Long time dynamics in two dimensional water waves

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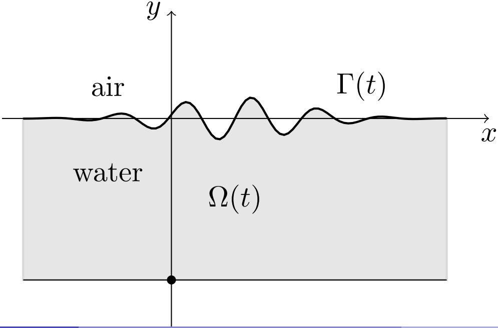
This is joint work with Mihaela Ifrim and, in part, with John Hunter and Benjamin Harrop-Griffith

Two dimensional fluids

The setting:

- inviscid incompressible fluid flow (Euler equations) in a fluid domain
  - governed by the incompressible Euler equations
  - with or without gravity
- finite or infinite bottom
  - no slip boundary condition for velocity on the bottom
- free boundary (the interface with air)
  - boundary condition for pressure on the top (surface tension, wind)



The standard formulation

Fluid domain: $\Omega(t)$, free boundary $\Gamma(t)$.
Velocity field u, pressure p, gravity g, surface tension σ.
Euler equations in $\Omega(t)$:

$$\begin{cases} u_t + u \cdot \nabla u = \nabla p - gj \\ \text{div } u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Boundary conditions on Γ_t:

$$\begin{cases} \partial_t + u \cdot \nabla \text{ is tangent to } \bigcup \Gamma_t & \text{(kinematic)} \\ p = -2\sigma \mathbf{H} \quad \text{on } \Gamma_t & \text{(dynamic)} \end{cases}$$

$\mathbf{H}$ = mean curvature of the boundary.

Irrotational flows (water waves)

Velocity potential

$$u = \nabla\phi, \qquad \Delta\phi = 0 \quad \text{in } \Omega_t$$

Dynamic boundary condition:

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + gy - \sigma\mathbf{H}(\eta) = 0 \qquad \text{on } \Gamma_t = \{y = \eta(x)\}$$

Equations reduced to the boundary in Eulerian formulation:

$$\begin{cases} \partial_t\eta - G(\eta)\phi = 0 \\ \partial_t\phi + g\eta - \sigma\mathbf{H}(\eta) + \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}\frac{(\nabla\eta\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = 0. \end{cases}$$

$$\mathbf{H}(\eta) = div\left(\frac{\nabla\eta}{1+|\nabla\eta|^2}\right), \quad G(\eta) = \text{ Dirichlet to Neumann operator}$$

Standard questions:

1. Obtain local well-posedness in Sobolev spaces
   - high regularity (via energy estimates)
   - low regularity (using also dispersion)
2. Understand asymptotic equations in various regimes
   - low frequency asymptotics
   - wave packet asymptotics
3. Study long time solutions (i.e. the stability of the trivial steady state) in two settings:
   - lifespan bounds for small data
   - global solutions for small localized data
4. Understand solitons and near soliton dynamics
   - existence of solitons
   - stability and asymptotic stability

Four interesting problems

1. **Gravity waves in deep water**[H,I,T](g)
   - infinite bottom, gravity, no surface tension (long waves)
   - no solitons
2. **Capillary waves in deep water**[I,T](t)
   - infinite bottom, surface tension, no gravity (short waves)
   - no small solitons
3. **Constant vorticity gravity waves in deep water**[I,T](v)
   - infinite bottom, no surface tension, gravity, constant vorticity (tides)
   - small solitons, Benjamin-Ono approximation
4. **Gravity waves in shallow water** [H-G,I,T](b)
   - finite bottom, no surface tension, gravity
   - small solitons, KdV approximation

Collaborators: Mihaela Ifrim (UC Berkeley), John Hunter (UC Davis), Benjamin Harrop-Griffiths (NYU)

Features

- Fully nonlinear evolutions of hyperbolic type
- either degenerate (g v b) or higher order (t)
- Nonlocal equations (due to Dirichlet to Neumann map)
- Hamiltonian structure
- conservation laws: energy (Hamiltonian), horizontal momentum, vertical momentum
- dispersive character (the speed of waves depends on the frequency)
- quadratic interactions
- few resonant quadratic interactions, with null structure at resonance
- symmetries: translation, scaling (g t)
- choice of coordinates = gauge freedom (e.g. Eulerian or Lagrangian)
- small solitons (v b)

