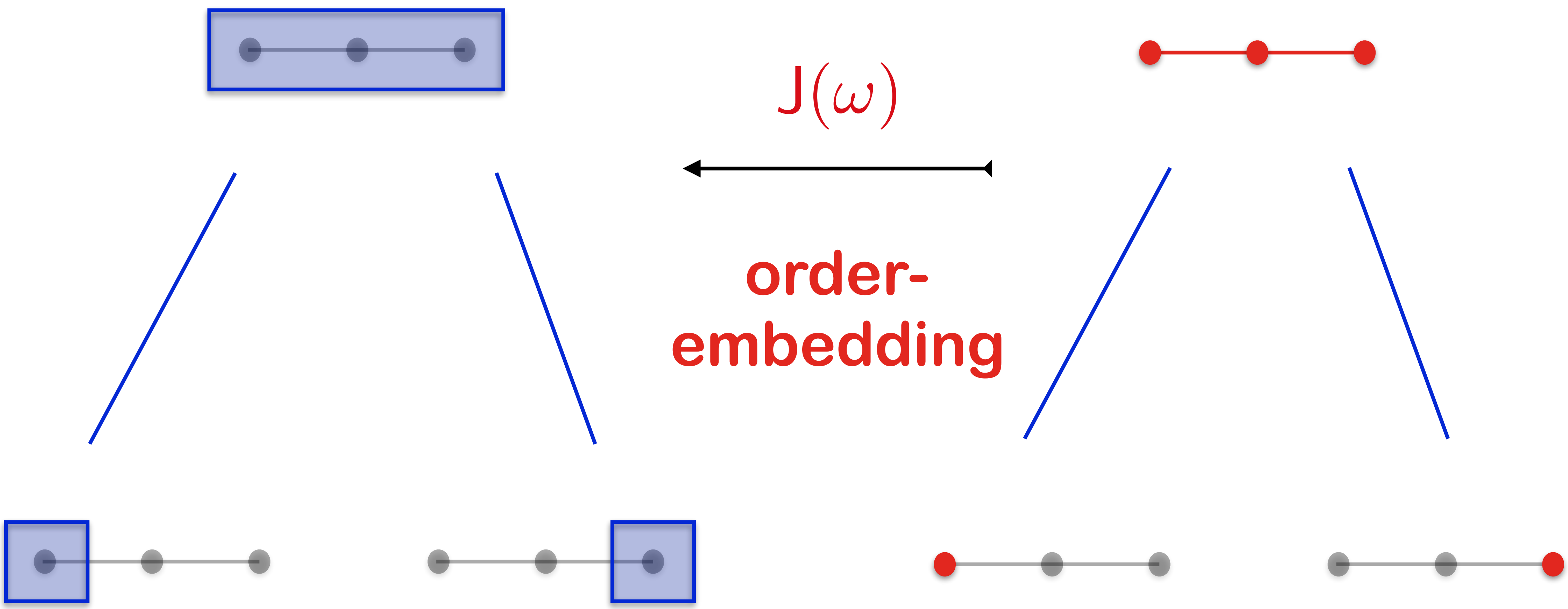
attractor lattice

$$\vee = \cup$$

$$\wedge = \omega \cap$$



Birkhoff's Representation Theorem: J induces a duality between finite distributive lattices and finite posets.



Conley form

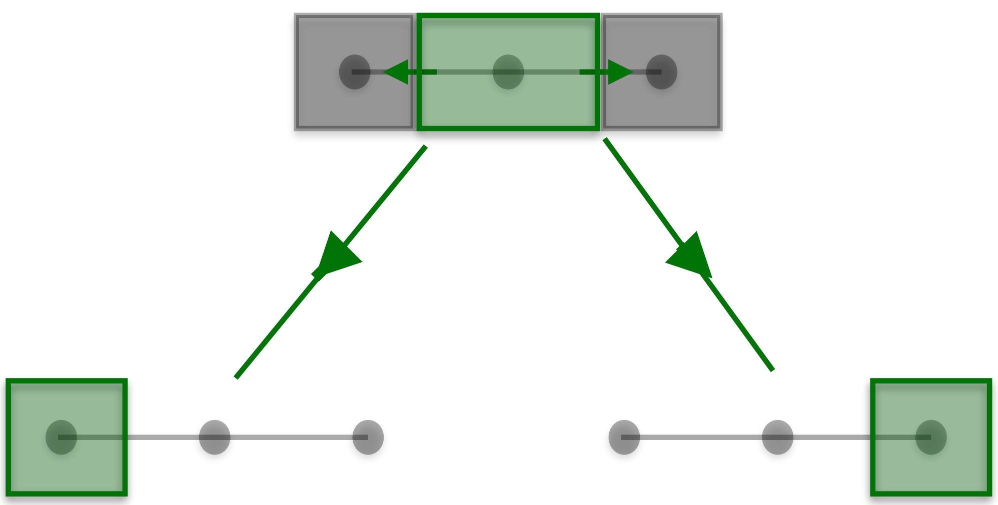
$$N_A \wedge (\overleftarrow{N_A})^{\#} = \mathrm{cl}(N_A \cap (\overleftarrow{N_A})^c) \qquad A \wedge (\overleftarrow{A})^* = A \cap (\overleftarrow{A})^*$$

poset of isolating neighborhoods

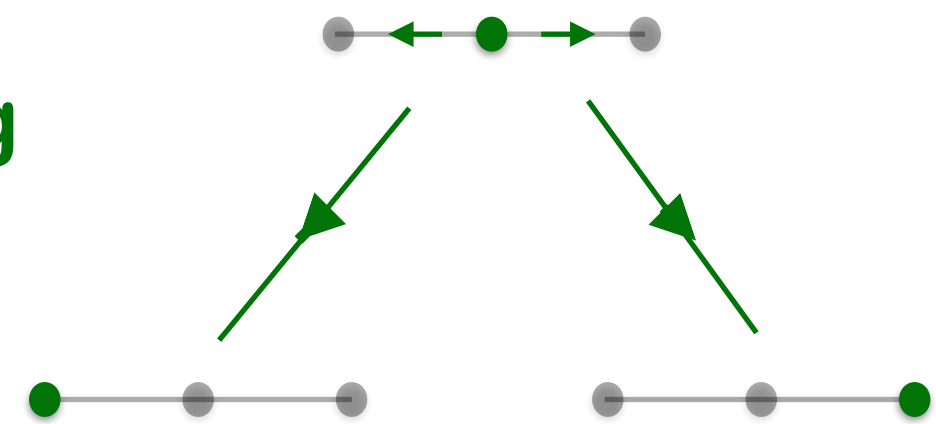
poset of invariant sets

π

order-embedding



T(N)