Main prior contributions:

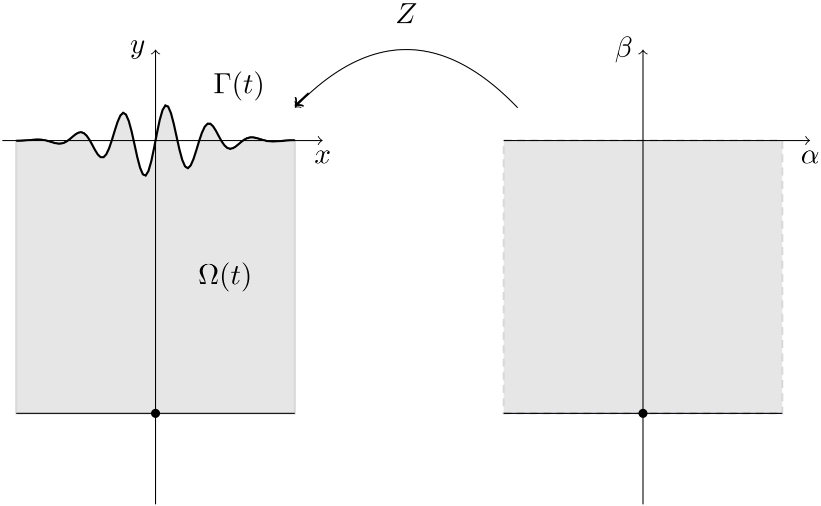
- Local well-posedness in smooth Sobolev spaces with nonzero vorticity: Lindblad-Christodoulou, Lannes, Lindblad, Coutand-Shkoller, Shatah-Zeng ...
- Local well-posedness for water waves: Nalimov '74, Yoshihara, Wu, Alazard-Burq-Zuily (low regularity)...
- Long time solutions for small localized data (g): Wu '09 (almost global) Ionescu-Pusateri '13, (global), Alazard-Delort '13 (global)
- Global solutions for small localized data in 3-d: Germain-Masmoudi-Shatah '10(g) , '12 (t), Wu '10 (g).

Three key ideas

A. Holomorphic coordinates. The aim is to work in coordinates where the problem is as simple as possible. Holomorphic coordinates (Nalimov '74, Zakharov, Wu '98) diagonalize the Dirichlet to Neumann map, and thus provide a simpler approach to the local problem.

B. The (quasilinear) modified energy method. This is a quasilinear improvement on Shatah's normal form method, which yields an easier route to long time solutions.

C. Wave packet testing. The aim here is to develop a more robust and simpler way, based on wave-packets, in order to capture modified scattering asymptotics and to obtain global solutions.



The conformal map

Holomorphic coordinates:

Holomorphic coordinates:

$$Z : \{\Im z \leq 0\} \to \Omega_t, \qquad \alpha + i\beta \to Z(\alpha + i\beta)$$

Boundary condition at infinity:

$$Z(\alpha) - \alpha \to 0 \ \ (nonperiodic) \qquad Z(\alpha) - \alpha \ periodic \ \ (periodic)$$

Free boundary parametrization:

$$Z : \mathbb{R} \to \Omega_t, \qquad \alpha \to Z(\alpha)$$

Perturbation of steady state:

$$W = Z - \alpha$$

Holomorphic velocity potential ($v = \nabla \phi$):

$$Q = \phi + i\psi$$

Water wave equations in holomorphic coords.

- P - Projection onto negative wavenumbers

Fully nonlinear equations for *holomorphic* variables $(W = Z - \alpha, Q)$:

$$\begin{cases} W_t + F(1 + W_\alpha) = 0, \\ Q_t + FQ_\alpha + P\left[\dfrac{|Q_\alpha|^2}{J}\right] - igW + i\sigma P\left[\dfrac{W_{\alpha\alpha}}{J^{1/2}(1+W_\alpha)} - \dfrac{\bar{W}_{\alpha\alpha}}{J^{1/2}(1+\bar{W}_\alpha)}\right] = 0. \end{cases}$$

where

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J = |1 + W_\alpha|^2.$$

Conserved energy (Hamiltonian):

$$E(W,Q) = \int \Im(Q\bar{Q}_\alpha) + \frac{1}{2}g\left(|W|^2 - \Re(\bar{W}^2 W_\alpha)\right) + \frac{1}{4}\sigma(J^{\frac{1}{2}} - 1 - \Re W_\alpha)\, d\alpha$$