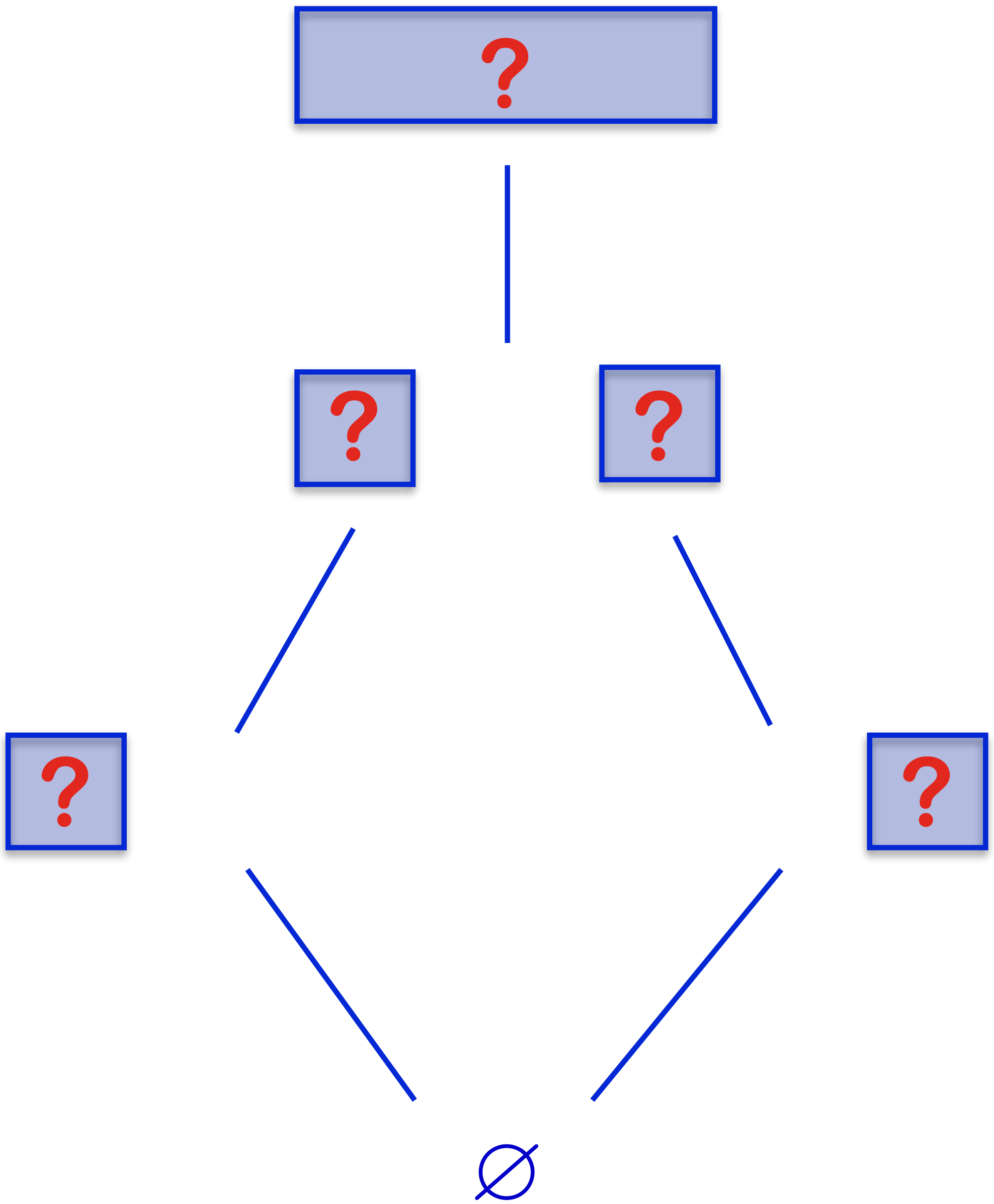


M(A)

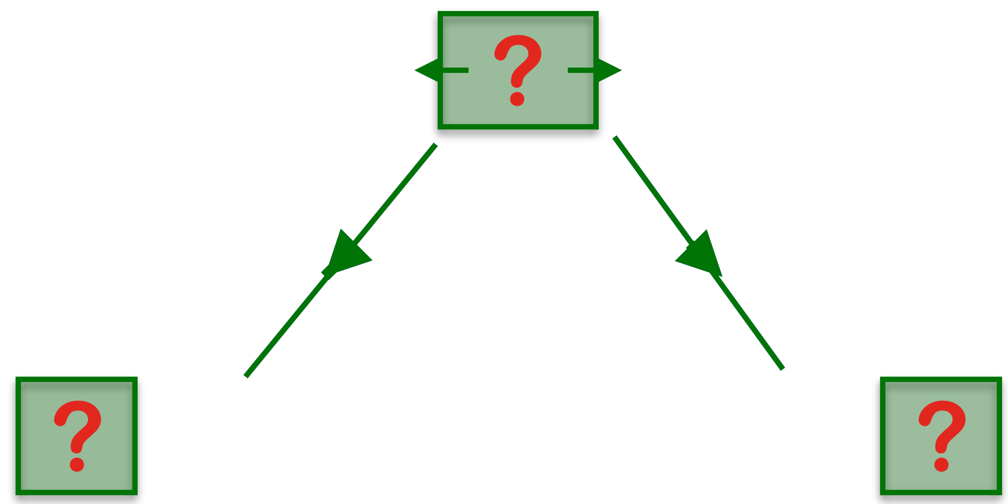
(tesselated) Morse decomposition

$$\mathsf{M(A)} \hookrightarrow \mathsf{T(N)}$$

attracting block lattice



poset of isolating neighborhoods



These structures are robust!

EXTENDED Birkhoff's Representation Theorem:

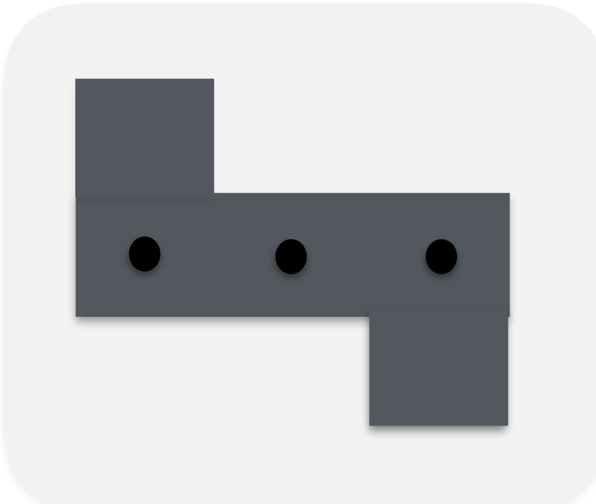
(K.,Kasti, Vandervorst)

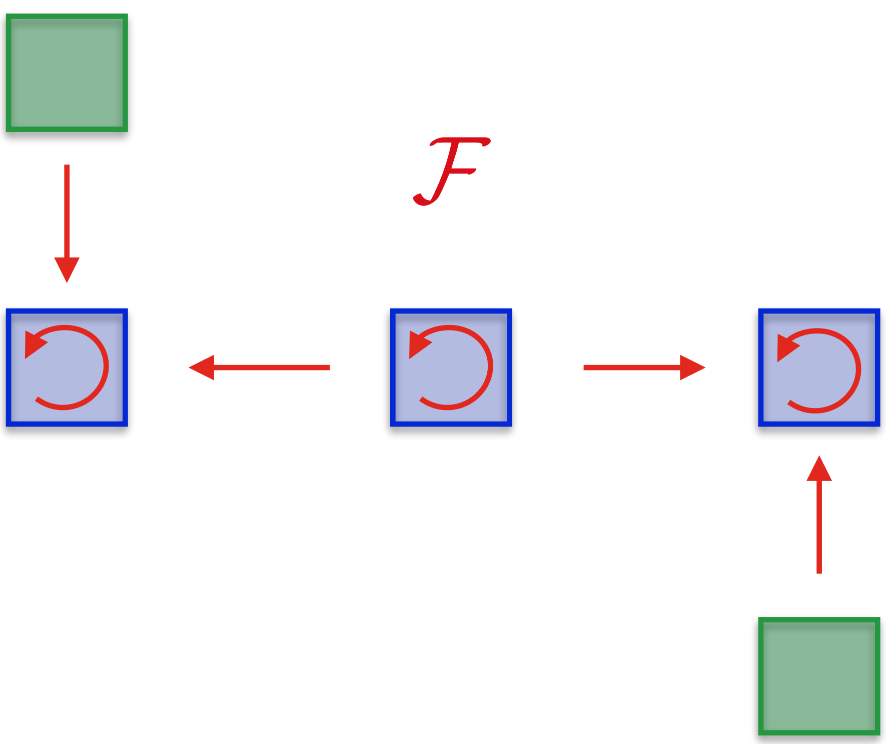
J induces a duality between surjective lattice homomorphisms on finite distributive lattices and finite binary relations / directed graphs / combinatorial multivalued maps— up to condensation and transitivity.

$$\begin{array}{ccc} \mathsf{K} & \longleftrightarrow & \mathsf{Invset}^{+}(\mathcal{F}) = \{\mathcal{U} \mid \mathcal{F}(\mathcal{U}) \subset \mathcal{U}\} \\ \Big\downarrow h & & \Big\downarrow \omega \\ \mathsf{L} & \longleftrightarrow & \mathsf{Att}(\mathcal{F}) = \{\mathcal{A} \mid \mathcal{F}(\mathcal{A}) = \mathcal{A}\} \end{array}$$

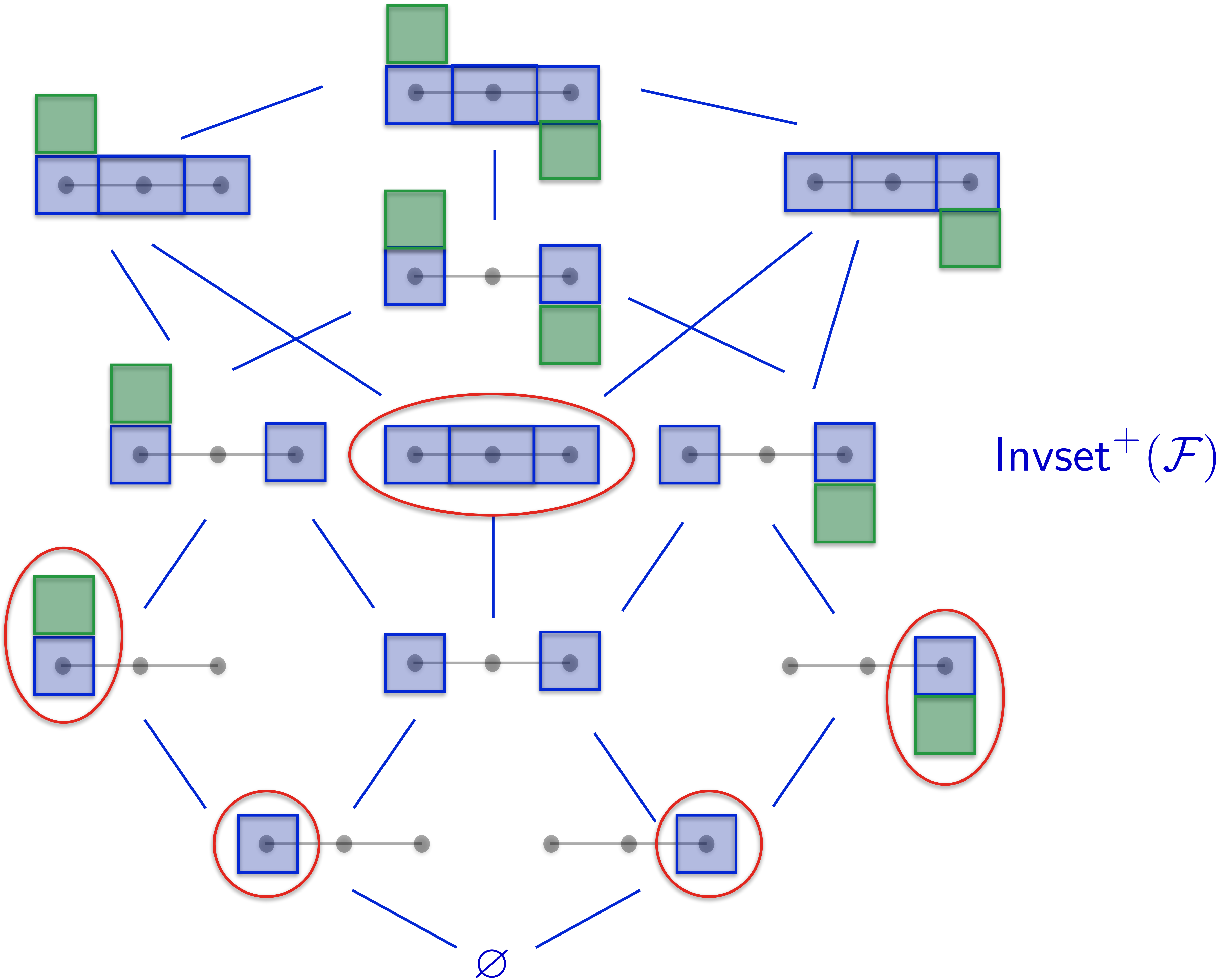
$$\omega(\mathcal{U}) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \mathcal{F}^{m}(\mathcal{U})$$

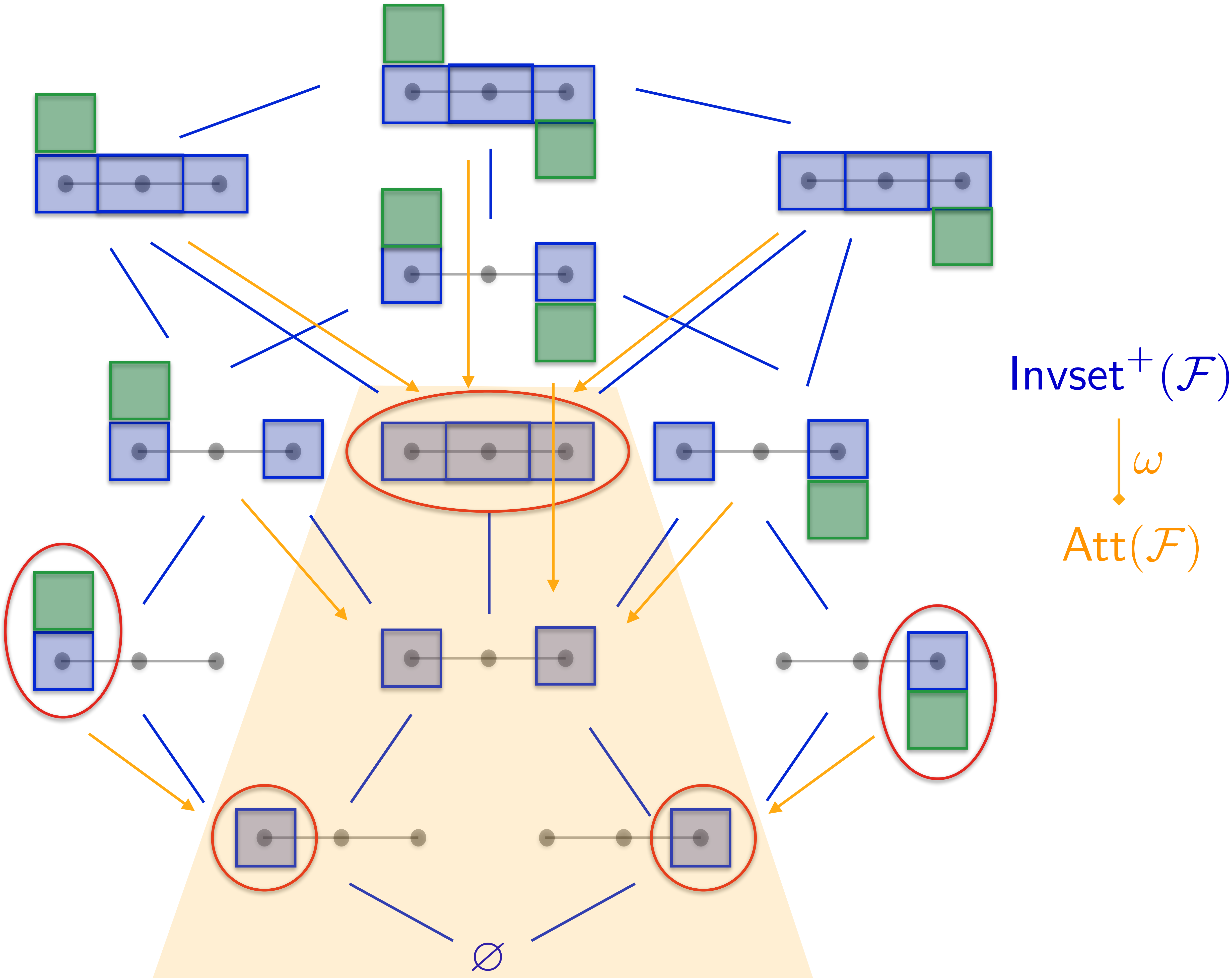
Computationally: try to represent $\mathcal{F}$ as a state transition graph on regular closed subsets of X