Symmetries:

- Translations in α and t.
- Scaling (g)(t): $(W(t,\alpha), Q(t,\alpha)) \to (\lambda^{-2}W(\lambda^3 t, \lambda^2\alpha), \lambda^{-3}Q(\lambda^3 t, \lambda^2\alpha))$

Physical parameters:

- $R = \dfrac{Q_\alpha}{1+W_\alpha}$ - velocity field on the free boundary
- $b = 2\Re P\left[\dfrac{R}{1+\bar{W}_\alpha}\right]$ - advection coeff. (high freq. velocity limit)
- $a = 2\Im P[R\bar{R}_\alpha] > 0$ - normal derivative of the pressure $= g + a$

Alternate quasilinear system for diagonal variables ($\mathbf{W} = W_\alpha, R$):

$$\begin{cases} \mathbf{W}_t + P[b\mathbf{W}_\alpha] + P\left[\dfrac{(1+\mathbf{W})R_\alpha}{1+\bar{\mathbf{W}}}\right] = G \\ R_t + P[bR_\alpha] + iP\left[\dfrac{g+a}{1+\mathbf{W}}\right] + i\sigma \dfrac{1}{1+\mathbf{W}} P\left[\dfrac{\mathbf{W}_\alpha}{J^{1/2}(1+\mathbf{W})}\right]_\alpha = K, \end{cases}$$

where

$$G = (1+\mathbf{W})P\left[\frac{\bar{R}_\alpha}{1+\mathbf{W}} + \frac{R\bar{\mathbf{W}}_\alpha}{(1+\bar{\mathbf{W}})^2}\right] + [P, \mathbf{W}]\left(\frac{R_\alpha}{1+\mathbf{W}} + \frac{\bar{R}\mathbf{W}_\alpha}{(1+\mathbf{W})^2}\right)$$

$$K = -P\left[R\bar{R}_\alpha\right] + i\sigma P\left[\frac{\bar{\mathbf{W}}_\alpha}{J^{1/2}(1+\bar{\mathbf{W}})}\right]_\alpha$$

represent perturbative terms in the equation.

Low regularity local well-posedness

Theorem

a) The equations (g), (v) and (b) are locally well-posed for data $(\mathbf{W}_0, R_0) \in H^1 \times H^{\frac{3}{2}}$*.*
b) The equation (t) is locally well-posed for data $(\mathbf{W}_0, R_0) \in H^2 \times H^{\frac{3}{2}}$*.*

- Earlier results with more regularity by Wu, Alazard-Burq-Zuily
- Both results are 1/2 derivative above scaling.
- Very recent improvements by Alazard-Burq-Zuily using dispersion, based on ideas developed for nonlinear wave equations by T., Bahouri-Chemin, Smith-T.
- The proof is based on sharp, scale invariant energy estimates both for the equations and for their linearizations, plus an iterative scheme to construct solutions.
- Key difficulty: The equations are degenerate hyperbolic, and all analysis must be carried out for diagonal variables $(\mathbf{W}, R)$ (Alihnac, Lannes).

Normal forms and long time existence

Question: Obtain improved lifespan estimates for small data solutions.

(i) Equations with quadratic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\| E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan $T_\epsilon \approx \epsilon^{-1}$

(ii) Equations with cubic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\|^2 E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan $T_\epsilon \approx \epsilon^{-2}$

(iii) Normal form method (Shatah '85): transform an equation with a quadratic nonlinearity into one with a cubic one via a normal form transformation,

$$u \to v = u + B(u,u) + \text{higher}$$

Normal forms for quasilinear equations

Key difficulty: The normal form transformation $v = u + B(u,u)$ involves the leading part of the equation, and thus is unbounded and not invertible. One is left with two seemingly incompatible estimates:
(i) A quadratic energy bound

$$\frac{d}{dt} E_k^{nl}(u) \lesssim A E_k^{nl}(u)$$

(ii) A cubic normal form bound

$$\frac{d}{dt} E_k^{lin}(v) \lesssim A^2 E_{k+1}(u)$$

where A is a good linear lower order control norm for u. Further, the norms $E_k^{nl}(u)$ and $E_k^{lin}(v)$ are likely not equivalent.

Question: How to combine the two in a favorable way ?