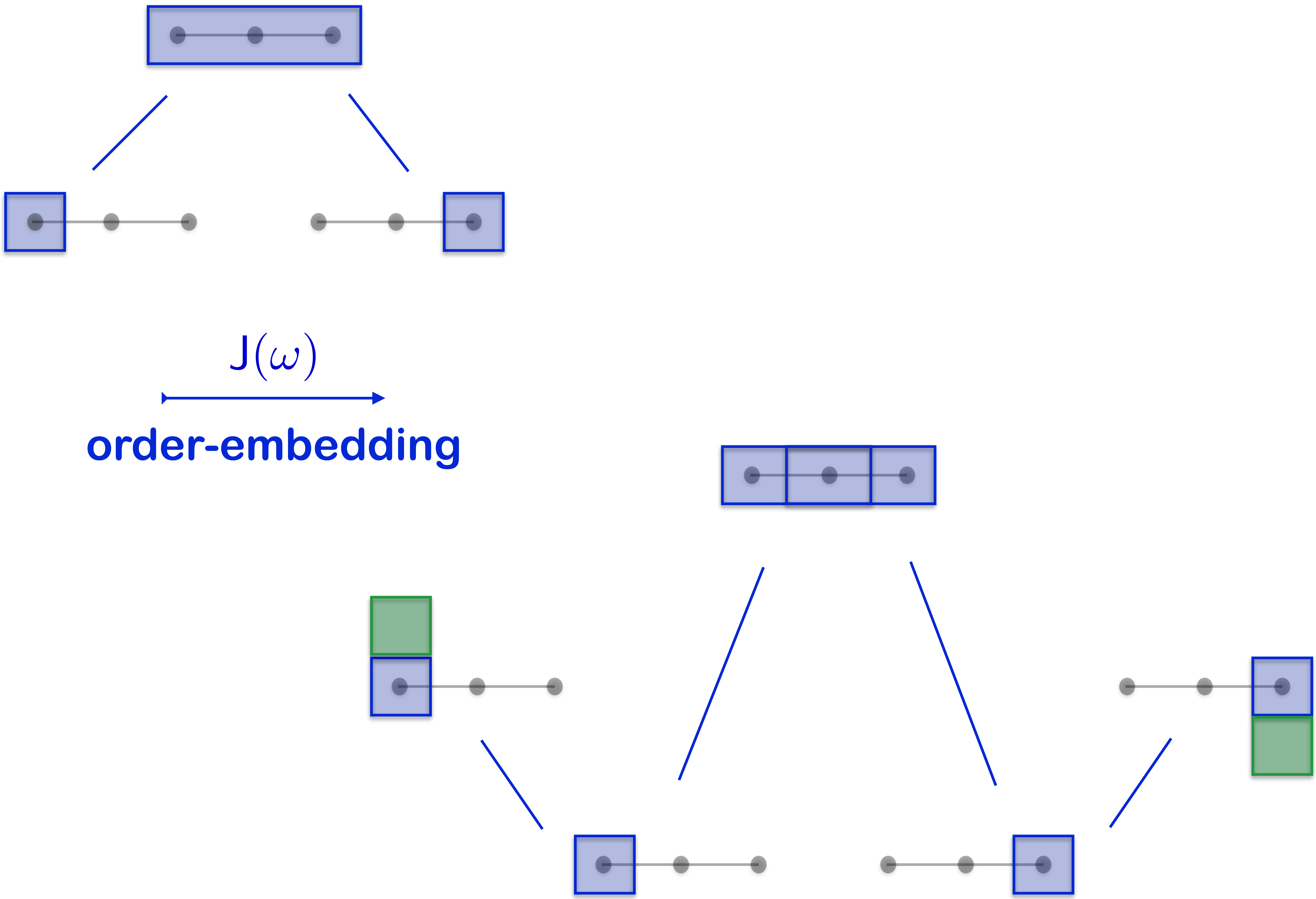


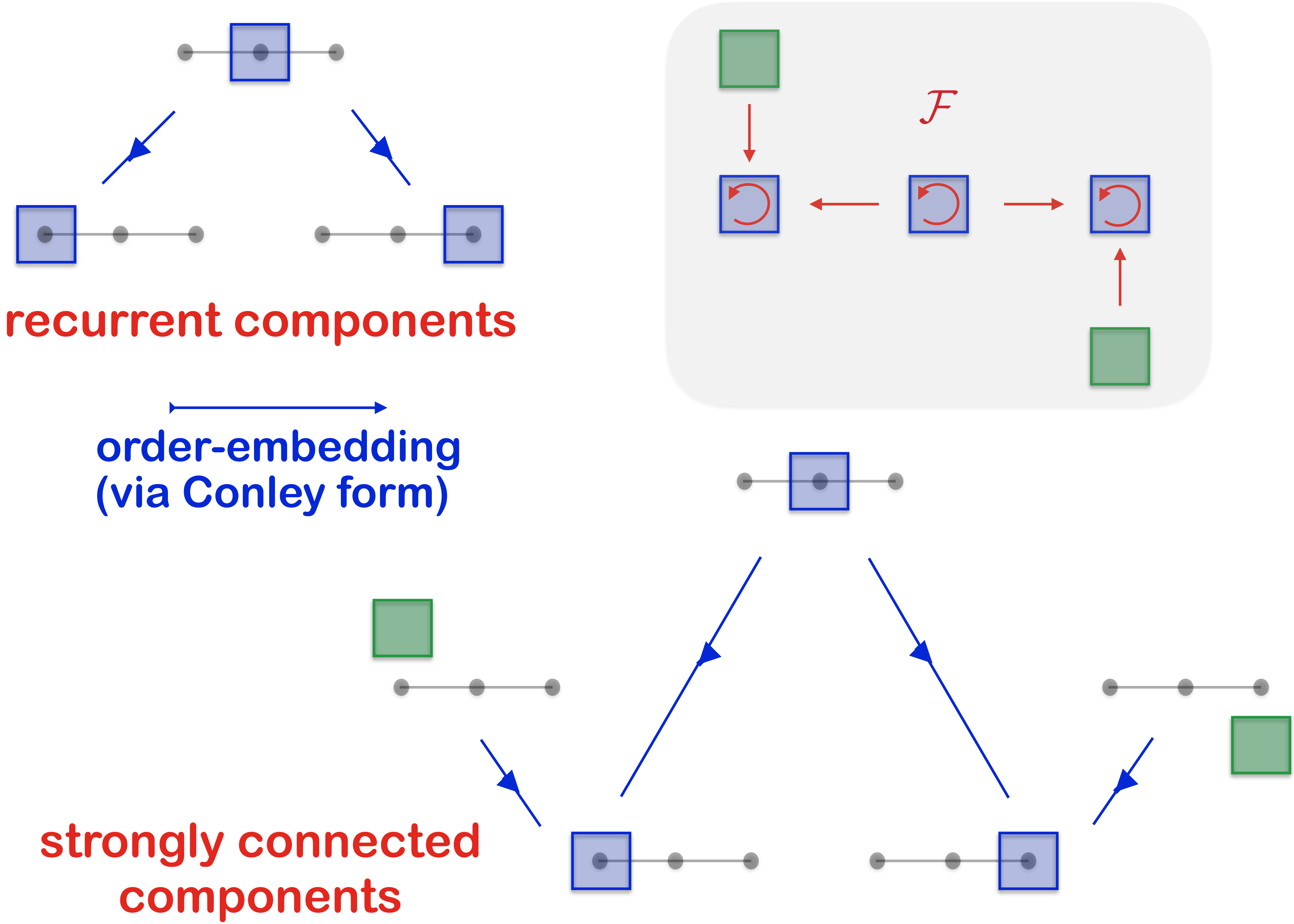
$$
\begin{array}{ccc}
\mathsf{Invset}^+(\mathcal{F}) & \overset{?}{\longleftrightarrow} & \mathsf{N} \hookrightarrow \mathsf{ABlock}_{\mathsf{R}}(X, f) \\
\downarrow \omega & & \downarrow \omega \\
\mathsf{Att}(\mathcal{F}) & \overset{?}{\longleftrightarrow} & \mathsf{A} \hookrightarrow \mathsf{Att}(X, f)
\end{array}
$$


Invset$^+(\mathcal{F})$
∅



$\mathsf{Invset}^+(\mathcal{F})$
ω
$\mathsf{Att}(\mathcal{F})$
$\varnothing$

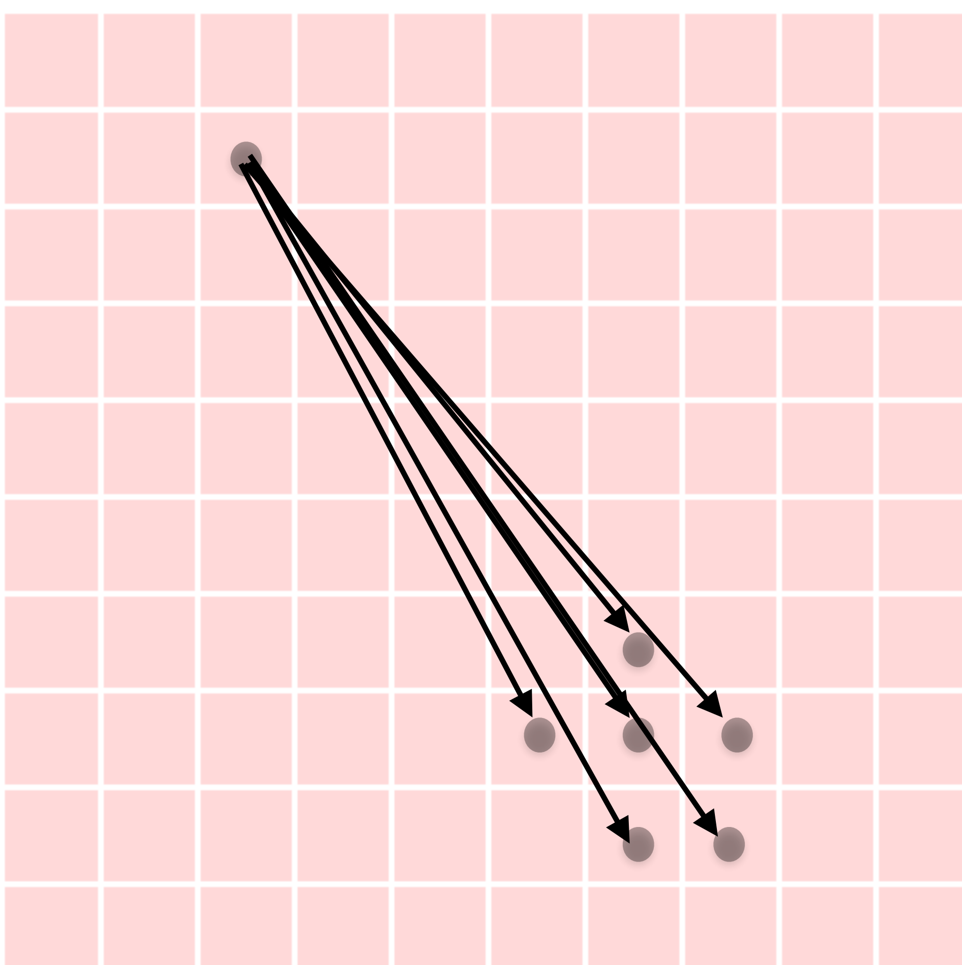

J(ω)
order-embedding

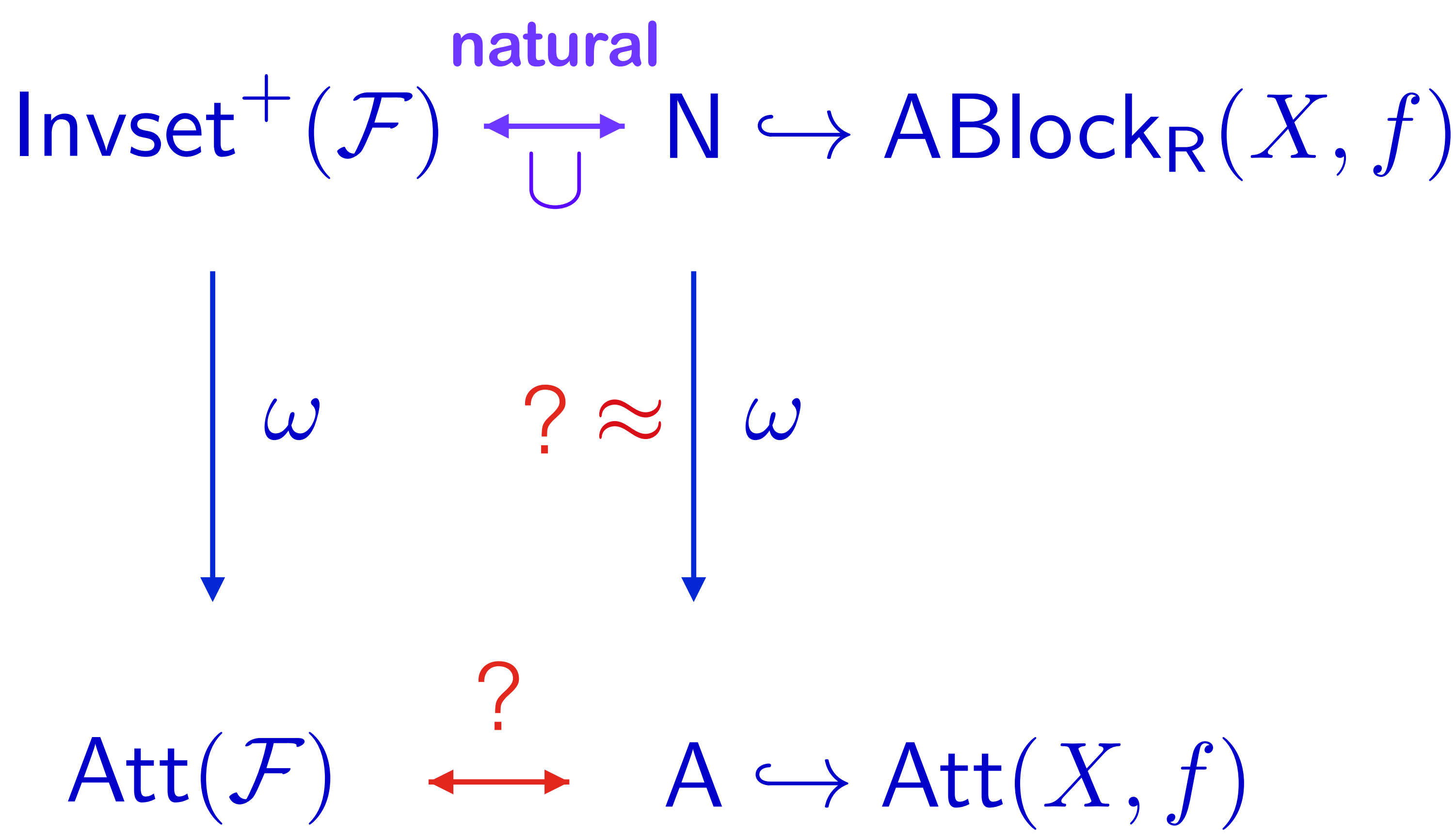


$\mathcal{F}$
recurrent components
order-embedding
(via Conley form)
strongly connected
components

There are linear time graph algorithms for computing the recurrent and strong components.