The modified energy method

Goal: Rather than using a normal form to eliminate quadratic terms in the equation, construct a modified energy for the original equation, compatible with both the quasilinear and the normal form structure,

$$\frac{d}{dt} E_k^{nl,3}(u) \lesssim A^2 E_k^{nl,3}(u)$$

Algorithm to construct the modified energy:

STEP 1: Start with a cubic normal form energy,

$$E_k^{NF,3}(u) = (\text{quadratic} + \text{cubic}) E_k^{lin}(v)$$

STEP 2: Split $E_k^{NF,3}(u)$ into a high frequency part and lower order terms,

$$E_k^{NF,3}(u) = E_{k,high}^{NF,3}(u) + E_{k,low}^{NF,3}(u)$$

STEP 3: Match to cubic order $E_{k,high}^{NF,3}(u) = E_k^{nl}(u) + \text{quartic}$, and define

$$E_k^{nl,3}(u) = E_k^{nl}(u) + E_{k,low}^{NF,3}(u)$$

Cubic lifespan bounds

Theorem

Consider the two dimensional differentiated water wave equation with initial data of size ϵ. Then the solutions have a lifespan of at least

$$T_\epsilon \approx \epsilon^{-2}$$

- The result applies to all four models (g), (v) (t) and (b).
- The result applies equally in periodic and non-periodic setting.
- The regularity of the data is same as in the LWP result.
- Proof idea: *quasilinear modified energy method*
- Bounds for all higher norms propagate on same timescale.
- Additional difficulty for water waves, due to the fact that the system is degenerate hyperbolic. Because of this, the modified energy needs to be in the diagonal variables.

Normal forms for water waves

Resonant interactions: two waves interact to produce a third wave. In general, Shatah's normal form exists if there are no resonant two wave interactions. Else, the normal form will have singularities at resonances.

Null condition: The symbol of the quadratic nonlinearities vanishes at resonances. This may allow the normal form to still be nonsingular.

Resonant interactions in water wave models: all four models (g), (v) (t) and (b) allow resonant interactions iff one of the three frequencies (two inputs, one output) is zero.

Null condition in water wave models: Full fledged for (g), (t). Partial only for (v) (b). In the latter case, the normal form has some singularity at frequency zero (i.e. ∂^{-1}); however, there is enough structure so that the singularity cancels in the normal form energies.

Example: Normal forms for gravity waves (g)

Linearization around zero:

$$\begin{cases} \partial_t w + r_\alpha = 0 \\ \partial_t r - igw = 0 \end{cases}$$

Dispersion relation:

$$\tau = \pm\sqrt{g|\xi|}, \qquad \xi \leq 0$$

Group velocity of waves:

$$v = \pm\frac{\sqrt{g}}{2\sqrt{|\xi|}}$$

Bilinear resonant interactions:

$$|\xi|^{\frac{1}{2}} \pm |\eta|^{\frac{1}{2}} \pm |\xi+\eta|^{\frac{1}{2}} = 0 \quad \Longrightarrow \quad \xi\eta(\xi+\eta) = 0$$

No resonances away from frequency zero ! Normal form:

$$\tilde{W} = W - 2P[\Re W W_\alpha],$$
$$\tilde{Q} = Q - 2P[\Re W Q_\alpha].$$

Resonance analysis for capillary waves (t)

Linearization around zero:

$$\begin{cases} \partial_t w + r_\alpha = 0 \\ \partial_t r + i\sigma w_{\alpha\alpha} = 0 \end{cases}$$

Dispersion relation:

$$\tau = \pm\sqrt{\sigma}|\xi|^{\frac{3}{2}}, \qquad \xi \leq 0$$

Group velocity of waves:

$$v = \pm\frac{3}{2}\sqrt{\sigma|\xi|}$$

Bilinear resonant interactions:

$$|\xi|^{\frac{3}{2}} \pm |\eta|^{\frac{3}{2}} \pm |\xi+\eta|^{\frac{3}{2}} = 0 \quad \Longrightarrow \quad \xi\eta(\xi+\eta) = 0$$