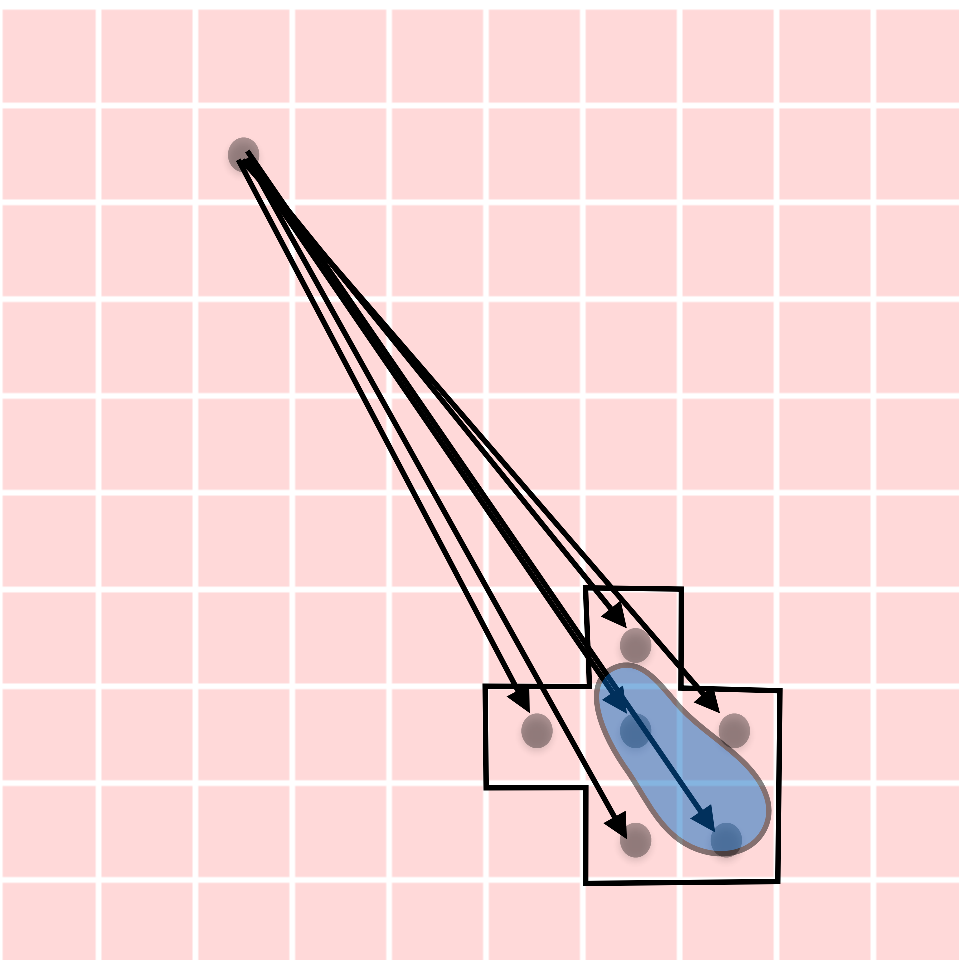
SO — reverse the question — if we start with an appropriate state transition graph, can we recover a lattice of attracting blocks that is isomorphic to a lattice of attractors?





A state transition graph $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ is an outer approximation of $f : X \to X$ if

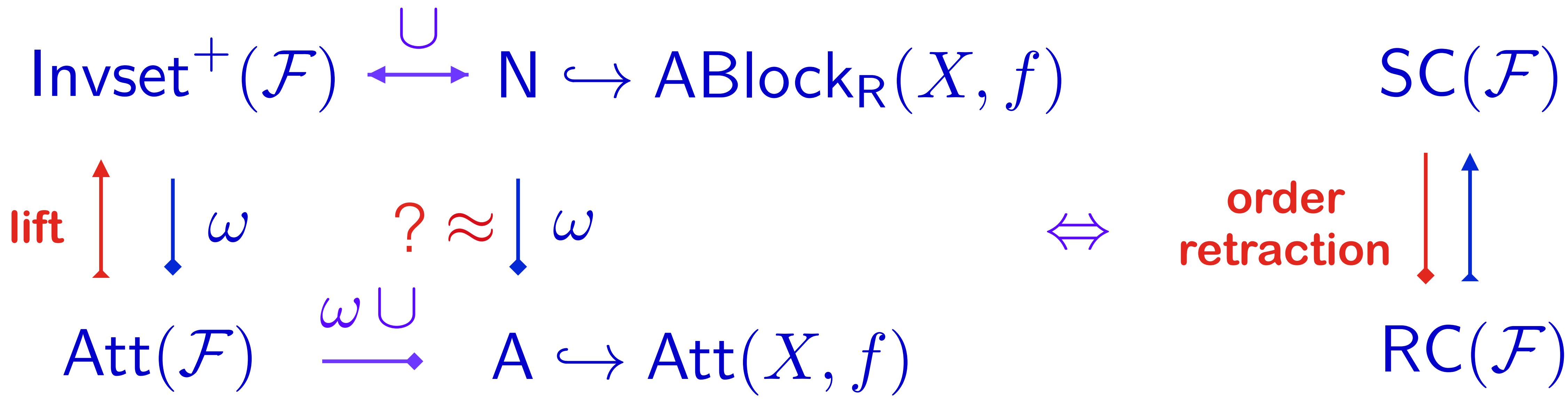
$$f(G) \subset \operatorname{int}(|\mathcal{F}(G)|) \ \ \forall G \in \mathcal{X}$$



If x_n is an orbit of f, then there exists a walk G_n of $\mathcal{F}$ with

$$x_n \in G_n$$

Hence an outer approximation does not mask any recurrent behavior.



Generally N is quite large compared to A. Is A isomorphic to a sublattice of N ?
Such an index lattice is equivalent to the existence of a isomorphic tessellated Morse decomposition $\mathsf{M}(\mathsf{A}) \leftrightarrow \mathsf{T}(\mathsf{N})$

Strategy: combine states to obtain a smaller sublattice via an order retraction.

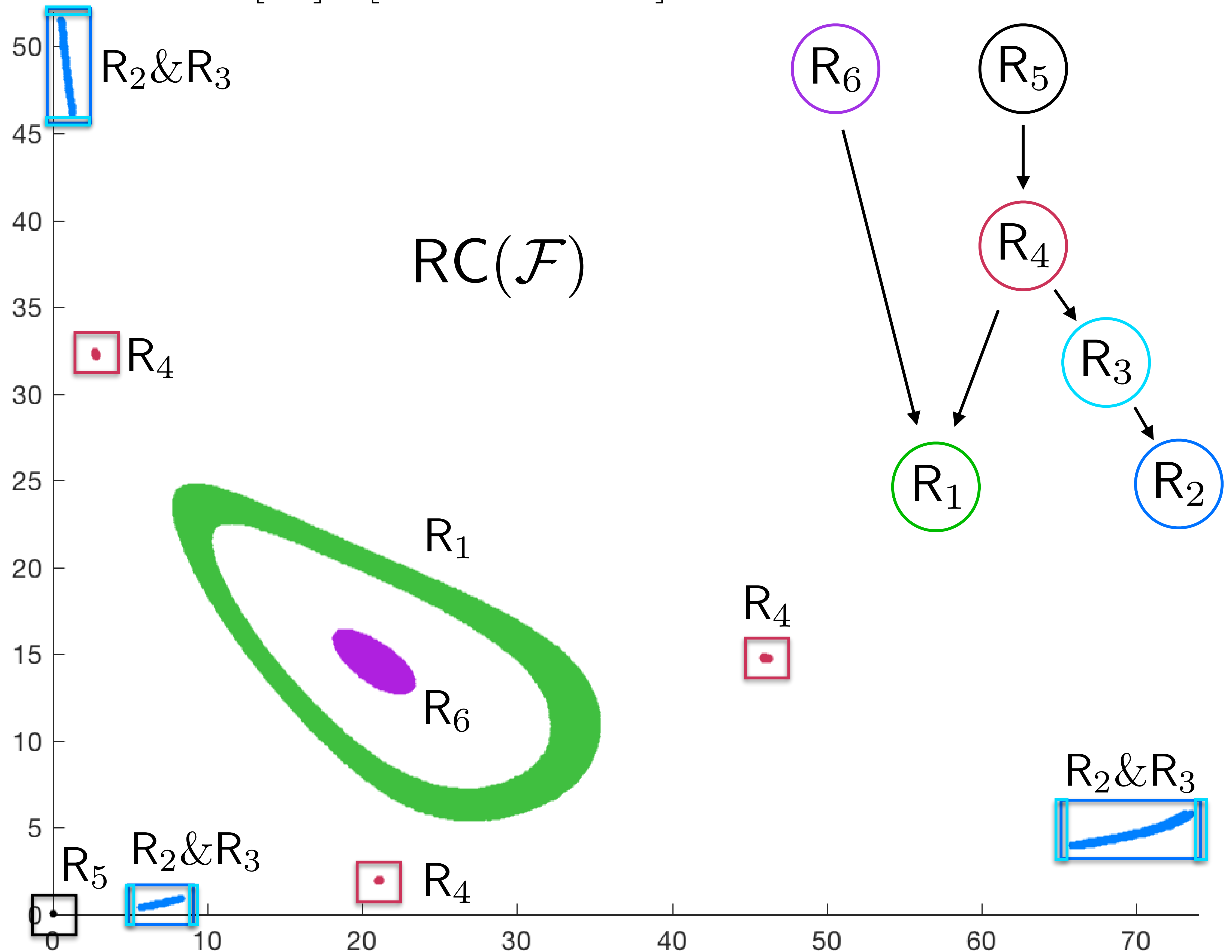
For a specific computation an order retraction / lift may not exist.

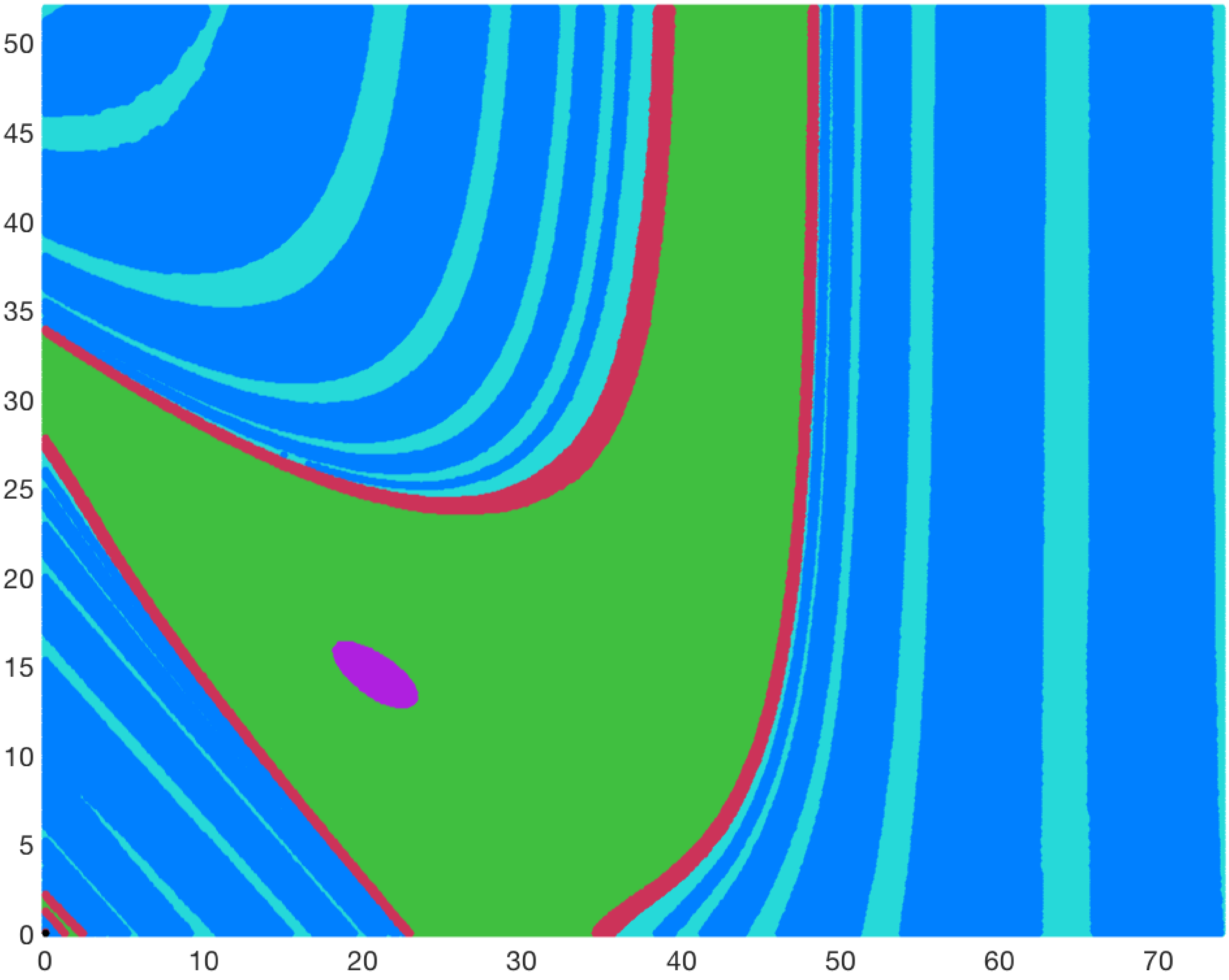
We have developed an algorithm to determine existence and the compute of an order retraction. (K. Kasti, Vandervorst)

Also, theoretically if the state grid is fine enough and the outer approximation is close enough to f, then an order retraction / lift exists.
(K., Mischaikow, Vandervorst)

A **may not be known!** $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N}) \approx \mathsf{RC}(\mathcal{F})$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x_1 + \theta_2 x_2)e^{-\phi(x_1+x_2)} \\ p x_1 \end{bmatrix} \quad \theta_1 = 20.0,\ \theta_2 = 20.0,\ \phi = 0.1, \text{ and } p = 0.7.$$





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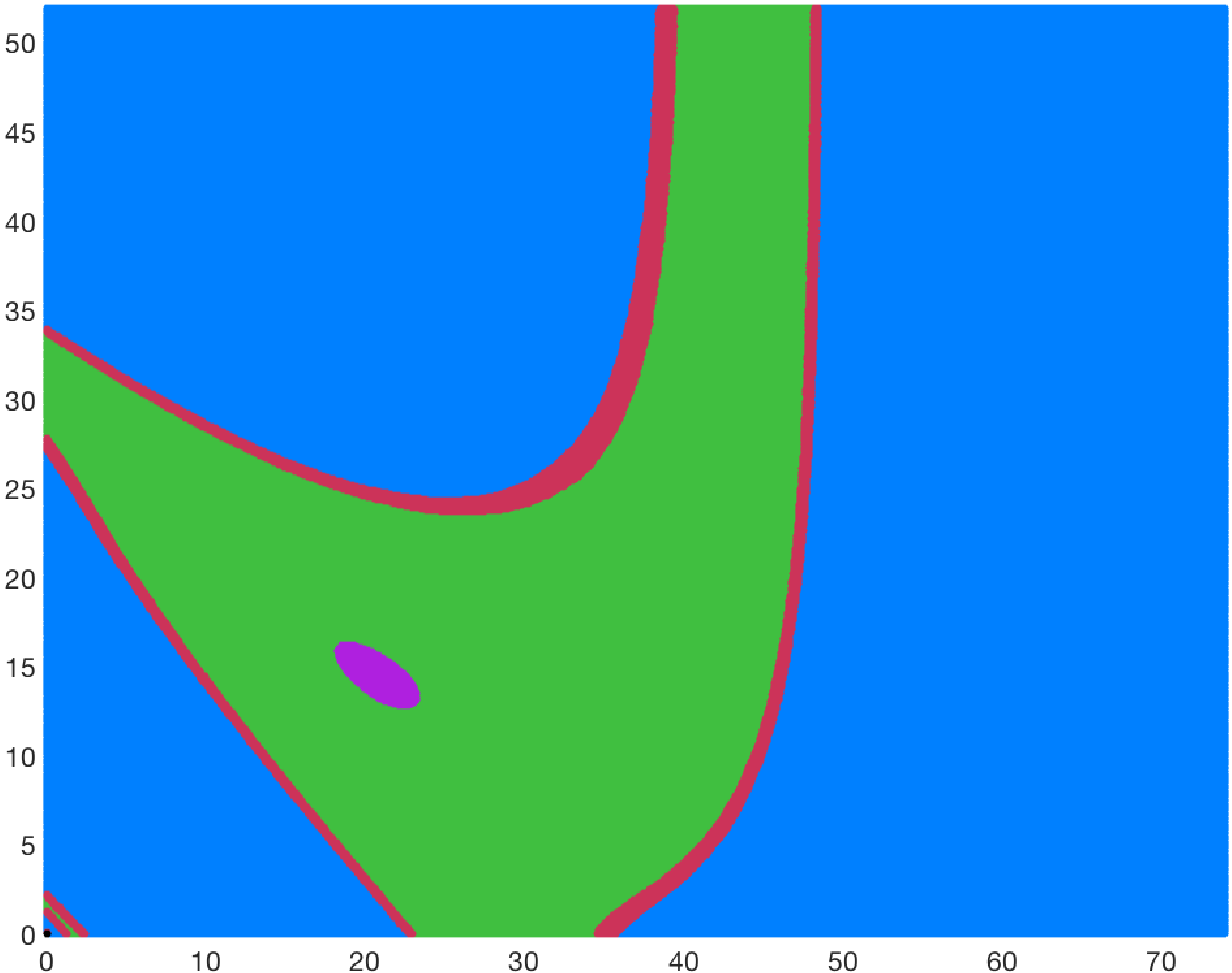