No resonances away from frequency zero ! Normal form:

$$\tilde{W} = W + L_1(W,W) + L_0(Q,Q)$$
$$\tilde{Q} = Q + L_1(W,Q)$$

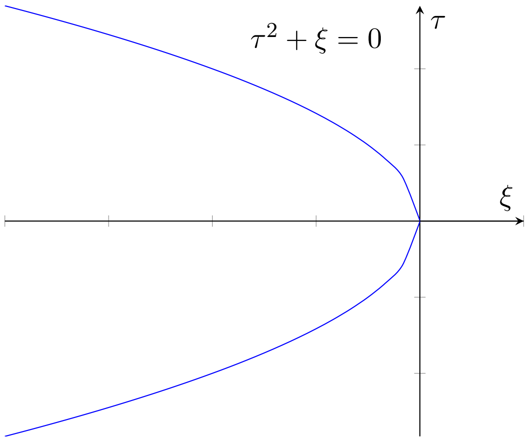


Figure : Dispersion relation (g)

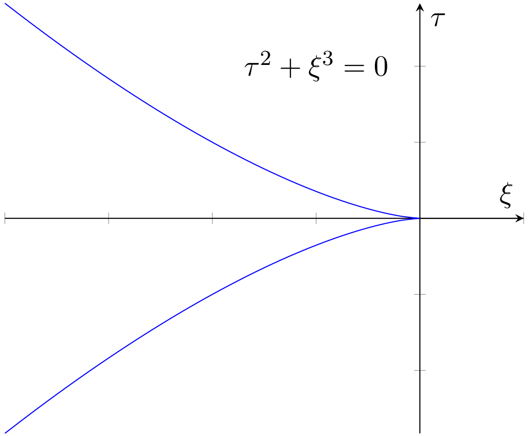


Figure : Dispersion relation (t)

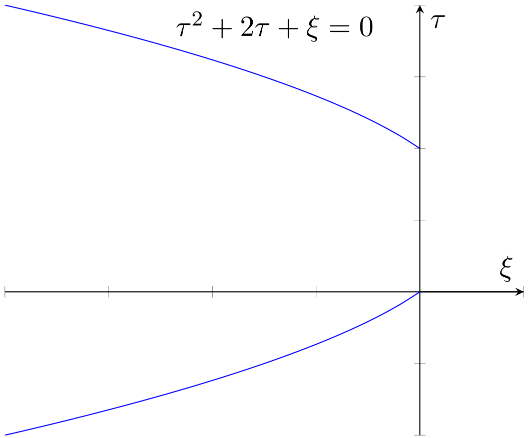


Figure : Dispersion relation (v)

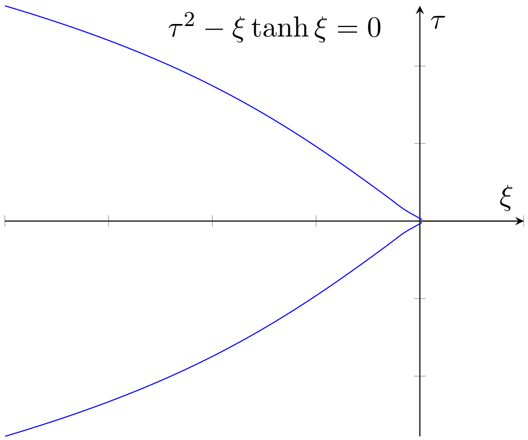


Figure : Dispersion relation (b)

Energy estimates for gravity waves

We use the control norms:

$$A = \|\mathbf{W}\|_{L^\infty} + \|\mathbf{W}(1+\mathbf{W})^{-1}\|_{L^\infty} + \|D^{\frac{1}{2}} R\|_{L^\infty}, \qquad \text{(scale invariant)}$$

$$B = \|D^{\frac{1}{2}}\mathbf{W}\|_{BMO} + \|R_\alpha\|_{BMO} \qquad \text{(controlled by } \|(\mathbf{W}, R)\|_{\dot{\mathcal{H}}^1}\text{)}$$

and construct energy functionals $E^n(\mathbf{W}, R)$ with these properties:

(i) Energy equivalence:

$$E^n(\mathbf{W}, R) = (1 + O(A))\|(\mathbf{W}, R)\|^2_{\dot{\mathcal{H}}^n}$$

(ii) Cubic scale invariant energy estimate:

$$\frac{d}{dt} E^n(\mathbf{W}, R) \lesssim_A AB E^n(\mathbf{W}, R)$$

Similar $\dot{\mathcal{H}}^0$ bound for linearized eqn $(w, r) = (w, q - Rw)$.