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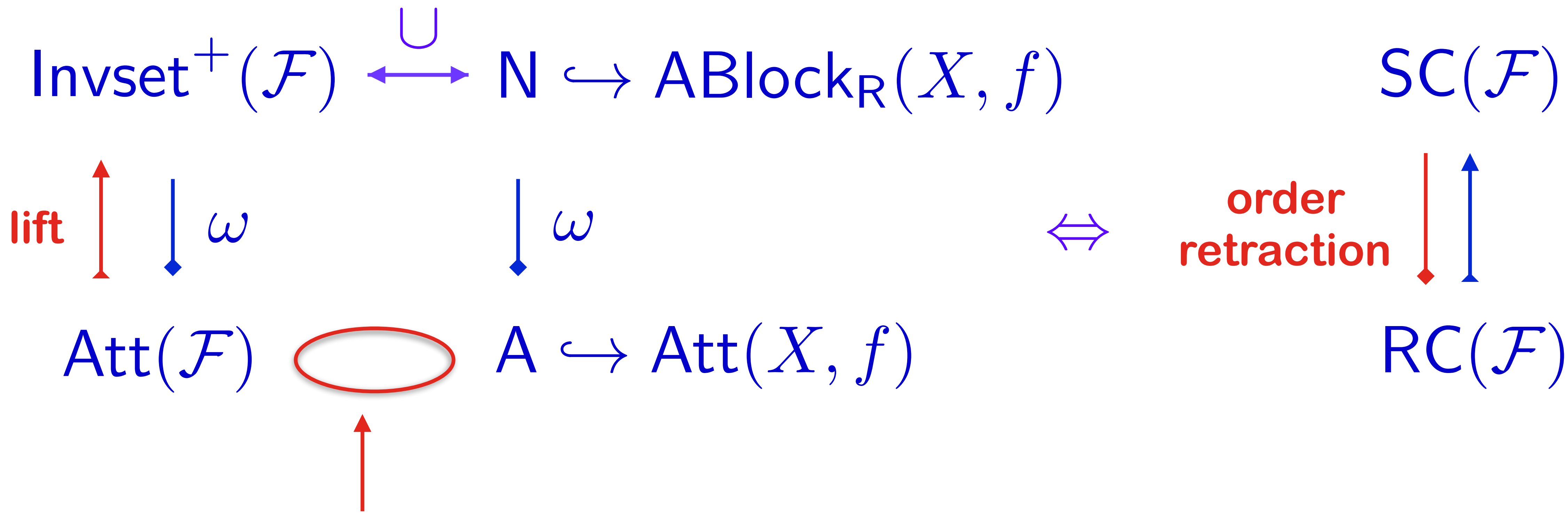
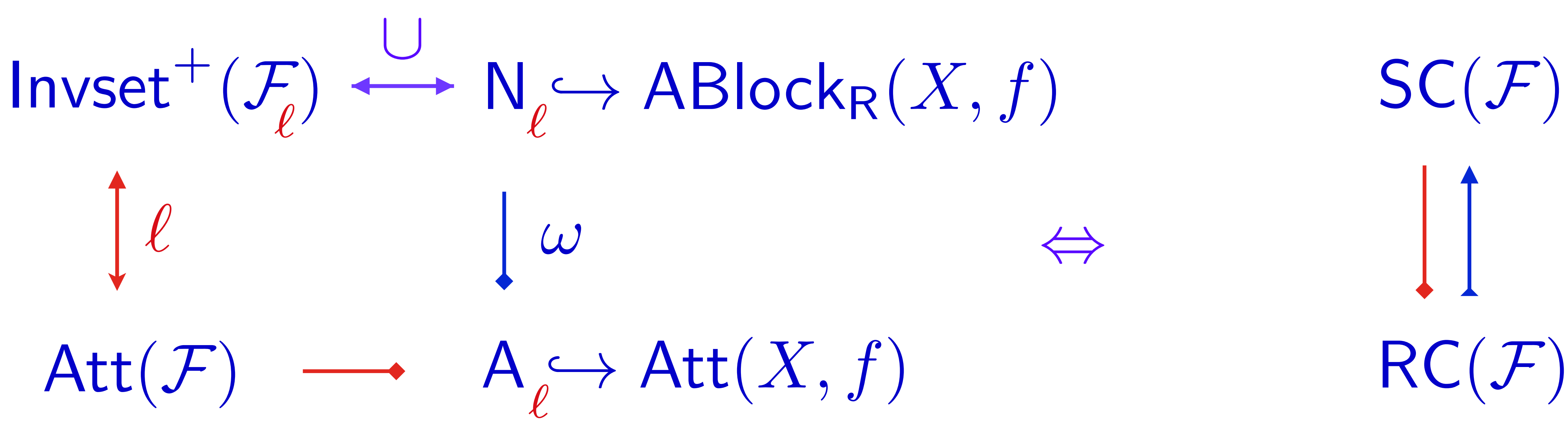


Suppose $\mathcal{F}$ is not a outer approximation.



$$\mathsf{M}(\mathsf{A}_\ell) \hookrightarrow \mathsf{T}(\mathsf{N}_\ell) \approx \mathrm{RC}(\mathcal{F})$$

Example: polygonal grid where vector field is transverse to the boundaries of the grid elements
(Bozcko, K., Mischaikow)

Parabolic recurrence vector fields
(Ghrist, van den Berg, Vandervorst)

Future work

Extract these structures from data?

Computational Conley theory

An algorithmic approach to chain recurrence (FoCM 2005)

Konstantin Mischaikow
Robert Vandervorst

An computational approach to Conley's decomposition theorem (JCND 2006)

Hyunju Ban

A database schema for the analysis of global dynamics of multi parameter systems (SIADS 2009)

Zin Arai, Hiroshi Kokubu, Konstantin Mischaikow,
Hiroe Oka, and Pawel Pilarczyk

Lattice structures

Lattice structures of attractors I - (J. Comp. Dyn. 2014)

Lattice structures of attractors II - (FoCM 2016)

Lattice structures of attractors III - (in preparation)

Konstantin Mischaikow
Robert Vandervorst

Dynamics and order theory - (in preparation)

Dinesh Kasti
Robert Vandervorst

Efficient computation of Lyapunov functions for Morse decompositions - (DCDS 2015)

Arnaud Goullet, Shaun Harker, Dinesh Kasti,
and Konstantin Mischaikow

Software

CHomP — http://chomp.rutgers.edu

Konstantin Mischaikow, Shaun Harker, …

CDS - Computational Dynamics Software

Kalies

Thank You!

Konstantin Mischaikow (Rutgers)
Robert Vandervorst (VU Amsterdam)
Dinesh Kasti (FAU)