Energy estimates for capillary waves

We use the control norms:

$$A = \|\mathbf{W}\|_{L^\infty} + \|\mathbf{W}(1+\mathbf{W})^{-1}\|_{L^\infty} + \|D^{-\frac{1}{2}} R\|_{L^\infty}, \qquad \text{(scale invariant)}$$

$$B = \|D^{\frac{3}{2}}\mathbf{W}\|_{BMO} + \|R_\alpha\|_{BMO} \qquad \text{(controlled by } \|(\mathbf{W}, R)\|_{\dot{\mathcal{H}}^2}\text{)}$$

and construct energy functionals $E^n(\mathbf{W}, R)$ with these properties:

(i) Energy equivalence:

$$E^n(\mathbf{W}, R) = (1 + O(A))\|(\mathbf{W}, R)\|^2_{\dot{\mathcal{H}}^n}$$

(ii) Cubic scale invariant energy estimate:

$$\frac{d}{dt} E^n(\mathbf{W}, R) \lesssim_A AB E^n(\mathbf{W}, R)$$

Water waves on the real line: heuristics #1

Dispersive decay for the linear equation

$$\begin{cases} W_t + Q_\alpha = 0 \\ Q_t - iW = 0 \end{cases}$$

with smooth localized data of size ϵ:

$$|W| + |Q| \lesssim \epsilon t^{-\frac{1}{2}}$$

This shows that one would expect our control norms to decay like

$$A, B \approx \epsilon t^{-\frac{1}{2}}$$

Hence

$$\frac{d}{dt} E \lesssim \epsilon^2 t^{-1} E$$

which provides uniform bounds up to an exponential time

$$T_\epsilon = e^{c\epsilon^{-2}}$$

Water waves on the real line: heuristics #2

$$\begin{cases} \tilde{W}_t + \tilde{Q}_\alpha = cubic(\tilde{W}, \tilde{Q}) + higher \\ \tilde{Q}_t - i\tilde{W} = cubic(\tilde{W}, \tilde{Q}) + higher \end{cases}$$

Based on the dispersion relation and the group velocity of waves

$$\tau = \pm\sqrt{|\xi|}, \qquad v = \pm \frac{1}{2\sqrt{|\xi|}}$$

for small data localized near 0, one expects asymptotics

$$(\tilde{W}, \tilde{Q}) \approx \gamma(t, \alpha/t) t^{-\frac{1}{2}} e^{\frac{it^2}{4\alpha}} \left(\frac{t}{2\alpha}, 1 \right)$$

with a better function γ. Substituting in the above equations, we get the asymptotic evolution (modified scattering)

$$\dot{\gamma}(t, v) = ic(v) t^{-1} \gamma(t, v) |\gamma(t, v)|^2 + O(\epsilon t^{-1-}), \qquad \gamma(1, \alpha) = O(\epsilon)$$

The global boundedness of gamma follows.

New tool: scaling symmetry

Scaling symmetry for gravity waves:

$$(W(t,\alpha), Q(t,\alpha)) \to (\lambda^{-2} W(\lambda t, \lambda^2 \alpha), \lambda^{-3} Q(\lambda t, \lambda^2 \alpha))$$

Generator:

$$\mathbf{S}(W,Q) = ((S-2)W, (S-3)Q), \qquad S = t\partial_t + 2\alpha\partial_\alpha$$

$\mathbf{S}(W,Q)$ solves the linearized equation, good energy estimates.

Scaling symmetry for capillary waves:

$$(W(t,\alpha), Q(t,\alpha)) \to (\lambda^{-1} W(\lambda^{\frac{3}{2}} t, \lambda\alpha), \lambda^{-\frac{1}{2}} Q(\lambda^{\frac{3}{2}} t, \lambda\alpha)).$$

Generator:

$$\mathbf{S}(W,Q) = ((S-1)W, (S-\frac{1}{2})Q), \qquad S = \frac{3}{2} t\partial_t + \alpha\partial_\alpha.$$

$\mathbf{S}(W,Q)$ solves the linearized equation, good energy estimates.

Global water waves for small localized data

Theorem

Assume that the initial data for the water wave equation (g) or (t) has size

$$\|(W,Q)(0)\|_{\mathcal{WH}} \lesssim \epsilon$$

Then the solution exists globally in time, with energy bounds

$$\|(W,Q)(t)\|_{\mathcal{WH}} \lesssim \epsilon t^{C\epsilon^2}$$

and pointwise decay

$$A(t) + B(t) \lesssim \frac{\epsilon}{\sqrt{t}}$$

- $\mathcal{WH}$ is a (time dependent) weighted localized Sobolev norm which contains 6 respectively 10 derivatives and one momentum.
- (g): Original almost global result by Wu, and global result by Ionescu-Pusateri and Alazard-Delort (with longer, more involved proofs and with much more regularity).

Bootstrap argument, starting with $t^{-\frac{1}{2}}$ decay

1. Energy estimates:

$$\|(W,Q)(t)\|_{\mathcal{WH}} \lesssim e^{\int_0^T AB(s)ds}\|(W,Q)(0)\|_{\mathcal{WH}} \lesssim \epsilon t^{C\epsilon^2}$$

- based on the cubic energy functionals

2. Initial pointwise bounds:

$$(A+B)(t) \lesssim t^{-\frac{1}{2}}\omega(t,\alpha/t)\|(W,Q)(t)\|_{\mathcal{WH}}, \quad \omega(t,v) = t^{-\frac{1}{18}} + (v+v^{-1})^{-\frac{1}{2}}$$

- akin to the Klainerman-Sobolev inequalities, only harder
- uses the normal form variables
- better decay for v away from 1, reduces problem to $t^{-\delta} < v < t^{\delta}$.

3. Final pointwise bounds:

$$A(t) + B(t) \lesssim \epsilon t^{-\frac{1}{2}}$$

- Uses the method of *testing with wave-packets* to derive an asymptotic equation.

Asymptotic equations for NLS: $iu_t + \Delta u = \pm u|u|^2$

A. Hayashi-Naumkin, refined by Kato-Pusateri; derive an asymptotic equation for the Fourier transform of the solutions,

$$\frac{d}{dt}\hat{u}(t,\xi) = \lambda i t^{-1}\,\hat{u}(t,\xi)|\hat{u}(t,\xi)| + O_{L^\infty}(t^{-1-\epsilon}).$$

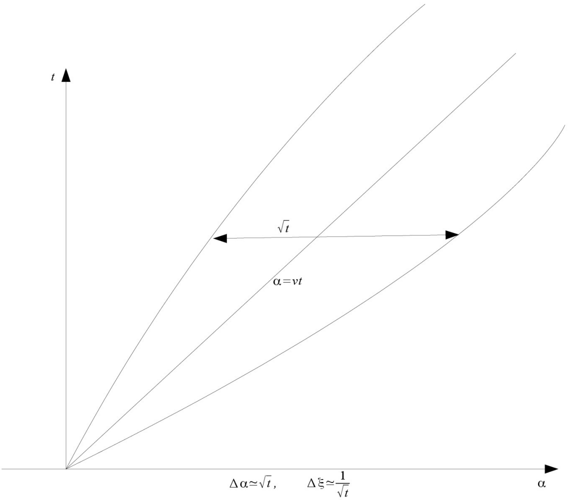
B. Lindblad-Soffer; derive an asymptotic equation in the physical space along rays $x = vt$,

$$(t\partial_t + x\partial_x)u(t,x) = \lambda i t u(t,x)|u(t,x)|^2 + O_{L^\infty}(t^{-\epsilon}).$$

C. Deift-Zhou used complete integrability and the inverse scattering method to obtain long range asymptotics

D. Ifrim-T. *wave packet testing*: test the NLS solution with an approximate wave packet type linear wave.

Wave Packets



Testing by wave packets

Wave packets:

$$\mathbf{q} = v\chi\left(\frac{v(\alpha - vt)}{\sqrt{t}}\right)e^{\frac{it^2}{4\alpha}}, \qquad \mathbf{w} = -i\mathbf{q}_t$$

The γ function:

$$\gamma(t, v) = \langle (\tilde{W}, \tilde{Q}), (\mathbf{w}, \mathbf{q}) \rangle_{\dot{\mathcal{H}}_0}$$

Good approximation (also for derivatives):

$$(\tilde{W}, \tilde{Q}) = \gamma(t, \alpha/t)t^{-\frac{1}{2}}e^{\frac{it^2}{4\alpha}}\left(\frac{t}{2\alpha}, 1\right) + O(t^{-\frac{5}{8}})$$

Asymptotic equation for γ:

$$\dot{\gamma}(t, v) = ic(v)t^{-1}\gamma(t, v)|\gamma(t, v)|^2 + O(\epsilon t^{-1-}), \qquad \gamma(1, \alpha) = O(\epsilon)$$

Thank you !